The uncertainty principle for energy and time

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(Received 29 January 1996; accepted 10 June 1996)

It is generally thought desirable that quantum theory entail an uncertainty relation for time and energy similar to the one for position and momentum. Nevertheless, the existence of such a relation has still remained problematic. Here, it is shown that the problem is due to a confusion between the position coordinates of a point particle (a material system) and the coordinates of a point in space: The time coordinate should be put on a par with the space coordinates, not with the canonical position coordinates of a material system. Whereas quantum mechanics incorporates a Heisenberg uncertainty relation between the canonical position coordinates and their conjugate momenta, there is no reason why a Heisenberg relation should hold between the space coordinates and the canonical momenta, or between the time coordinate and the energy of the system. However, uncertainty relations of a different kind exist between the space coordinates and the total momentum of the system and between the time coordinate and the total energy. These relations are completely similar and may be taken together to form a relativistically covariant set of uncertainty relations. The relation between the time coordinate and the energy implies the well-known relation between the lifetime of a state and its energy spread.

I. INTRODUCTION

Ever since Heisenberg published his famous article[1] on the Uncertainty Principle, the uncertainty relation between energy and time has been puzzling. It is the purpose of the present paper to solve this puzzle by exposing the confusions from which it arose and by showing how a generally valid uncertainty relation for energy and time can be derived.

In his article, Heisenberg sought to clarify the physical meaning of the quantum mechanical commutation relations, in particular the one between position and momentum, by analyzing how these quantities are measured in an actual physical experiment. He thus arrived at the first formulation of the uncertainty principle for position and momentum. In the same article Heisenberg also dealt with the case of energy and time and one finds the following statement[2]: "In a definite 'state' of the atom, the phases are in principle indeterminate, as one can see as a direct consequence of the familiar equations

$$\mathbf{E}t-t\mathbf{E}=-i\hbar \quad \text{or} \quad \mathbf{J}\mathbf{w}-\mathbf{w}\mathbf{J}=-i\hbar, \tag{1}$$

where $\mathbf{J}$ is the action variable and $\mathbf{w}$ is the angle variable." This single sentence carries with it much of the confusion about the uncertainty principle for energy and time. For, first, it suggests that there exists or should exist an operator for time in quantum mechanics, and, second, it mixes up energy and time with action and angle variables.

The extent of this confusion may be read off from several passages in the book of John von Neumann, the man who gave quantum mechanics a sound mathematical basis. Whereas, as we shall see, there is no reason why (coordinate) time should be an operator in quantum mechanics, von Neumann considers the treatment of time as a parameter as a real if not the main weakness of that theory.[3]

In Sec. II we shall start by recalling the Hamiltonian formalism of classical mechanics on which nonrelativistic quantum mechanics was based. We shall argue that the evolution parameter t occurring in that formalism should not be put on a par with the dynamical variables of the system. In particular, the evolution parameter and the Hamiltonian, contrary to what is often said, do not form a canonical pair. Accordingly, a relation like the first equation in (1) does not occur in quantum mechanics. However, "timelike" canonical variables, in the form of angle variables, do sometimes exist, and they are observables in quantum mechanics.[4] The argument of Sec. II will be much strengthened by analyzing, in Sec. III, the distinction as well as the relation between the canonical variables and the coordinates of space and time. Much of the confusion about the uncertainty principle for energy and time is caused by mixing up the canonical position coordinates of a point particle and the coordinates of a point in space. The ubiquity of point particles in fundamental physics has helped to propagate this confusion, but, from the point of view of general mechanics, point particles are only very special mechanical systems. We shall conclude that the partners of the evolution parameter t are not to be found among the canonical variables: The partners of t are the three coordinates of three-dimensional space. Just as the coordinates of a point in space are not turned into operators in quantum mechanics, so neither should coordinate time be turned into an operator. Hence, an uncertainty relation for t similar to the one between the canonical coordinates and momenta, does not exist. Nevertheless, there are many instances in physics where an "uncertainty" relation between time and energy does hold, the relation between the lifetime and the linewidth of a quantum state being the prime example. In Sec. IV we shall show how this relation can be derived. A completely similar relation involving the space coordinates and the total momentum also exists.

