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Noether's theorem for local gauge transformations

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(Received 2 June 1988; accepted for publication 14 March 1989)

The variational methods of classical field theory may be applied to any theory with an action that is invariant under local gauge transformations. What is the significance of the resulting Noether current? This article examines such currents for both Abelian and non-Abelian gauge theories and provides an explanation for their form and limited range of physical significance on a level accessible to those with a basic knowledge of classical field theory. Several of the more subtle aspects encountered in the application of the residual local gauge symmetry found by Becchi, Rouet, Stora, and Tyutin are also considered in detail in a self-contained manner.

I. INTRODUCTION

The continuous symmetries of classical field theories along with the equations of motion for the fields imply the existence of conserved currents from which one can construct conserved charges. This is usually called Noether's theorem,[1–11] which in both its classical and operator forms is very important for classifying the general physical characteristics of quantum field theories.[5–11] A brief review of the theorem is given in Sec. II. Nevertheless, the theorem does not seem to apply in a straightforward manner when the symmetry of interest is local gauge invariance.[7,11,12]

Why this is so is not explored in any of the standard texts on field theory that almost exclusively confine themselves to global gauge transformations when applying Noether's theorem.[13] Brandt[14] and Jackiw,[11] for example, have determined the Noether currents for Abelian and non-Abelian local gauge transformations, but their treatments are not readily accessible to students at the introductory level of field theory. In the present article, we present a simple and self-contained discussion of the implications of Noether's theorem for local gauge transformations associated with Abelian and non-Abelian internal symmetries. This is intended to supplement the standard expositions available in introductory works on quantum field theory. Several unusual features of the general Noether currents and charges corresponding to local gauge transformations are pointed out in Sec. III that are of interest even though the physics implicit in a gauge theory can be extracted only after introducing constraints that destroy the gauge symmetry.

In the classical case, gauge fixing is required in order to integrate the equations of motion for the electromagnetic potentials. The quantization of a gauge field theory can only be achieved after the gauge degrees of freedom are suitably restrained. In this last instance, however, a remarkable residual local gauge symmetry[15] has been a useful technical device for the analysis of the physical content of such theories. Noether's theorem will, of course, yield the current and charge corresponding to what is referred to as Becchi, Rouet, Stora, and Tyutin (BRST) symmetry.[15]

The implications of Noether's theorem in the case of BRST symmetry are certainly well known.[12,14,16] However, the application of the relevant formalism in this case is sufficiently subtle as to warrant the introductory expositions we carry out in Secs. IV and V. We point out several unusual, but important, aspects of both the Noether and Lagrangian formalisms in this case. We also compare the BRST current with the usual current resulting from gauge symmetry and the source current in the field-tensor equation of motion. A summary is presented in Sec. VI.

II. NOETHER'S THEOREM

Let us consider a classical field theory characterized by a Lagrangian density $\mathscr{L}[\phi_a(x),\partial_\mu\phi_a(x)]$ involving the fields $\phi_a(x)$ at the space-time point $x=(x^0,\mathbf{x})$ and their first-order space-time derivatives. For the sake of simplicity, we ignore any explicit dependence of $\mathscr{L}$ on x. Here, the index a enumerates the different field types including reference to their transformation properties with respect to the Lorentz group (scalar, spinor, vector, etc.). Also, $\partial_\mu=\partial/\partial x^\mu$, where $\mu=0,1,2,3$, and we employ the diagonal metric $g_{00}=-g_{jj}=+1, j=1,2,3$ to raise and lower the vector index μ.

A signature of the symmetry of a classical field theory is the invariance of the action integral

$$S_{21}[\phi]=\int_1^2 d^4x\,\mathscr{L}[\phi_a(x),\partial_\mu\phi_a(x)], \tag{1}$$

taken between two spacelike surfaces under the associated transformations of the fields. Hamilton's principle then implies that the equations of motion are also invariant under these transformations. Noether's theorem refers to the local implications of a symmetry and these are determined by exploring the consequences of the invariance

$$\delta S_{21}[\phi]=0, \tag{2}$$

under the infinitesimal transformations

$$\phi_a(x)\to\phi_a(x)+\delta\phi_a(x). \tag{3}$$

The variations are assumed to vanish on the boundary surfaces 1 and 2.

Corresponding to (3) we have the variation in the Lagrangian density

$$\delta\mathscr{L}=\frac{\partial\mathscr{L}}{\partial\phi_a(x)}\,\delta\phi_a(x)+\frac{\partial\mathscr{L}}{\partial[\partial_\mu\phi_a(x)]}\,\delta[\partial_\mu\phi_a(x)], \tag{4}$$

where summation over any repeated index is implied. If we suppose that we can interchange the δ and ∂_μ operations, (2) and (4), together with the equations of motion

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{[\partial\partial_\mu\phi_a(x)]}\right)-\frac{\partial\mathscr{L}}{\partial\phi_a(x)}=0, \tag{5}$$

imply that

$$0=\int_1^2 d^4x\,\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial[\partial_\mu\phi_a(x)]}\,\delta\phi_a(x)\right). \tag{6}$$

Since the integrand of (6) vanishes on the boundary surfaces and involves otherwise arbitrary variations of the fields induced by the symmetry transformations, it follows that

$$\partial^\mu j_\mu(x)=0, \tag{7}$$

where[17]

$$j_\mu(x)=\frac{\partial\mathscr{L}}{\partial[\partial_\mu\phi_a(x)]}\,\delta\phi_a(x) \tag{8}$$

is the conserved Noether current. Evidently we can always define a Noether current corresponding to (3) whether or not (2), and therefore (7), is realized.

