In general relativity gravitational effects (as seen near a massive object or in cosmology) are inextricably mixed with local motions. As Synge observes, a truly gravitational effect should involve the Riemann tensor, which the formulas (41) or (42) do not. Thus, in principle, we could have $z \neq 0$ from these formulas even in flat space–time scenarios. For example, the Robertson–Walker model with $k=-1$ and $a(t) \propto t$ is flat but has a cosmological redshift.

This result derived by Synge is, unfortunately, not well known even among the community of general relativists today. With spectral shifts being used so often in different contexts, it is worth appreciating the underlying theme that Synge had highlighted. The purpose of this communication was to bring the result to the notice of modern workers in relativity with the help of explicit examples familiar to them.

[1] N. T. Bishop and P. T. Landsberg, "Equivalence principle: 60 years of a misuse?," Nature **252**, 459–460, (1974)

[2] N. T. Bishop and P. T. Landsberg, "Gravitational redshift and the equivalence principle," Found. Phys. **6**, 727–737 (1976).

[3] J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).

[4] J. V. Narlikar, *Introduction to Cosmology* (Cambridge University Press, Cambridge, U.K., 1993).

Dynamic interpretation of Maxwell's equations

Harry Soodak and Martin S. Tiersten

Physics Department, The City College, City University of New York, New York 10031

(Received 16 September 1993; accepted 12 May 1994)

Conventional discussions of Maxwell's equations in free space have for many years taken a historical approach starting with electrostatics and magnetostatics, and have taught us that the sources of **E** are electric charge and $\dot{\mathbf{B}}$, and the sources of **B** are electric current and $\dot{\mathbf{E}}$. However, a direct dynamic reading of Maxwell's differential equations leads unquestionably to the surprisingly different conclusions that the sources of **E** are electric current and curl **B**, and the single source of **B** is curl **E**. In this dynamic reading of Maxwell's equations, electric field is generated locally by electric current, and fields propagate away from the current source by the dual mechanisms of curl **E** generating **B** locally and curl **B** generating **E** locally.

I. INTRODUCTION

Conventional discussions of Maxwell's differential equations in free space

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}, \tag{1}$$

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \dot{\mathbf{E}}), \tag{2}$$

follow the historical development of electromagnetism, proceeding from electrostatics (Coulomb) and magnetostatics (Ampere and Biot–Savart) to Faraday's induction and finally to Maxwell's displacement current and field propagation. It seems to follow naturally from electrostatics and magnetostatics, that charge and current are the sources, respectively, of electric and magnetic field. It again seems natural to interpret the contributions of Faraday and Maxwell by saying that electric field is generated also by time varying magnetic field, and that magnetic field is generated also by time varying electric field. Finally, recognizing the neat fit between these interpretations and the mathematics of the Helmholtz theorem, which shows that a vector field is determined by its divergence and its curl as sources of the field, it is no wonder that most physicists trained in this tradition have had no reason to question these "natural" teachings, or to look for alternative interpretations. The Helmholtz theorem[1]

$$\mathbf{V}(\mathbf{r},t) = \int \operatorname{div} \mathbf{V}(\mathbf{r}',t) \frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r}' + \int \operatorname{curl} \mathbf{V}(\mathbf{r}',t) \times \frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r}' \tag{3}$$

expresses a vector field $\mathbf{V}(\mathbf{r},t)$ as a sum of an irrotational (Coulomb-type) field with source density div **V**, and a solenoidal (Biot–Savart-type) field with source density curl **V**. The integrals in Eq. (3) extend over all space.

In this paper we describe a surprisingly different interpretation, one that follows naturally by reading Maxwell's differential equations directly as a set of local dynamic field equations. In this reading, the instantaneous state of the electromagnetic field is described by the values of $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ at all points of space, and the rate of change of the state, described by $\dot{\mathbf{E}}$ and $\dot{\mathbf{B}}$, is determined by the instantaneous values of the fields and of the current distribution **j**, through the two curl equations of Maxwell,

$$\dot{\mathbf{E}} = -\frac{1}{\epsilon_0} \mathbf{j} + \frac{1}{\epsilon_0 \mu_0} \operatorname{curl} \mathbf{B}, \tag{4}$$

$$\dot{\mathbf{B}} = -\operatorname{curl} \mathbf{E}. \tag{5}$$

We can imagine numerically integrating these equations, time step by time step, using

$$\mathbf{E}(\mathbf{r},t+\delta t) = \mathbf{E}(\mathbf{r},t) + \mathbf{S}_E(\mathbf{r},t)\,\delta t,$$

$$\mathbf{B}(\mathbf{r},t+\delta t) = \mathbf{B}(\mathbf{r},t) + \mathbf{S}_B(\mathbf{r},t)\,\delta t,$$

where $\mathbf{S}_E(\mathbf{r},t)$ is the right side of Eq. (4) and generates $\mathbf{E}$ locally, and $\mathbf{S}_B(\mathbf{r},t)$ is the right side of Eq. (5) and generates $\mathbf{B}$ locally.