II. THE HAMILTONIAN FORMALISM

In this section the Hamiltonian formalism of classical mechanics will be briefly recalled. In this formalism the state of a system is given by a set of $2n$, time-dependent, so-called "canonical" variables:

$$q_1(t),\ldots,q_n(t),\ p_1(t),\ldots,p_n(t).$$

The q_i are called (generalized) coordinates, the p_i are the conjugate momenta. We can think of these variables as defining a point in a $2n$-dimensional phase space. Depending on the system, the physical meaning of the canonical variables may vary widely, a point to be remembered in the following. The time evolution of the system is determined by the Hamiltonian H, a function of the canonical variables. If time-dependent external forces act on the system H may also depend explicitly on t: $H=H(q,p,t)$. The time evolution of an arbitrary function $A=A(q,p,t)$ of the canonical variables is given by

$$\frac{dA}{dt}=\frac{\partial A}{\partial t}+\{A,H\}. \tag{2}$$

The symbol on the right-hand side of this equation is the Poisson bracket, defined as

$$\{A,B\}=\sum_{i=1}^{n}\left\{\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right\}.$$

If H does not depend explicitly on t, it is a constant of the motion and $H=E$ is the *energy* of the system. For the canonical variables themselves (2) reduces to the familiar Hamiltonian equations of motion:

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}.$$

The canonical variables satisfy

$$\{q_i,p_j\}=\delta_{ij}. \tag{3}$$

The transition to quantum mechanics is made[5] by replacing the canonical variables by operators on a Hilbert space satisfying the canonical commutation relations:

$$[\mathbf{q}_i,\mathbf{p}_j]=i\hbar\,\delta_{ij}. \tag{4}$$

From this the well-known Heisenberg uncertainty relations:

$$\Delta\mathbf{q}_i\Delta\mathbf{p}_j\geqslant\tfrac{1}{2}\hbar\,\delta_{ij} \tag{5}$$

can be derived. It should be remarked right away that the transition from arbitrary canonical variables to quantum mechanical operators is fraught with problems and that relations (4) and (5) are relatively unproblematic only for Cartesian coordinates and momenta. We shall come back to this later in this section.

A canonical transformation is a transformation from one set of canonical variables to another:

$$q'=q'(q,p,t),\quad p'=p'(q,p,t),$$

where we have dropped indices for convenience. The new variables have the same Poisson brackets (3) as the old ones and obey the same equations of motion with respect to a new Hamiltonian function $H'(q',p',t)$.

As has been remarked already, the physical meaning of the generalized coordinates and the conjugate momenta may vary widely. If the system is a set of point particles, the coordinates q are usually taken to be the Cartesian position coordinates of the particles but polar coordinates may equally well be used. If the system is an extended rigid body, some of the coordinates may be angles defining the orientation of the body. Furthermore, the coordinates and momenta may become mingled by a canonical transformation. The roles of the q's and the p's may even be completely interchanged by such a transformation.

Quite often one finds the statement that in the Hamiltonian formalism energy and time are canonical conjugates.[3,6] Since, initially, t is just a parameter in the theory this must be taken to mean first, that t is now considered as a new independent canonical variable and, second, that this new variable and (minus) the Hamiltonian H form an $(n+1)$th canonical pair in an enlarged $(2n+2)$-dimensional phase space. The first step to accomplish this is to introduce a new evolution parameter on which all variables including t depend. The final step would be to find a new Hamiltonian which generates the evolution of the system with respect to the new parameter. However, it turns out that this program cannot be carried through: It is not possible to find a Hamiltonian that generates the motion in the enlarged phase space.[7] That incorporating t and H in the set of canonical variables spells trouble may be guessed already from the fact that H is not independent of the other canonical variables.