Actually, since arbitrary variations of the fields are involved, Eq. (8) defines an entire family of currents as well as the charges

$$Q(x_0)\equiv\int d^3x\, j_0(x), \tag{9}$$

which are also conserved,

$$\frac{dQ(x_0)}{dx_0}=0, \tag{10}$$

as a consequence of (7) provided $j_\mu(x)$ vanishes sufficiently quickly in spacelike directions at infinity.[18] This uninteresting diversity is usually eliminated by parametrizing the variations $\delta\phi_a(x)$ by a space-time independent infinitesimal parameter ϵ so that

$$\delta\phi_a(x)=\epsilon f_a[\phi(x)], \tag{11}$$

where $f_a[\phi]$ is some function of all the $\phi_a(x)$'s. The content of Noether's theorem can then be stated in a form that reflects the intrinsic character of the symmetry transformation rather than factoring in irrelevant information about the parameters that particularize it. That is, equations of the same form as (7), (8), and (10) hold as before but now in terms of what may be called the *intrinsic* Noether current,

$$j_\mu(x)_{\rm N}\equiv\frac{\partial\mathscr{L}}{\partial[\partial_\mu\phi_a(x)]}f_a[\phi(x)], \tag{12}$$

with which is associated the Noether charge

$$Q(x_0)_{\rm N}\equiv\int d^3x\, j_0(x)_{\rm N}. \tag{13}$$

If $j_\mu(x)_{\rm N}$ is conserved, the time-independent charge $Q_{\rm N}$ given by (13) can be regarded as an intrinsic observable of the system. On the other hand, given any tensor field $R_{\mu\nu}(x)$ that falls off sufficiently rapidly in spacelike directions, the conserved current

$$j'_\mu(x)_N = j_\mu(x)_N + \partial^\nu(R_{\mu\nu} - R_{\nu\mu}) \qquad (14)$$

determines the same charge Q_N because

$$\int d^3x\, \partial^\nu(R_{0\nu} - R_{\nu 0}) = \int d^3x\, \partial^i(R_{0i} - R_{i0}) = 0.$$

This device of adding the divergence of an antisymmetric tensor[19] can be used to "improve" the original canonical Noether current so as to attain some other property, such as the indicial symmetry of the energy-momentum tensor while maintaining current conservation and the same value of the Noether charge.[10,11]

We will see in Sec. IV that local gauge transformations are unusual in two respects with regard to Noether's theorem. Generally, one cannot disentangle the parameters that define the transformations from the local gauge Noether current. Furthermore, we find that the canonical Noether current in this case is itself the divergence of an antisymmetric tensor and so meaningful Noether charges can arise only from long-ranged contributions in spacelike directions.

III. LOCAL GAUGE TRANSFORMATIONS

By definition, gauge transformations of any sort do not represent physical symmetry operations. Rather, they probe the phase relationships within the model space of a physical theory. The intrinsic conserved charges that accompany the gauge invariance of a theory generally are observable provided they are themselves gauge invariant. These two aspects do not conflict because the range of values of the charges do not represent different possible states of a particular system, as would be the case for angular momentum or energy, for example. Rather, the values of the charges define different classes of possible systems, namely, all those with definite values of the charges.

A. Abelian gauge groups

Several properties of general gauge theories in regard to Noether's theorem are already present in the trivial example of a free electromagnetic field with

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad (15)$$

where the antisymmetric electromagnetic field tensor $F_{\mu\nu}$ is expressed in terms of the gauge fields $A_\mu(x)$ in the usual manner,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (16)$$

Evidently (16) and thus (15) are invariant under the U(1) group of Abelian gauge transformations,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x). \qquad (17)$$

For infinitesimal $\theta(x)$, the Noether current (8) is

$$j_\mu(x)_N = \partial^\nu[F_{\nu\mu}(x)\theta(x)], \qquad (18)$$

which is manifestly conserved by virtue of the antisymmetry of $F_{\mu\nu}(x)$.

We see that because of the derivative operation on the gauge parametrizing function $\theta(x)$, we generally cannot factor out the incidental attributes of the transformation. For constant $\theta(x)$, namely, global, or rigid, gauge transformations, we see that (18) vanishes by the equations of motion and we always have zero intrinsic charge. The charges one would try to infer from (18) are not defined except for special gauge functions θ.

The point is that the time-independent charge integral

$$Q[\theta] = \int d^3x\, \partial^i[F_{i0}\theta(x)] \qquad (19)$$

will not exist for an arbitrary free-field tensor F_{i0} unless $\theta(x)$ is suitably well behaved. Then the three-dimensional spatial integral (19) can be converted into a surface integral whose value depends upon $\theta(x)$. If θ is a constant, then obviously $Q(\theta) = 0$, but otherwise the value of $Q[\theta]$ depends both on the particular gauge and field functions that enter into (19). Thus the infinity of conserved charges (19) do not represent *useful* observables of the system. This is because they fail in what is the signal function of a conserved charge, namely, to provide a labeling of equivalence classes of systems governed by the same underlying dynamics. The explicit gauge dependence of (19) is, of course, a direct reflection of this failure.

Another way of looking at the failure of local gauge invariance to provide any new observables is to realize that one has introduced into the problem redundant degrees of freedom by treating all four components of A_μ as independent fields in the variations. In classical field theory, one is able to specify uniquely the functional form of the gauge field only after this arbitrariness is removed by some gauge-fixing condition. Imposing such a condition is equivalent to adding a gauge-fixing term to the Lagrangian, a type of term that is not locally gauge invariant. Therefore, when one wishes to solve the equations of motion describing the gauge field, the local gauge invariance of the Lagrangian, and so the action, is destroyed along with the possibility of additional observables. On the other hand, global gauge invariance still holds and one is left with the corresponding intrinsic conserved current and charge. So, while the demand for local gauge invariance motivates both the introduction of the gauge fields and their coupling to matter, it yields no new observable since one must eventually break the symmetry to solve for the gauge field uniquely. This is consistent with the requirement that all observables be gauge invariant.