Although the consequences of the local dynamic view of Maxwell's equations were a surprise to us when we first realized them, we discovered later that the view had already been adopted in a textbook by Pieter B. Visscher,[2] although from a somewhat different perspective.

In this reading of Maxwell's equations, electric field is generated locally by the current (density) $\mathbf{j}$ and by curl $\mathbf{B}$ (multiplied by coefficients), and magnetic field is generated locally by $-$curl $\mathbf{E}$; these are the right sides of Eqs. (4) and (5). Electric field $\mathbf{E}$ is *initiated* locally by the current $\mathbf{j}$, which is external to the field, and field propagates away from the current source by the dual field mechanisms of $-$curl $\mathbf{E}$ generating $\mathbf{B}$ locally and curl $\mathbf{B}/\epsilon_0\mu_0$ generating $\mathbf{E}$ locally.[3] We will use the natural terminology that $\mathbf{j}$ and curl $\mathbf{B}$ (multiplied by coefficients) are the sources of $\mathbf{E}$, and that $-$curl $\mathbf{E}$ is the source of $\mathbf{B}$.

Also, both of Maxwell's divergence equations are direct consequences of the two curl equations and of charge conservation. The vanishing of div $\mathbf{B}$ is a consequence of the fact that the source of $\mathbf{B}$ has zero divergence. The proportionality of div $\mathbf{E}$ and ρ is a consequence of the fact that currents generate both electric field and charge so as to satisfy div $\mathbf{E}=\rho/\epsilon_0$. Additionally, electrostatics and magnetostatics are regarded simply as equilibrium situations in which $\dot{\mathbf{E}}$ and $\dot{\mathbf{B}}$ are both zero.

It is to be expected that the Helmholtz theorem plays no role in a local dynamic interpretation of Maxwell's equations, because it can be regarded simply as a mathematical identity that describes a static and nonlocal relation between a vector field at one position and its divergence and curl at other positions, all at the same time.

The direct dynamic interpretation is discussed further in Sec. II, along with some examples and consequences. In Sec. III, the historical evolution from electrostatics and magnetostatics to Faraday's induction and finally to Maxwell's displacement current and field propagation is illuminated by discussing two approximations to Maxwell's equations, called the quasistatic and the Faraday approximations. Section IV presents an indirect dynamic interpretation in which the electric field is resolved into a Coulomb part and the remainder, and the magnetic field into a Biot–Savart part and the remainder. Finally, in Sec. V we compare briefly the ideas of this paper with interpretative comments made by other authors.

It is of course true that Maxwell's equations need not be "interpreted" at all, and that their quantitative consequences follow directly from the equations themselves and are independent of whatever interpretation is adopted. It is also true, however, that interpretations, viewpoints, and ways of thinking about the equations, are useful for both qualitative and quantitative understanding.[4]

II. DYNAMIC INTERPRETATION

A. The divergence equations from the dynamic curl equations

As stated in Sec. I, the two Maxwell "curl equations," in the forms of Eqs. (4) and (5), describe the local dynamics of the electromagnetic field. The electromagnetic sources of electric field are the terms on the right side of Eq. (4), $\mathbf{j}$ and curl $\mathbf{B}$ (aside from coefficients); and, the single source of magnetic field is $-$curl $\mathbf{E}$.

Taking the divergence of Eq. (5) gives $(\partial/\partial t)$div $\mathbf{B}=0$, which states that the electromagnetic source of $\mathbf{B}$ (which is $-$curl $\mathbf{E}$) generates only divergence-free magnetic field. If we then imagine no sources of field other than those on the right sides of Eqs. (4) and (5), it follows[5] that div $\mathbf{B}=0$.