Why would one *want* to regard t as an $(n+1)$th canonical coordinate? This seems to be inspired by the wish to arrive at a relativistically covariant description.[3,7,8] If the system consists of a single particle and the coordinates q_i are taken to be the three Cartesian position coordinates of the particle, it is tempting to try and treat the time parameter as a fourth coordinate to arrive at a relativistically covariant description. This can be done up to a point but ultimately leads out of the normal Hamiltonian scheme.[9] Furthermore, for a system consisting of more than one particle this route is clearly blocked; in fact, it has been shown that a relativistically covariant Hamiltonian formalism for a system of interacting point particles does not exist.[10] For an *arbitrary* mechanical system the physical meaning of the generalized coordinates may be rather less simple than it is for a system of point particles and the desire to treat the evolution parameter as a canonical variable alongside the q's in this general case[8] to obtain a relativistic description does not seem to be well motivated.

We shall not go into the details of the complicated matter of relativistic particle dynamics.[10] The point we want to make is that one cannot fulfill the demand of relativistic covariance by adding t to the canonical position variables of the system. The arena of the Hamiltonian formalism is the $2n$-dimensional phase space of t-dependent canonical variables. In the following we shall argue that the true partners of t are not the n canonical coordinates of the material system but the three space coordinates of a point in four-dimensional space time.

We conclude that in the Hamiltonian formalism t is a parameter, not a dynamical variable. The relationship between t and H is given by Eq. (2); that is, they are the evolution parameter and the generator of the time evolution, respectively. In particular, there is no Poisson bracket defined between t and H. Consequently, in quantum mechanics, one does not have a relation like the first equation in (1). Accordingly, there is no natural analog for energy and time of the "canonical" uncertainty relations (5).

We have argued that in the Hamiltonian formalism time is not a dynamical variable. There may, however, exist true dynamical variables which behave very much like the time parameter. Consider, for example, the position of the tip of a hand of a clock. This position represents time for us because it depends on the time parameter in a very simple manner. But this position is *not* the time parameter. It is a dynamical variable of the system exhibiting a particularly simple t dependence. A clockmaker's shop is full of such variables!

A general class of "timelike" canonical variables is formed by the so-called *angle variables*. They occur in multiperiodic systems. As an example, one may think of the angles defining the direction of the hands of a clock. The corresponding canonical momenta are called *action variables*. We denote the angle variables by w_i and the action variables by J_i. In these variables the Hamiltonian takes the form $H=H(J_1,...,J_n)$, i.e., it does not depend on the angle variables. Then

$$\{w_i,J_j\}=\delta_{ij}; \quad dJ_i/dt=\{J_i,H\}=0;$$

$$dw_i/dt=\{w_i,H\}=\nu_i.$$

The ν_i are constants depending on the J's which themselves are constants of the motion. So, the solution of the equations of motion is

$$J_i=\text{const}$$

$$w_i=\nu_i t+\delta_i,$$

where δ_i is an arbitrary constant. This solves the dynamical problem, but the canonical transformation leading from the original variables to the angle and action variables is generally hard to find.

The w_i and J_i must be sharply distinguished from t and H; the former are pairs of canonically conjugate dynamical variables and there may be as many as n such pairs. Although the angle variables behave in much the same way as t there is an important difference. Whereas t is linear, running from $-\infty$ to $+\infty$ on the real axis, the w_i are periodic in the sense that the points $(w_1,...,w_n,J_1,...,J_n)$ and $(w_1+2\pi,...,w_n+2\pi,J_1,...,J_n)$ label the same phase point. The ν_i are the frequencies of the periodic motions.

In quantum mechanics the canonical variables w_i and J_i become operators for which one would naively put $[\mathbf{w}_i,\mathbf{J}_j]=i\hbar\delta_{ij}$. Uncertainty relations of type (5) between the **w**'s and **J**'s would then seem to follow. Identifying the "state" determining quantities in the quotation preceding equations (1) with **J**'s and the "phases" with **w**'s we may now appreciate Heisenberg's statement. At the same time we see clearly how confusing the "or" in (1) really is.

We have noted already that the transition from canonical variables to quantum mechanical operators is not straightforward. The action and angle variables form a case in point. In classical physics the action variables take on positive values only. However, the commutation relation $[\mathbf{w}_i,\mathbf{J}_j]=i\hbar\delta_{ij}$ cannot be satisfied if $\mathbf{J}_j$ is a positive operator. Remarkably, this problem had been recognized already at the time when Heisenberg wrote his article.[11] The problem is closely related to the problem of defining a phase operator for the quantized electromagnetic field. For the purpose of the present article we may ignore this difficulty since we only want to show that from the point of view of the Hamiltonian formalism the first equation in (1) is unfounded whereas the second is not.