In order to see that the situation is the same when minimally charged matter fields are included, let us consider the Lagrangian

$$\mathscr{L} = \mathscr{L}_{\text{gauge}} + [(\partial_\mu + ieA_\mu)\phi]^*[(\partial^\mu + ieA^\mu)\phi] + m^2|\phi|^2 + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi. \qquad (20)$$

Here, $\mathscr{L}_{\text{gauge}}$ is given by the free electromagnetic Lagrangian (15). The complex scalar field $\phi(x)$ represents charged scalar particles while the four-component spinor $\psi(x)$ corresponds to charged Dirac particles. The slash notation designates a contraction with respect to the gamma matrices γ^μ, for example, $A\!\!\!/ = A_\mu\gamma^\mu$. Also $\bar{\psi} = \psi^\dagger\gamma^0$, where the Hermitian adjoint operation includes a transposition with respect to the spinor indices, $\alpha = 1,2,3,4$, in addition to complex conjugation, where we keep in mind that we are dealing with classical fields.

It should be noted that because of the asymmetrical structure of the Dirac term in (20), $\mathscr{L}$ is not real. Symmetrization leads to a real Lagrangian that differs from (20) by an action-preserving total divergence. One subtlety in this occurs in connection with calculations of variations such as (4) where all of the independent fields enter. When using (20), in effect, only the $\psi_\alpha(x)$ are regarded as independent fields. If the symmetrized $\mathscr{L}$ were to be used, then the $\psi_\alpha^*(x)$ fields as well would have to be taken to be independent. In the first case, one obtains a real contribution to $\delta\mathscr{L}$ that is identical to the sum of the contributions in the second instance. Another subtlety concerns the Grassmann character of ψ in the quantized case; we remark about this in Sec. IV.

Local gauge transformations now entail phase changes in the charged fields to accompany (17), namely,

$$\phi(x) \to e^{-ie\theta(x)}\phi(x), \tag{21a}$$

$$\psi(x) \to e^{-ie\theta(x)}\psi(x). \tag{21b}$$

One easily sees that for infinitesimal $\theta(x)$, the Noether current is

$$j_\mu(x) = \theta(x)J_\mu(x) - F_{\mu\nu}(x)\partial^\nu\theta(x), \tag{22}$$

where

$$J^\mu(x) = ie(\phi^* \overleftrightarrow{\partial}^\mu\phi + 2ieA^\mu|\phi|^2) + e\bar{\psi}\gamma^\mu\psi \tag{23}$$

is the gauge-invariant source current in the equation of motion for the electromagnetic field

$$\partial_\mu F^{\mu\nu}(x) = J^\nu(x). \tag{24}$$

We can then rewrite (22) as

$$j_\mu(x) = \partial^\nu[F_{\mu\nu}(x)\theta(x)], \tag{25}$$

which is just (18) again.

For constant gauge functions θ,

$$j_\mu(x) = \theta J_\mu(x), \tag{26}$$

so that $J_\mu(x)$ is the intrinsic Noether current. In contrast to the discussion of the free electromagnetic field, the Noether charge

$$Q = \int d^3x\, J_0(x) \tag{27}$$

is a useful observable because it does provide a means of classifying equivalence classes of systems. It is important to note that

$$Q = \int d^3x\, \partial^i F_{i0}(x), \tag{28}$$

so that for fields generated by localized charges and currents (28) can be converted into a surface integral at infinity that is finite and nonzero in general (Gauss' law). As in the noninteracting case, the infinite family of conserved charges associated with the general current (25) represents, in essence, redundant observables and serves no practical purpose.

We note that any infinitesimal function $\theta(x)$ can be written in the form

$$\theta(x) = \epsilon[1 + f(x)], \tag{29}$$

where ϵ is an infinitesimal parameter and $f(x)$ is an arbitrary ϵ-independent function. Then

$$j_\mu(x) = \epsilon\{J_\mu(x) + \partial^\nu[F_{\mu\nu}f(x)]\}, \tag{30}$$

so that the terms within the curly brackets resemble what we referred to as an intrinsic Noether current.[7] The form (30), while it seems to show explicitly the form of the redundancy in the Noether currents that exists for local gauge transformations, actually provides little insight over (25) as may be seen by taking $f(x)$ to be a constant. Equation (30) is, perhaps, more incisive if $f(x)$ is restricted to those functions that vanish at infinity in all spacelike directions.

It is due to the very special properties of minimally coupled gauge theories in generating interactions that the Noether current associated with a gauge symmetry transformation of all the fields can be expressed in terms of the field strengths with no explicit reference to the matter fields. In the electromagnetic case, the underlying reason for this to happen is the validity of identities such as

$$\frac{\partial X(\phi,\phi^*,A_\nu)}{\partial A_\mu(x)} = ie\phi(x)\frac{\partial X(\phi,\phi^*,A_\nu)}{\partial[\partial_\mu\phi(x)]} - ie\phi^*(x)\frac{\partial X(\phi,\phi^*,A_\nu)}{\partial[\partial_\mu\phi^*(x)]}, \tag{31}$$

where X is any function that involves only the matter fields $\phi(x)$ and $\phi^*(x)$ minimally coupled electromagnetically. Equation (31) is a variational differential form of the minimal coupling prescription. It is Eq. (31) and a similar one involving ϕ, ϕ^*, and ψ_α that are responsible for the connection between the source current of the gauge fields and the associated Noether current.

B. Non-Abelian gauge group

Non-Abelian gauge theories are distinguished from the Abelian ones by a multiplicity of gauge fields, self-interactions, and gauge transformations that involve the gauge fields in addition to the functions that parametrize them. This adds to the technical complexity of determining the consequences of gauge invariance. Nonetheless, for the Noether currents and charges the results are essentially identical to the Abelian case except for trivial multiplicities produced by the group indices so long as any matter fields that enter into the theory are coupled to the gauge fields minimally.

It is convenient to follow Cheng and Li[20] in their conventions for the phases and coupling constants for the gauge fields $A_\mu^a(x)$ as well as the gauge transformations that characterize them. We consider the least complicated situation corresponding to a gauge group G generated by a compact, simple Lie group with N Hermitian generators T^a, $a = 1,\dots,N$ that satisfy

$$[T^a,T^b] = iC_{abc}T^c. \tag{32}$$

The structure constants C_{abc} are completely antisymmetric in their indices.