The divergence of Eq. (4) results in $(\partial/\partial t)$div $\mathbf{E}=-(1/\epsilon_0)$div $\mathbf{j}$. Using charge conservation,

$$\dot{\rho}=-\text{div }\mathbf{j} \tag{6}$$

then leads to $\partial(\text{div }\mathbf{E}-\rho/\epsilon_0)/\partial t=0$, which states[5] that the electric field generated by $\mathbf{j}$ satisfies Maxwell's divergence equation, div $\mathbf{E}=\rho/\epsilon_0$. The current density $\mathbf{j}$ generates both $\mathbf{E}$ and ρ in such a way that the divergence equation is automatically satisfied. The conclusion is that the Maxwell divergence equations are consequences of the two dynamic curl equations and of charge conservation.

B. Static fields as equilibrium fields

An equilibrium electromagnetic field is one for which $\dot{\mathbf{E}}$ and $\dot{\mathbf{B}}$ are zero at all $\mathbf{r}$. Equating them to zero in Eqs. (4) and (5), or in Eqs. (1) and (2), results in curl $\mathbf{E}=0$ and curl $\mathbf{B}=\mu_0\mathbf{j}$. Taking the time derivative of the div $\mathbf{E}$ equation in Eq. (1) shows that $\dot{\rho}=0$, and therefore from Eq. (6), that div $\mathbf{j}=0$. The static, or equilibrium fields, which we denote by $\mathbf{E}_C$ and $\mathbf{B}_{BS}$ (for Coulomb and Biot–Savart) are then determined through Helmholtz's theorem by their divergences and curls,

$$\begin{aligned} &\text{div }\mathbf{E}_C=\rho/\epsilon_0, &&\text{curl }\mathbf{E}_C=0,\\ &\text{div }\mathbf{B}_{BS}=0, &&\text{curl }\mathbf{B}_{BS}=\mu_0\mathbf{j}, \end{aligned} \tag{7}$$

resulting in the following fields, in which the integrals extend over all space:

$$\mathbf{E}_C(\mathbf{r})=\int\frac{\rho(\mathbf{r}')}{\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}', \tag{8}$$

$$\mathbf{B}_{BS}(\mathbf{r})=\int\mu_0\mathbf{j}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'. \tag{9}$$

C. Dynamics of a current delta pulse

Imagine an initial condition with no field, no charges, and no currents, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$, and ρ vanishing over all space. Then imagine a "delta" pulse of current at time zero of the form

$$\mathbf{j}(\mathbf{r},t)=\sigma(\mathbf{r})\delta(t), \tag{10}$$

where $\delta(t)$ is the Dirac delta function. Integrating Eqs. (4), (5), and (6) over the duration of the delta pulse, from $t=0^-$ immediately before the pulse to $t=0^+$ immediately after the delta pulse, gives

$$\begin{aligned} &\mathbf{E}(\mathbf{r},0^+)=-\frac{1}{\epsilon_0}\sigma(\mathbf{r}), &&\mathbf{B}(\mathbf{r},0^+)=0,\\ &\rho(\mathbf{r},0^+)=-\text{div }\sigma(\mathbf{r}). \end{aligned} \tag{11}$$

Also, using Eqs. (4) and (5) at time $t=0^+$, at which $\mathbf{j}=0$ and $\mathbf{E}$ and $\mathbf{B}$ are given by Eq. (11), leads to

$$\dot{\mathbf{E}}(\mathbf{r},0^+)=0, \qquad \dot{\mathbf{B}}(\mathbf{r},0^+)=-\text{curl }\mathbf{E}(\mathbf{r},0^+). \tag{12}$$

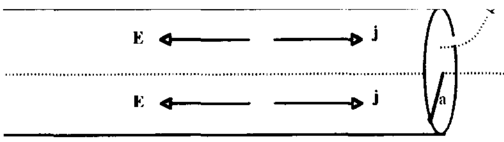

Fig. 1. A tube current pulse generating a uniformly charged disk and a uniform tube electric field.

Suppose first that div $\sigma = 0$, in which case no charge is generated, and $\rho=0$. Then after the delta pulse, for $t>0^+$, both $\mathbf{j}$ and ρ are zero, and the fields $\mathbf{E}$ and $\mathbf{B}$ satisfy Maxwell's free field equations and thus propagate away to infinity at the wave speed