Summarizing this section, we have seen that in the Hamiltonian formalism a mechanical system is described by n generalized coordinates and n conjugate momenta depending on an evolution parameter t and forming n canonically conjugate pairs. The evolution of the system is governed by the Hamiltonian function. In quantum mechanics the canonical pairs are turned into operators obeying the canonical commutation relations. From these relations uncertainty relations between the operators may, in principle, be derived. From the point of view of this quantization procedure there is no ground to turn the evolution parameter into an operator. Consequently, there is no analogous uncertainty relation involving t. In certain cases there exist canonical variables which depend linearly on t and which, therefore, resemble t very much. In quantum mechanics these variables become quantum mechanical observables. They should, however, be sharply distinguished from the evolution parameter. In the next section these conclusions will be further reinforced.

III. SPACETIME

In the previous section we have seen that the arena of the Hamiltonian formalism is the $2n$-dimensional phase space of the canonical variables. It is true that mechanical systems are situated in three-dimensional space but for the Hamiltonian formalism this fact is generally not relevant: In many applications the coordinates x_1,x_2,x_3 of three-dimensional space need not even be mentioned. (The same is true for the Lagrange formalism.) Of course, the canonical variables usually have a direct interpretation as quantities in three-dimensional space. This is particularly so in the theoretically all-important case of a system of point particles. The canonical coordinates q_i then are the Cartesian position coordinates of the particles in three-dimensional space, and the conjugate momenta p_i are the three-dimensional momenta. We shall denote this particular set of canonical variables by (q_k^α,p_k^α), where $k=1,2,3$ and $\alpha=1,...,N$ is the particle index. The variables q much resemble the coordinates x of three-dimensional space but they should be clearly distinguished. For example, the same location x in three-dimensional space may be occupied by any of the N particles, i.e., for any α we may have $q_k^\alpha=x_k$. In fact, ignoring this distinction between the position of a particle (a material body) and of an abstract point in space has been an important cause of the confusion mentioned in the preceding section. We have seen how for a one-particle system people have tried to obtain a Lorentz-covariant description by considering the evolution parameter t as a fourth canonical coordinate alongside the particle's position q_k. However, the wish to include t in the set of canonical variables rests on an optical illusion, for the companions of the evolution parameter are not to be found among the canonical coordinates; the true companions of t are the three space coordinates x_1,x_2,x_3. Likewise, the covariant partners of the Hamiltonian H are not the conjugate canonical momenta but the three generators P_k of translations in space.

If we think of a system as being situated in three-dimensional space, we may consider the effect on the dynamical variables of a translation in space. Just as the dynamical variables may explicitly depend on t they may also explicitly depend on x. This may be the case in the presence of an inhomogeneous external field. For a general dynamical variable one would then have to write $A=A(q,p,x,t)$ which indicates explicitly that t goes with x, not with q. By analogy with (2):

$$\frac{dA}{dx_k}=\frac{\partial A}{\partial x_k}+\{A,P_k\}.$$

In particular,

$$\frac{dq_i}{dx_k}=\frac{\partial P_k}{\partial p_i}, \quad \frac{dp_i}{dx_k}=-\frac{\partial P_k}{\partial q_i}, \quad (i=1,...,n; \quad k=1,2,3).$$

Usually the x dependence of the canonical variables is not very interesting from a physical point of view and in many cases it will be rather trivial. For example, a space translation adds a constant to all canonical variables which denote positions in space, whereas their conjugate momenta remain unchanged. But canonical variables may be much more complicated than that (remember the freedom of performing canonical transformations) and their behavior under translations in space may be as complicated as under translations in time.

For a closed system (no external fields) the translations in space and time are symmetry transformations and the generators P_k and H are conserved quantities having the meaning of the total momentum and total energy of the system, respectively. [The use of the name "momentum" for both the generator P_k of space translations and the canonical conjugate momenta p_i is apt to cause confusion. This is particularly true in the case of point particles (cf. below). But it should be clear from the foregoing that the two notions are conceptually very different.]