The gauge fields transform globally according to the adjoint representation of G and so are enumerated in the same manner as the generators. In the absence of matter fields the dynamics of the gauge fields are generated by the Lagrangian

$$\mathscr{L}_{\text{gauge}} = -\tfrac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}, \tag{33}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gC_{abc}A_\mu^b A_\nu^c \tag{34}$$

is the field strength. Since

$$\frac{\partial\mathscr{L}_{\text{gauge}}}{\partial(\partial_\mu A_\nu^a)} = -F_a^{\mu\nu}, \tag{35a}$$

$$\frac{\partial\mathscr{L}_{\text{gauge}}}{\partial A_\nu^a} = -gF_b^{\nu\lambda}C_{bad}A_\lambda^d, \tag{35b}$$

the equations of motion are

$$\partial_\mu F_a^{\mu a} = gF_b^{\nu\lambda}C_{bad}A_\lambda^d. \tag{36}$$

Under an infinitesimal local gauge transformation parametrized by the (infinitesimal) functions $\theta^a(x)$, the change in A^a_μ is

$$\delta A^a_\mu = C_{abc}\theta^b A^c_\mu - (1/g)\partial_\mu \theta^a. \tag{37}$$

With (35) and (37) it is now straightforward to show that the Noether current corresponding to an infinitesimal local gauge transformation is

$$j_\mu(x)_{\mathrm{N}} = (-1/g)\partial^\nu[F^a_{\nu\mu}(x)\theta^a(x)]. \tag{38}$$

This differs from our Abelian result [cf. Eq. (18)] only in the appearance of the factor $(-1/g)$ that results from the different phase and coupling constant conventions used in the two instances. If we make the transformations $g\to -g$ and $\theta\to g\theta$ in Eqs. (34)–(38) we obtain a convention consistent with the one used in the Abelian case.

We next show that (38) is still valid when matter fields are introduced provided that their coupling to the gauge fields is minimal. The scalar (ϕ) and fermion (ψ) matter fields are presumed to transform globally according to arbitrary finite-dimensional irreducible representations of G that correspond to matrix realizations L^a of the generators T^a. The matrix elements L^a_{ij} as well as the field components ϕ_i and ψ_i are labeled by indices with ranges appropriate to the individual representations; the spinor indices are suppressed. It is only necessary to suppose that there is one scalar and one spinor multiplet and that they both transform according to the same representation of G. Then the complications entailed in the generalizations to arbitrary numbers of multiplets of either Lorentz types are trivially indicial.

Generally we have a Lagrangian

$$\mathscr{L} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{matter}}, \tag{39}$$

where $\mathscr{L}_{\text{gauge}}$ is given by (33) and

$$\mathscr{L}_{\text{matter}} = (D^\mu\phi)^\dagger(D_\mu\phi) + \bar{\psi} i\gamma^\mu D_\mu\psi - V. \tag{40}$$

The potential $V(\phi^*,\phi,\bar{\psi},\psi)$ includes all mass terms and the nongauge couplings of the scalar and fermion fields. The minimal gauge coupling is manifested by the covariant derivative

$$D_\mu = \partial_\mu - igL^a A^a_\mu, \tag{41}$$

which is a matrix in the finite-dimensional representation space.

The Noether current corresponding to local gauge transformations for the system represented by $\mathscr{L}$ is

$$j_\mu(x)_{\mathrm{N}} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu A^a_\nu)}\delta A^a_\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i^*)}\delta\phi_i^* + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi_i)}\delta\psi_i. \tag{42}$$

Here, δA^a_ν is given by (37) as before while

$$\delta\phi_j = -iL^a_{jk}\theta^a\phi_k \tag{43}$$

and

$$\delta\psi_j = -iL^a_{jk}\theta^a\psi_k \tag{44}$$

are the infinitesimal changes of the matter fields. We have suppressed the spinor indices in the last term on the right-hand side of (42).

The demonstration that (42) reduces to (38) follows immediately from the following two identities. The first merely generalizes the calculation employed in the pure gauge case:

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu A^a_\nu)}\delta A^a_\nu = \left(-\frac{1}{g}\right)\partial_\nu(F^{\nu\mu}_a\theta^a) - \frac{\partial\mathscr{L}_{\text{matter}}}{\partial A^a_\mu}\left(\frac{\theta^a}{g}\right). \tag{45}$$

The second corresponds to (31) and is a variational differential manifestation of minimal gauge coupling in the present context:

$$\left(\delta\phi_i\frac{\partial}{\partial(\partial_\mu\phi_i)} + \delta\phi_i^*\frac{\partial}{\partial(\partial_\mu\phi_i)} + \delta\psi_i\frac{\partial}{\partial(\partial_\mu\psi_i)}\right)\mathscr{L} = \frac{\partial\mathscr{L}_{\text{matter}}}{\partial A^a_\mu}\left(\frac{\theta^a}{g}\right). \tag{46}$$

IV. BRST SYMMETRY

In order to facilitate our discussions, we ignore matter fields and employ a well-known compact index-free notation with respect to the adjoint representation of the simple group G similar to that employed by Frampton,[16] e.g., whose treatment we loosely follow. The field strength is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gA_\mu\wedge A_\nu, \tag{47}$$

where the cross product is defined as

$$(B\wedge K)_a \equiv C_{abc}B_b K_c, \tag{48}$$

where B_b and K_c are any two objects ("vectors") that transform according to the regular representation of G. Then if a dot is used to denote the scalar product of vectors, the gauge Lagrangian (33) becomes

$$\mathscr{L}_{\text{gauge}} = -\tfrac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}. \tag{49}$$

We will need only the covariant derivative D^{ab}_μ in the regular representation,

$$D_\mu\xi = \partial_\mu\xi + gA_\mu\wedge\xi, \tag{50}$$

where ξ^a is a vector.