$$c=(\mu_0\epsilon_0)^{-1/2}. \tag{13}$$

When div $\sigma \neq 0$, a charge density $\rho=-\text{div}\,\sigma$ is generated by the current delta pulse, and this charge density remains stationary as long as no further currents act. If the electric field generated by the delta pulse, $\mathbf{E}(\mathbf{r},0^+)$, is by chance or design the Coulomb field, $\mathbf{E}_C$, associated with the charge density generated by the pulse, then nothing further happens after the pulse; the situation immediately after the pulse is one of static equilibrium, with $\mathbf{B}=0$, $\rho=-\text{div}\,\sigma$, and $\mathbf{E}=\mathbf{E}_C$ where div $\mathbf{E}_C=-\text{div}\,\sigma/\epsilon_0$ and curl $\mathbf{E}_C=0$. When the generated electric field is different from $\mathbf{E}_C$, then field dynamics results in creation of magnetic field, and field propagation takes place in such a way that the electric and magnetic fields settle down to the equilibrium values, $\mathbf{B}_{BS}$ and $\mathbf{E}_C$, associated with $\mathbf{j}=0$ and $\rho=-\text{div}\,\sigma$, where $\mathbf{B}_{BS}=0$, and $\mathbf{E}_C$ is determined by curl $\mathbf{E}_C=0$ and div $\mathbf{E}_C=-(\text{div}\,\sigma/\epsilon_0)$ (and the condition that it vanishes at infinity).

D. Examples

As a first example, consider a current delta pulse with $\sigma(\mathbf{r})=-[Q/(4\pi r^2)]\hat{\mathbf{r}}$. This pulse generates a point charge Q at the origin, and an electric field $\mathbf{E}=[Q/(4\pi\epsilon_0 r^2)]\hat{\mathbf{r}}$, which is precisely the equilibrium static field for the charge Q. There is then no further change. This inverse square current pulse generates the electrostatic situation of a point charge and its equilibrium field.

Now consider a delta pulse with $\sigma(\mathbf{r})=(Q/\pi a^2)\,\bar{\eta}(x)\,\bar{\eta}(r'-a)\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector in the positive x direction, r' is distance perpendicular to the x axis, and $\bar{\eta}$ is the complement of the step function, equal to unity when the argument is less than zero, and zero when the argument is positive. This pulse consists of a uniform current density of magnitude $Q/\pi a^2$ in the x direction within the tube of radius a centered on a negative x axis, as indicated in Fig. 1. The pulse generates a uniformly charge disk of total charge Q, centered at the origin and perpendicular to the x axis. It also generates an electric field $\mathbf{E}=-(1/\epsilon_0)\sigma(\mathbf{r})$ which is a uniform field within the tube directed leftward out from the charged disk. This electric field is not the equilibrium field for the disk of charge; although it satisfies the divergence equation (having the correct flux emanating from the charge), its curl is not zero. How the field starts changing toward the equilibrium electrostatic field will be considered shortly.

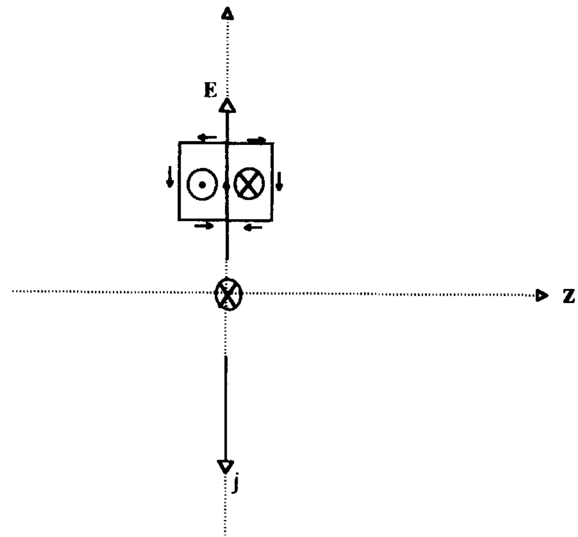

Fig. 2. A sheet current pulse in the x,y plane generates a sheet $\mathbf{E}$ field. Because $\mathbf{E}$ vanishes off the plane, its circulation is in the $\hat{\mathbf{x}}$ direction on the positive z side of the plane the—$\hat{\mathbf{x}}$ direction on the negative z side. The paths for the line integrals are indicated by the arrows.

An electric dipole consisting of positive and negative charged disks can be generated by a current pulse in a tube of finite length. The electric field generated by this pulse is again not the equilibrium field.