For a system of point particles the behavior under space translations of the special canonical variables (q_k^α, p_k^α) is very simple:

$$x_k \to x_k + a_k \Rightarrow q_k^\alpha \to q_k^\alpha + a_k, \quad p_k^\alpha \to p_k^\alpha.$$

In this case, the generator of space translations is given by

$$P_k = \sum_\alpha p_k^\alpha.$$

If there is only one particle, P_k coincides with the canonical momentum p_k providing another example of the deceptive simplicity of this system.

The simple behavior under *space* translations of a system of point particles may be compared with the simple behavior under *time* translations of the angle and action variables of a multiperiodic system:

$$t \to t + a \Rightarrow w_i \to w_i + \nu_i a, \quad J_i \to J_i.$$

Just as the canonical position coordinates q_k of point particles resemble the space coordinates x_k in their behavior under space translations, so the w_i, or rather the w_i/ν_i, of multiperiodic systems resemble the time coordinate t in their behavior under time translations. This may easily cause confusion, especially if there is only one particle or only one angle variable. However, also in this case, the canonical position q_k remains conceptually different from the space coordinate x_k and the canonical angle variable w remains conceptually different from the time coordinate t.

Confusing the space coordinates with the canonical position variables of point particles led von Neumann to suppose that in a relativistic quantum mechanics time must be an operator and that it would even be desirable to have as many times as there are particles.[3] However, in present-day elementary particle physics, particles are described by relativistic quantum *fields*. Fields allow for a much more natural covariant description than do particles. The space-time coordinates appear as arguments of the fields. There is no time operator. What von Neumann had in mind would correspond to what may be called a "clock particle"; a point particle provided with a very small "point" clock of unit frequency. Such a clock particle would be characterized by a position variable q_k and an angle variable w satisfying the numerical equations $q_k = x_k$ and $w = t$. Under a Lorentz transformation of the space-time frame (x_k, t) and (q_k, w) transform as four vectors. On the other hand, q_k and t are quantities of a different kind and should not be considered to form a four vector.

The relation between the q's and the w's on the one hand, and the x's and t on the other, is quite interesting. It seems plausible that the notions of space and time are derived from the properties of material bodies. The q's and w's would then correspond to more primitive notions of space and time than the x's and t, the latter being abstractions from the former. We may speculate that the concrete notions of time as they are connected with periodic changes would have led to the abstract notion of a single linear time extending from minus to plus infinity. Similarly, the local notions of space as derived from the behavior of material bodies would have led to the abstract notion of an infinitely extended linear space. Thus there would have originated the idea of an empty, infinitely extended linear space time, the stage on which the drama of nature unfolds and the starting point of most considerations in theoretical physics since Newton. In fact, the whole of physics, apart from general relativity, is based on this notion. In particular, the concept of space-time *symmetries*, leading up to important conservation laws, rests on it.

Just as spacetime is the stage for classical mechanics, so it is for quantum mechanics. In quantum mechanics the dynamical variables q_i, p_i, q_k^α, p_k^α, w_i, J_i, H, and P_k become quantum mechanical observables. By contrast, the space-time coordinates x_k and t remain what they are: c-number labels of space-time points. No more should t be made an operator than should x_k. In particular, the unitary operators which represent space-time symmetries in quantum mechanics depend on the c-number parameters of ordinary space and time (see next section). Paraphrasing von Neumann, the fact that quantum mechanics presupposes for its formulation the existence of an ordinary space-time frame could perhaps be considered as its main weakness!

Summarizing this section, we have seen that a sharp distinction must be made between the canonical variables denoting particle positions and the space and time coordinates labeling points in spacetime. The latter are not turned into operators in quantum mechanics. In particular, for a system of particles, one should not demand a commutation relation between t and H as a complement to the ones between q and p, nor could there be such a commutation relation.[12] Again, one sees that there is no uncertainty relation analogous to (5) between H and t.