One method for quantizing the classical theory based on (49) leads to a (covariantly) gauge-fixed Lagrangian that contains so-called Faddeev–Popov ghosts that are denoted by c_a and c_a^+ and are independent mutually anticommuting (Grassmann) variables:

$$\mathscr{L} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{GF}} + \mathscr{L}_{\text{FPG}}, \tag{51}$$

where

$$\mathscr{L}_{\text{GF}} = -(1/2\alpha)(\partial^\mu A_\mu)^2, \tag{52}$$

$$\mathscr{L}_{\text{FPG}} = \partial_\mu c^+\cdot D^\mu c, \tag{53}$$

and α is a fixed parameter whose significance is immaterial for our consideration.

Obviously no statement can be made about the behavior of $\mathscr{L}$ under an infinitesimal local gauge transformation,

$$\delta A_\mu = -(1/g)D_\mu\theta, \tag{54}$$

unless the transformation properties of c and c^+ are defined and the gauge parameters $\theta^a(x)$ suitably restricted. This is indeed the case for the BRST transformation appropriate to $\mathscr{L}$ defined by choosing

$$\delta_{\text{BRST}}\theta = gc\,\delta\lambda, \tag{55a}$$

so

$$\delta_{\text{BRST}}A_\mu = -(D_\mu c)\delta\lambda, \tag{55b}$$

with

$$\delta_{\mathrm{BRST}} c \equiv -(g/2)(c \wedge c)\delta\lambda, \tag{55c}$$

$$\delta_{\mathrm{BRST}} c^{+} \equiv -\tfrac{1}{2}(\partial^{\mu} \cdot A_{\mu})\delta\lambda. \tag{55d}$$

Here, $\delta\lambda$ is an infinitesimal Grassmann parameter that satisfies the anticommutation relations

$$\{\delta\lambda, c\} = \{\delta\lambda, c^{+}\} = 0. \tag{56}$$

Otherwise $\delta\lambda$ behaves like an ordinary number, namely,

$$\partial_{\mu}\, \delta\lambda = 0,$$

$$[A_{\mu}, \delta\lambda\,] = 0.$$

The role of $\delta\lambda$ is to render δ_{BRST} a bosonic variation that satisfies the usual form of the Liebnitz rule when acting on a product

$$\delta_{\mathrm{BRST}}(O_1 O_2) = (\delta O_1) O_2 + O_1(\delta O_2). \tag{57}$$

The calculation of $\delta_{\mathrm{BRST}} \mathscr{L}$ is simplified with the aid of vector identities that are straightforward generalizations of those of vector analysis in three-dimensional Euclidean space.

A. Vector identities

The structure constants of the Lie algebra, C_{abc} are totally antisymmetric with respect to permutations of their indices. This attribute along with the fact that they also satisfy the group algebra (regular representation) implies that

$$C_{abk} C_{lpk} = \delta_{al}\, \delta_{bp} - \delta_{ap}\, \delta_{bl}. \tag{58}$$

Antisymmetry and (58) suffice for the proof of all of the identities listed next in terms of the arbitrary vectors A, B, K.

Clearly,

$$A \wedge B = \mp B \wedge A, \tag{59}$$

with the upper (lower) sign if all the components of A and B are commuting (anticommuting).

There are three independent types of triple-scalar products:

(i) All components of A, B, K commute:

$$A \cdot (B \wedge K) = K \cdot (A \wedge B). \tag{60}$$

(ii) $[A^{a}, B^{b}] = [A^{a}, K^{k}] = 0$; $\{B^{b}, K^{k}\} = 0$:

$$A \cdot (B \wedge K) = -K \cdot (A \wedge B). \tag{61}$$

(iii) All components of A, B, K anticommute:

$A \cdot (B \wedge K)$ is independent of ordering. (62)

There are two independent triple-vector products:

(i) $[B^{b}, K^{k}] = 0$:

$$A \wedge (B \wedge K) = (A \cdot K) B - (A \cdot B) K, \tag{63}$$

(ii) $\{B^{b}, K^{k}\} = 0$:

$$A \wedge (B \wedge K) = -(A \cdot K) \cdot B - (A \cdot B) \cdot K. \tag{64}$$

Two further identities that are repeatedly used in subsequent calculations are[16]

$$A \wedge (B \wedge B) = 2(A \wedge B) \wedge B \tag{65}$$

and

$$\partial_{\mu}(B \wedge B) = 2(\partial_{\mu} B) \wedge B, \tag{66}$$

where B is a Grassmann vector,

$$\{B^{b}, B^{a}\} = 0. \tag{67}$$

Equation (65) follows trivially from (64), (59), and (63). Only (59) and the fact that ∂_{μ} obeys the usual Liebnitz rule suffice to prove (66). We use (60)–(67) repeatedly in what follows.

B. $\delta_{\mathrm{BRST}}\, \mathscr{L}$

Since (55b) is a local gauge transformation, we must have

$$\delta_{\mathrm{BRST}} \mathscr{L}_{\mathrm{gauge}} = 0, \tag{68}$$

as can be seen from

$$\delta_{\mathrm{BRST}} F_{\mu\nu} = \theta \wedge F_{\mu\nu}$$

and

$$\delta_{\mathrm{BRST}}(F_{\mu\nu} \cdot F^{\mu\nu}) = -\theta \cdot (F_{\mu\nu} \wedge F^{\mu\nu}) = 0.$$

The remainder of the calculation is straightforward and one finds

$$\delta_{\mathrm{BRST}}\, \mathscr{L} = \partial^{\mu}[\tfrac{1}{2}(\partial^{\nu} A_{\nu}) \cdot (D_{\mu} c)]\delta\lambda, \tag{69}$$

namely, the BRST variation of $\mathscr{L}$ is a total divergence that is transformed into a surface term in the action and so has no effect on the dynamics, which are BRST invariant. We demonstrate this in a more direct manner in the Appendix. Another implication of the nonvanishing of the right-hand side of (69) is that the BRST–Noether current derived from $\mathscr{L}$ is not conserved, but one that is can be constructed simply by subtracting the square-bracketed term in (69) from the original current.

In the course of establishing (69), one exploits the fact, which follows from (55), that

$$\delta_{\mathrm{BRST}}(D_{\mu} c) = 0, \tag{70}$$

which is just one of the nilpotency relations that characterize the BRST transformation:

$$\delta^{2}_{\mathrm{BRST}}(A_{\mu}) = 0, \tag{71a}$$

$$\delta^{2}_{\mathrm{BRST}} c = 0, \tag{71b}$$

$$\delta^{2}_{\mathrm{BRST}} c^{+} = 0. \tag{71c}$$

We remark that (71c) follows using the equations of motion for c; it is interesting to note, however, that $\delta^{3}_{\mathrm{BRST}} c^{+}$ vanishes independently of the equations of motion. Since

$$\delta_{\mathrm{BRST}} \mathscr{L}_{\mathrm{gauge}} = \frac{\partial \mathscr{L}_{\mathrm{gauge}}}{\partial(\partial_{\mu} A_{\nu})} \cdot \partial_{\mu}(\delta_{\mathrm{BRST}} A_{\nu}) + \frac{\partial \mathscr{L}_{\mathrm{gauge}}}{\partial A_{\nu}} \cdot \delta_{\mathrm{BRST}} A_{\nu}, \tag{72}$$

it is seen that (68) follows from the validity of the useful identity

$$F_{\mu\nu} \cdot \partial^{\mu}(D^{\nu} c) = g(A^{\lambda} \wedge F_{\lambda\nu}) \cdot D^{\nu} c. \tag{73}$$

Of course, (73) is valid independently of (68), (55), or the equations of motion that have as yet to be exploited. It should be possible to use instead of $\mathscr{L}$ a Lagrangian that is BRST invariant. It is easily shown that

$$\delta_{\mathrm{BRST}} \mathscr{L}_{i} = 0, \tag{74}$$

where the invariant (i) Lagrangian,

$$\mathscr{L}_{i} = \mathscr{L}_{\mathrm{gauge}} - (1/2\alpha)(\partial^{\mu} \cdot A_{\mu})^{2} - c^{+} \cdot (\partial^{\mu} D_{\mu} c), \tag{75}$$

differs from $\mathscr{L}$ by a total divergence. The problem with (75) is that it contains second derivatives that we examine in more detail within Sec. V since it then requires a modification of the usual Noether formalism.

Recall, however, that we have demonstrated here only one quite conventional method of gauge fixing the Lagrangian, and that one may also consider other prescriptions. At the expense of introducing auxiliary scalar commuting bosonic fields f_a, for example, one may define a covariant gauge-fixing prescription that automatically leads to a vanishing BRST variation of $\mathscr{L}$. We thank the anonymous referee of this work for pointing out to us that for

$$\mathscr{L}_{\rm GF} = \partial_\mu f \cdot A^\mu + (\alpha/2) f \cdot f, \tag{76}$$

$$\mathscr{L}_{\rm FPG} = \partial_\mu c^+ \cdot D^\mu c, \tag{77}$$

one finds that the Lagrangian

$$\mathscr{L} = \mathscr{L}_{\rm gauge} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FPG}$$

is invariant under the following set of transformations:

$$\delta_{\rm BRST}\theta = gc\,\delta\lambda, \tag{78a}$$

$$\delta_{\rm BRST}A_\mu = -(D_\mu c)\delta\lambda, \tag{78b}$$

$$\delta_{\rm BRST}c = -(g/2)(c\wedge c)\delta\lambda, \tag{78c}$$

$$\delta_{\rm BRST}c^+ = -f\delta\lambda, \tag{78d}$$

$$\delta_{\rm BRST}f = 0. \tag{78e}$$

This Lagrangian, we note, suffers from no higher derivative problem, and moreover that the nilpotency relations of the BRST transformations, Eq. (71) plus $\delta^2_{\rm BRST}f = 0$, formulated in this way are found to be satisfied without recourse to the ghost equations of motion.

V. BRST NOETHER CURRENTS

The Noether formalism for the particular realizations of BRST symmetry considered in Sec. IV has two complications not encountered previously. One is the higher-derivative problem pointed out at the end of Sec. IV. The other has to do with the fact that c and c^+ are anticommuting Grassmann quantities. So that for any variation δ of the fields we have, e.g.,

$$\delta\,\mathscr{L} = \delta c^+ \cdot \frac{\partial\,\mathscr{L}}{\partial c^+} + \delta c \cdot \frac{\partial\,\mathscr{L}}{\partial c} + \cdots, \tag{79a}$$

$$= \frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial c^+} \cdot \delta c^+ + \frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial c} \cdot \delta c + \cdots, \tag{79b}$$

where $\partial\,\mathscr{L}/\partial c$ and $\overleftarrow{\partial}\,\mathscr{L}/\partial c$, e.g., denote the left- and right-handed variational derivatives,[21] respectively, of $\mathscr{L}$ with respect to the Grassmann quantity c. Consequently, the relevant Noether currents are given by

$$j_\mu(x) = \delta c^+ \cdot \frac{\partial\,\mathscr{L}}{\partial(\partial_\mu c^+)} + \delta c \cdot \frac{\partial\,\mathscr{L}}{\partial(\partial_\mu c)} + \cdots, \tag{80a}$$

$$= \frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial(\partial_\mu c^+)} \cdot \delta c^+ + \frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial(\partial_\mu c)} \cdot \delta c + \cdots. \tag{80b}$$

The Lagrangian formalism holds as usual with respect to c^+ and c so long as either only right or only left variational derivatives are employed; evidently the same equations of motion are obtained either way.