IV. THE UNCERTAINTY PRINCIPLE FOR ENERGY AND TIME

Up till now we have tried to show that the wish to have a "canonical" commutation relation for energy and time rests on an optical illusion originating in classical mechanics. Nevertheless there are many instances in physics where an uncertainty principle of some sort for energy and time does hold. Foremost among these is the relation between the energy width and the lifetime of a quantum state. In this section we shall show that for this case an uncertainty relation can be derived on the basis of the ideas developed in the previous sections. A completely similar relation holds between the momentum width and the spatial width of a state.

Let us consider an arbitrary closed system. In quantum mechanics its states are represented by unit vectors $|\Psi\rangle$ in a Hilbert space. We employ the Heisenberg picture so that the vector representing a state does not change in time. If $|\Psi\rangle$ and $|\Phi\rangle$ are unit vectors, the scalar product $\langle\Psi|\Phi\rangle$ is called the *transition amplitude* of the states; the square of its absolute value is the probability of finding the system in the state

$|\Phi\rangle$ if it has been prepared in the state $|\Psi\rangle$. The system is supposed to be symmetric with respect to certain transformations of the inertial space-time frame such as translations in space and time, rotations, and possibly Lorentz transformations. In the present article we are only interested in translations in space and time. These symmetry transformations are represented by unitary operators in Hilbert space which we shall denote by $\mathbf{U}(a_k)$ and $\mathbf{U}(a)$, respectively. Thus $\mathbf{U}(a_k)|\Psi\rangle$ is the state which is displaced in space by a_k in the direction k with respect to the state $|\Psi\rangle$. Likewise, $\mathbf{U}(a)|\Psi\rangle$ is the state which is translated in time by a with respect to the state $|\Psi\rangle$. We may write

$$\mathbf{U}(a_k)=\exp(-ia_k\mathbf{P}_k), \quad \mathbf{U}(a)=\exp(ia\mathbf{H}). \tag{6}$$

The self-adjoint operators $\mathbf{P}_k$ and $\mathbf{H}$ are the generators of translations in space and time in quantum mechanics; they are the counterparts of P_k and H of the previous sections and correspond to the total momentum and energy of the system.[13] (We have put $\hbar=1$ for notational convenience.)

We suppose that $\mathbf{P}_k$ and $\mathbf{H}$ have complete sets of (improper) eigenstates which we denote by $|p_k\rangle$ and $|E\rangle$, respectively, where possible degeneracies are ignored to facilitate notation. Then $\int|p_k\rangle\langle p_k|dp_k=\mathbf{1}$ and $\int|E\rangle\langle E|dE=\mathbf{1}$, where the last integral may include a summation over discrete eigenstates. Using this and (6), we obtain

$$\langle\Psi|\mathbf{U}(a_k)|\Psi\rangle=\int e^{-ia_kp_k}|\langle p_k|\Psi\rangle|^2\,dp_k, \quad (k=1,2,3) \tag{7}$$

$$\langle\Psi|\mathbf{U}(a)|\Psi\rangle=\int e^{iaE}|\langle E|\Psi\rangle|^2\,dE.$$

For the purpose of the present section we shall confine attention to translations in time.

The transition amplitude $\langle\Psi|\mathbf{U}(a)|\Psi\rangle$ is called the *survival amplitude* of the state $|\Psi\rangle$. The square of its absolute value is the probability of finding the system, after a time a has passed, still in its original state $|\Psi\rangle$. This quantity may be taken as the starting point for a definition of the *lifetime* of a state. We define the time τ_β as the smallest time interval satisfying

$$|\langle\Psi|U(\tau_\beta)|\Psi\rangle|=\beta \quad (\beta<1). \tag{8}$$

For example, taking $\beta=\sqrt{1/2}$ in (8), $\tau_{\sqrt{1/2}}$ is the so-called half-life of the state, that is, $\tau_{\sqrt{1/2}}$ is the (smallest) time at which the probability of finding the system still in its original state has decreased to 50%. From Eq. (7) an "uncertainty" relation between the time τ_β and the width of the energy spectrum $|\langle E|\Psi\rangle|$ of the state $|\Psi\rangle$ may be derived. There are several definitions possible of the width of a probability distribution. The following is a suitable one: Define W_α to be the size of the smallest energy-interval W such that

$$\int_W|\langle E|\Psi\rangle|^2\,dE=\alpha.$$

W_α gives a reasonable measure for the uncertainty in energy if α is less than but close to 1. For example, if $\alpha=0.9$ then W_α is the smallest interval on which 90% of the energy distribution is situated. Then (inserting $\hbar$ again) one can show[14]

$$\tau_\beta W_\alpha\geqslant 2\hbar\,\arccos\left(\frac{\beta+1-\alpha}{\alpha}\right), \quad \text{for } \beta\leqslant 2\alpha-1. \tag{9}$$

This relation is valid for all states $|\Psi\rangle$.