Since

$$\frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial(\partial_\mu c^+)} = -\frac{\partial\,\mathscr{L}}{\partial(\partial_\mu c^+)}$$
$$= -D^\mu c, \tag{81a}$$

$$\frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial(\partial_\mu c)} = -\frac{\partial\,\mathscr{L}}{\partial(\partial_\mu c)}$$
$$= +\partial^\mu c^+, \tag{81b}$$

it follows that the BRST Noether current associated with $\mathscr{L}$ is[22]

$$[j_\mu(x)]_{\rm BRST} = F_{\mu\nu}\cdot D^\nu c - (g/2)(\partial_\mu c^+)\cdot(c\wedge c) + (2/\alpha)(\partial^\nu A_\nu)\cdot D_\mu c, \tag{82}$$

where we have dropped an overall $\delta\lambda$ factor on the right side of (82). A conserved BRST current is obtained from (82) by subtracting the divergence (69) yielding (again without the overall $\delta\lambda$ factor)

$$j_\mu(x)_{\rm BRST} = F_{\mu\nu}\cdot D^\nu c - (g/2)(\partial_\mu c^+)\cdot(c\wedge c) + (1/\alpha)(\partial^\nu A_\nu)\cdot(D_\mu c). \tag{83}$$

The verification that one also obtains (83) from the BRST-invariant Lagrangian $\mathscr{L}_i$ requires the use of a higher derivative Lagrangian and Noether formalism that we quote, for simplicity, in terms of a scalar, non-Grassmann field $\phi(x)$. If

$$\overline{\mathscr{L}} = \overline{\mathscr{L}}(\phi, \partial_\mu\phi, \partial^2\phi), \tag{84}$$

then the action principle implies the equations of motion

$$O_{\rm L}\,\mathscr{L} = 0, \tag{85}$$

where

$$O_{\rm L} = -\partial^2\left(\frac{\partial}{\partial(\partial^2\phi)}\right) + \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu\phi)}\right) - \frac{\partial}{\partial\phi} \tag{86}$$

is the Lagrangian operator for this case. For any variation δ, the Noether current is

$$j_\mu(x) = \left[\frac{\partial\,\mathscr{L}}{\partial(\partial^\mu\phi)} - \partial_\mu\left(\frac{\partial\,\mathscr{L}}{\partial(\partial^2\phi)}\right)\right]\delta\phi + \frac{\partial\,\mathscr{L}}{\partial(\partial^2\phi)}\,\partial_\mu(\delta\phi), \tag{87}$$

which is conserved if $\delta\,\mathscr{L} = 0$. The adaptation of (84)–(87) to the Grassmann situation encountered in connection with $\mathscr{L}$ and the BRST transformation (55) is clear from our treatment of ordering questions earlier in this section. One finds, for example, that

$$j_\mu(x)_{\rm BRST} = \frac{\partial\,\mathscr{L}_i}{\partial(\partial^\mu A_\nu)}\cdot\delta_{\rm BRST}A_\nu + J^c_\mu(x), \tag{88}$$

where

$$J^c_\mu(x) = \left[\frac{\overleftarrow{\partial}\,\mathscr{L}_i}{\partial(\partial^\mu c)} - \partial_\mu\left(\frac{\overleftarrow{\partial}\,\mathscr{L}}{\partial(\partial^2 c)}\right)\right]\cdot\delta_{\rm BRST}c + \frac{\overleftarrow{\partial}\,\mathscr{L}_i}{\partial(\partial^2 c)}\cdot\partial_\mu(\delta_{\rm BRST}c). \tag{89}$$

One then obtains (83), with the aid of some of our previously quoted identities, which demonstrates the consistency of the two approaches.

As a further check on the validity of our result (83), it is instructive to verify directly that

$$\partial^\mu j_\mu(x)_{\rm BRST} = 0, \tag{90}$$

using the explicit equations of motion[23]:

$$\partial^\mu F_{\mu\nu} = \mathscr{J}_\nu(x), \tag{91a}$$

$$\partial^\mu(D_\mu c) = 0, \tag{91b}$$

$$\partial^2 c^+ = g(\partial_\mu c^+) \wedge A^\mu. \tag{91c}$$

The conserved *source current* for the field strength,

$$\mathscr{J}_\nu(x) = -(1/\alpha)\partial_\nu(\partial\cdot A) + g(\partial_\nu c^+)\wedge c + gA^\lambda \wedge F_{\nu\lambda}, \tag{92}$$

is evidently quite distinct from the Noether current (83). Both of these currents, in turn, are each different from the conserved Noether current that is associated with the invariance

$$\delta_\theta \mathscr{L} = \delta_\theta \mathscr{L}_i = 0, \tag{93}$$

under the global (rigid) gauge transformation

$$\delta_\theta A_\mu = gA_\mu \wedge \theta, \tag{94a}$$

$$\delta_\theta c = g(c\wedge\theta), \tag{94b}$$

$$\delta_\theta c^+ = g(c^+ \wedge \theta), \tag{94c}$$

namely,

$$J_\mu(x) = F_{\mu\nu}\wedge A^\nu + (1/\alpha)(\partial^\nu A_\nu)\wedge A_\mu - \partial_\mu c^+ \wedge c + c^+ \wedge D_\mu c. \tag{95}$$

Now $J_\mu(x)$ corresponds to the conservation of "charge," although $J_\mu(x)$ and the conserved charge

$$Q = \int d^3x\, J_0(x) \tag{96}$$

are generally gauge variant.

VI. SUMMARY

We have elaborated upon the customary pedagogical treatments of Noether's theorem when it is applied to local gauge transformations. We have shown for both Abelian and non-Abelian gauge groups that the Noether currents associated with local gauge symmetry are expressed solely in terms of field strengths and are trivially conserved. The same form for the current is obtained whether or not the gauge fields are coupled to matter so long as that coupling is minimal. The resultant currents and their associated charges do not generally correspond to physical observables except for constant gauge parameters, in which case they reduce to the currents and charges implied by global (or rigid) gauge symmetry. This is reflective of the lack of gauge fixing.