For example, from (9) follows

$$\tau_{\sqrt{1/2}}W_{0.9}\geqslant 0.9\hbar.$$

A related relation can be derived if instead of W_α the standard deviation ΔE is chosen as a measure of the width of the energy spectrum (however, in many cases of physical interest the standard deviation is not a suitable measure of width). One then finds[14]

$$\tau_{\sqrt{1/2}}\Delta E\geqslant \pi\hbar/4.$$

We have thus obtained satisfactory and general expressions of the "uncertainty" relation between the energy spread (linewidth) of a quantum state and its lifetime. Note that the uncertainties in energy and time appearing in (9) are conceptually very different. This is only to be expected because time is not an operator. We refer to Refs. 15 and 16 for an interpretation of τ_β as an uncertainty.

Exactly the same procedure can be applied to translations in space leading to an uncertainty relation between the spread in the total momentum of a state and its so-called spatial translation width. A discussion of this relation is given in Refs. 15 and 17.

Thus are obtained four, completely similar, generally valid, uncertainty relations between the spreads in the total momentum and the total energy of a state on the one hand and its translation widths with respect to space and time on the other. Evidently, these relations fit easily in a relativistic formulation of quantum mechanics. Space and time then merge into space-time and $\mathbf{P}_k$ and $\mathbf{H}$ become the components of a four vector. In this way, the urge to unite time and space that has led people, wrongly, to include t in the set of canonical variables, has indeed been satisfied!

V. OTHER FORMULATIONS

There exist many other formulations of the uncertainty principle for energy and time on which we shall only comment briefly. Some formulations are simply wrong, such as the statement[3] that for a measurement of the energy with accuracy δE a time $\delta t>\hbar/\delta E$ is needed. This statement is wrong because it is an assumption of quantum mechanics that all observables can be measured with arbitrary accuracy in an arbitrarily short time and the energy is no exception to this. Indeed, consider a free particle; its energy is a simple function of its momentum and a measurement of the latter is, at the same time, a measurement of the former. Hence, if we assume that momentum can be accurately measured in an arbitrarily short time, so can energy. Other formulations are confined to special cases. For example, if the wave packet of a particle does not spread too fast, an uncertainty relation between the time at which the particle passes a certain point in space and the particle's energy can be derived from the well-known uncertainty relation between the spreads in the position and momentum of the particle. Still other formulations depend on first-order perturbation theory such as the statement that the conservation law of energy may be violated by an amount δE during a time $\delta t\cong\hbar/\delta E$.[18] This statement is misleading because it confuses the energy of the actual system with the energy of the unperturbed system. A class of generally valid uncertainty relations has been derived from the quantum version of the equation of motion (2) by Mandelstam and Tamm.[19] For each observable $\mathbf{A}$ one defines an uncertainty in time:

$$\tau_A := \frac{\Delta \mathbf{A}}{\left|\frac{d\langle \mathbf{A}\rangle}{dt}\right|}. \tag{10}$$

It is then possible to derive the relation $\tau_A \Delta E \geqslant \frac{1}{2}\hbar$, where Δ denotes the standard deviation. Note the difference between τ_A and the coordinate time t appearing in the *same* formula (10)! Note also that τ_A not only depends on **A** but also on the state of the system. Although this approach is conceptually very different from the one leading to (9), there still is an interesting connection.[14] For a recent comprehensive review the reader may consult Ref. 20. For a recent textbook we recommend Ref. 21.

ACKNOWLEDGMENTS

This article has benefited from valuable discussions with Jos Uffink. I am indebted to Tim Budden and Fred Muller for a critical reading of the manuscript.

[1]W. Heisenberg, "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik," Z. Phys. **43**, 172–198 (1927). For an English translation of this article see Ref. 2.

[2]J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton U.P., Princeton, NJ, 1983), p. 62.

[3]John von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, 1955), pp. 353, 354. Translated from the original German edition: Johann von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1932), pp. 187 and 188. These two pages provide an easy reference for the views we want to criticize in this article, but similar views are widespread in the literature.

[4]In view of the difficulties that beset the definition of action and angle variables as self-adjoint operators in quantum mechanics we prefer the more general term "observable" to distinguish them from c numbers.

[5]P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford U.P., Oxford, 1947), 3rd ed., Chap. IV.

[6]Niels Bohr, "Discussion with Einstein on epistemological problems in atomic physics," in *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp (Open Court, La Salle, IL, 1949), p. 214. Reprinted in Ref. 2, p. 22.

[7]O. D. Johns, "Canonical transformations with time as a coordinate," Am. J. Phys. **57**, 204–215 (1989).

[8]J. L. Synge, "Classical Dynamics," in Vol. 3, Part 1 of *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1960), p. 105. Somewhat disappointingly, Synge confines his treatment of relativistic particle dynamics to one-particle systems, a most atypical case.

[9]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, 2nd ed, 1980), Section 8-4. One first introduces a new evolution parameter θ of which the four coordinates q_i and t are functions. It is then possible to find an appropriate Lagrangian for the system. From this Lagrangian the momentum conjugate to t may be calculated and it turns out to be $-H$. This would then be a ground for considering t and $-H$ as forming a fourth canonical pair. However, the next step, the inversion of the Lagrangian to obtain the generalized velocities as functions of the generalized coordinates and momenta, turns out to be impossible, that is, the equations of motion cannot be written in the usual Hamiltonian form.

[10]E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974), Chap. 21.

[11]F. London, "Ueber die Jacobischen Transformationen der Quantenmechanik," Z. Phys. **37**, 915 (1926).

[12]Pauli has already remarked that if the spectrum of **t** is the whole real axis and **t** and **H** satisfy the commutation relation in (1), then the spectrum of **H** cannot contain discrete eigenvalues. In fact, the restrictions following from $\mathbf{H} \geqslant 0$ are even more severe.

[13]See Ref. 5, Sec. 25. For a modern treatment: L. E. Ballentine, *Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1990), Chap. 3.

[14]J. Uffink, "The rate of evolution of a quantum state," Am. J. Phys. **61**, 935–936 (1993).

[15]J. Hilgevoord and J. Uffink, "A new view on the uncertainty principle," in *Sixty-Two Years of Uncertainty, Historical and Physical Inquiries into the Foundations of Quantum Mechanics*, edited by A. I. Miller (Plenum, New York, 1990), pp. 121–139.

[16]J. Hilgevoord and J. Uffink, "Uncertainty in prediction and in inference," Found. Phys. **21**, 323–341 (1991).

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[18]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Volume 3 of *Course of Theoretical Physics* (Pergamon, Oxford, 1975), 2nd ed., Sec. 44.

[19]L. Mandelstam and I. Tamm, "The uncertainty relation between energy and time in nonrelativistic quantum mechanics," J. Phys. (USSR) **9**, 249–254 (1945).

[20]P. Busch, "On the energy-time uncertainty relation," Part I and II, Found. Phys. **20**, 1–43 (1990).

[21]A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).

DOES THE NORTH POLE EXIST

It is difficult to imagine that we could ever be in possession of final physical principles that have no explanation in terms of deeper principles. Many people take it for granted that instead we shall find an endless chain of deeper and deeper principles. ...

Popper and the many others who believe in an infinite chain of more and more fundamental principles might turn out to be right. But I do not think that this position can be argued on the grounds that no one has yet found a final theory. That would be like a nineteenth-century explorer arguing that, because all previous arctic explorations over hundreds of years had always found that however far north they penetrated there was still more sea and ice left unexplored to the north, either there was no North Pole or in any case no one would ever reach it. Some searches do come to an end.

Steven Weinberg, *Dreams of a Final Theory* (Pantheon Books, New York, 1992), pp. 230–231.