When the gauge fields are constrained, as is necessary to obtain unique solutions of the classical equations or to quantize the theory, the local symmetry is destroyed and the consequences of Noether's theorem are much more limited. A particularly interesting case is when the quantization is carried out using the device of anticommuting Faddeev–Popov ghosts. The gauge–fixed Lagrangian still possesses a restricted local gauge symmetry, BRST invariance. The application of Noether's theorem in this case involves subtleties in Grassmann calculus and the higher derivative Lagrangian formalism not usually explicated in introductory treatments of gauge theories. We have given a self-contained treatment that considers these points in detail and also considers the differences among the source current, and the two Noether gauge symmetry currents (global and BRST) that arise in a quantized gauge field theory. Two theorems concerning the nonuniqueness in the choice of field theory Lagrangians are proved in the Appendix.

ACKNOWLEDGMENTS

This work was supported by the U.S. National Science foundation and by the U.S. Department of Energy, Division of High Energy Physics, Contract W-31-100-ENG-38.

APPENDIX

It is often stated that Lagrangians that differ by total divergences yield equivalent equations of motion because such terms are converted to surface terms in the action principle that can be discarded. The result, of course, is actually independent of the consideration of surface integrals. We explore this question here because it takes on a slightly different form for Lagrangians containing higher derivatives such as $\mathscr{L}_i$.

The usual result is that if we have

$$\overline{\mathscr{L}} = \mathscr{L}(\phi,\partial_\mu\phi) + \partial_\mu f^\mu(\phi), \tag{A1}$$

then, because

$$\partial_\mu f^\mu(\phi) = \frac{\partial f^\mu}{\partial\phi}\,\partial_\mu\phi, \tag{A2}$$

we have

$$O_{\rm L}\overline{\mathscr{L}} = O_{\rm L}\mathscr{L}, \tag{A3}$$

where

$$O_{\rm L} = \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu\phi)}\right) - \frac{\partial}{\partial\phi}. \tag{A4}$$

From (A3) we see that the equations of motion will be the same for both $\mathscr{L}$ and $\overline{\mathscr{L}}$.

A number of generalizations of the preceding nonuniqueness (of the choice of Lagrangian) theorem to higher derivative Lagrangians are possible. The one we require, however, is where

$$\overline{\mathscr{L}} = \overline{\mathscr{L}}(\phi,\partial_\mu\phi,\partial_\lambda\partial_\rho\phi) = \mathscr{L}(\phi,\partial_\mu\phi) + \partial_\mu f^\mu(\phi,\partial_\nu\phi). \tag{A5}$$

Then, since

$$\partial_\mu f^\mu = \frac{\partial f^\mu}{\partial\phi}\,\partial_\mu\phi + \frac{\partial f^\mu}{\partial(\partial_\nu\phi)}\,\partial_\mu\partial_\nu\phi, \tag{A6}$$

one finds, after a somewhat lengthy calculation, that

$$\hat{O}_{\rm L}\overline{\mathscr{L}} = \hat{O}_{\rm L}\mathscr{L}, \tag{A7}$$

where

$$\hat{O}_{\rm L} = -\partial_\lambda\partial_\rho\frac{\partial}{\partial(\partial_\lambda\partial_\rho\phi)} + \partial_\mu\frac{\partial}{\partial(\partial_\mu\phi)} - \frac{\partial}{\partial\phi}. \tag{A8}$$

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[22]The corresponding expression obtained in Ref. 16 is incorrect.
[23]The fact that one obtains the same equations of motion from $\mathscr{L}$ and $\mathscr{L}_i$ is a special case of the nonuniqueness theorem proven in the Appendix.

A communication on electrical charge relaxation in metals

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(Received 4 April 1988; accepted for publication 13 March 1989)

The correction of an erroneous textbook derivation on electrical charge relaxation in conductors is discussed. The actual decay in a good conductor is damped oscillatory instead of the simple exponential decay that is often claimed, while short wavelength disturbances spread through the medium much like particles of mass $m = \hbar\omega_p/\langle v^2\rangle$, where ω_p is the plasmon frequency and $\langle v^2\rangle$ is the mean-square electron velocity.

A derivation that is found in many prominent electromagnetics textbooks for physicists[1] and engineers,[2] and at least one well-known optics text,[3] concerns the relaxation of electrical charge in a conductor. Its expressed purpose is to demonstrate that the free relaxation of a disturbance away from equilibrium in the charge density is an extremely rapid process, so that on the time scale of most physical events no electrical charge perturbance can prevail inside a good conductor. Even though it has been pointed out before[4,5] that both the proof and the result given in these books are seriously in error, many authors of new texts continue to include it. The error is due to the implicit assumption that the relaxation time of the charge density is a dc phenomenon, even though the time scale of the process is calculated to be as short as 10^{-19} s. This decay rate actually corresponds to the x-ray frequency region! It is desirable that proofs presented in undergraduate textbooks be within the grasp of the average student but educational objectives are poorly served by erroneous proofs, especially when widely adopted. Moreover, for those texts that are written at the graduate level, such "simplification" is even less appropriate.

This issue was apparently first attended to by Ashby,[4] who points out that the actual electrical decay time of a conductor should in fact be of the order of the electron collision time, which is about 2×10^{-14} s for copper. A more recent discussion, by Ohanian,[5] makes an excellent point of the fact that the relaxation actually proceeds in three stages. The first stage is the relaxation of electrical charge density. The second stage is the expulsion of the electric and magnetic fields to the exterior of the conductor and that of currents to the surface. During the third stage, the process terminates with the slower ohmic and radiative damping of the surface currents. Ohanian's discussion centers on the second and third stages of the process, while Ashby treats the first.

Physical consideration shows that a description of the first stage, that of charge relaxation, depends on whether or not the total charge is zero. In the former case, the relaxation of current and charge densities is accompanied by a transport of the surplus charge to the surface of the conductor, which requires special mathematical treatment. The time constant associated with this stage is then also related to the size and geometry of the conductor, as well as to the initial charge distribution. Moreover, the equations describing the transport of the uncompensated charge are nonlinear, the solution of which would be much more difficult. And finally, since there is no definitive decoupling