Field propagation may be illustrated by the delta pulse with $\sigma(\mathbf{r})=-\lambda\delta(z)\hat{\mathbf{y}}$. As schematized in Fig. 2, this describes a current pulse in the x,y plane, directed in the negative y direction, with current λ per unit length along the x direction. The pulse has zero divergence and consequently generates no charge. From Eqs. (11) and (12), the fields generated by the pulse satisfy

$$\mathbf{E}(\mathbf{r},0^+)=\frac{\lambda}{\epsilon_0}\delta(z)\hat{\mathbf{y}}, \quad \dot{\mathbf{E}}(\mathbf{r},0^+)=0, \tag{14}$$

$$\mathbf{B}(\mathbf{r},0^+)=0, \quad \dot{\mathbf{B}}(\mathbf{r},0^+)=-\text{curl}\,\mathbf{E}(\mathbf{r},0^+)=\frac{\lambda}{\epsilon_0}\delta'(z)\hat{\mathbf{x}}. \tag{15}$$

The generated electric field is a sheet of $\mathbf{E}$ in the x,y plane, directed in the positive y direction, which is then propagated in both directions outward from the sheet toward infinity by the dual mechanism of $-$curl $\mathbf{E}$ generating $\mathbf{B}$ and c^2 curl $\mathbf{B}$ generating $\mathbf{E}$. The fields, $\mathbf{E}=E_y\hat{\mathbf{y}}$ and $\mathbf{B}=B_x\hat{\mathbf{x}}$ at times $t>0^+$ are given by[6]

$$E_y=\frac{1}{2}\frac{\lambda}{\epsilon_0}[\delta(z-ct)+\delta(z+ct)]$$

$$B_x=\frac{1}{2c}\frac{\lambda}{\epsilon_0}[-\delta(z-ct)+\delta(z+ct)]. \tag{16}$$

Half of the initial $\mathbf{E}$ sheet amplitude propagates in the positive z direction and half in the negative z direction, each half propagating along with its associated $\mathbf{B}$ field. The fields propagating in the positive z direction are sheets given by $E_y=(\lambda/2\epsilon_0)\delta(z-ct)$ and $B_x=-(\lambda/2c\epsilon_0)\delta(z-ct)$.

The dual mechanism that gives rise to the propagation, $-$curl $\mathbf{E}$ generating $\mathbf{B}$ and c^2 curl $\mathbf{B}$ generating $\mathbf{E}$, may be described qualitatively as follows. The curl of the initially

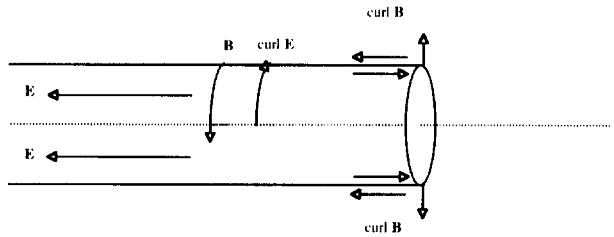

Fig. 3. The **B** field induced by $-$curl **E** of the field in Fig. 1, and the curl of that **B** field, which induces new **E** field.

generated **E**, of Fig. 2, is a double layer[7] centered at the x,y plane, directed in the $+\hat{\mathbf{x}}$ direction on the positive z side and the $-\hat{\mathbf{x}}$ direction on the negative z side. Consequently, $-$curl **E** generates a double layer of **B** directed in the $-\hat{\mathbf{x}}$ direction on the positive z side and the $+\hat{\mathbf{x}}$ direction on the negative z side. Note that the direction of **B** on both sides is such that **E×B** is outward from the x,y plane. This is the mechanism for initiation of the two outwardly propagating **B** sheets. In similar fashion, each **B** sheet generates a double **E** layer, with the **E** at the side closer to the x,y plane in the $-\hat{\mathbf{y}}$ direction, thus canceling the previous **E** on the closer side, and the **E** on the side further from the x,y plane in the initial **E** direction, that of $+\hat{\mathbf{y}}$, thus propagating the initial **E** outward (1/2 in each direction). In this fashion each of the propagating **E** sheets generates a double **B** layer that cancels the previous **B** sheet on the side closer to the source plane, and generates the new **B** sheet on the side further from the source plane. In the same way, each of the propagating **B** sheets generates a double **E** layer that cancels the previous **E** sheet on the side closer to the source plane, and generates the new **E** sheet on the side further from the source plane. (For a particle of matter to propagate in this fashion, it would have to continually be destroyed at its current position and recreated at its next position.)

A similar analysis shows how the tube of **E** lines emanating leftward out of the charged disk of the second example above, Fig. 1, starts to change into the electrostatic equilibrium field. As indicated in Fig. 3, the curl of the tube **E** field is a cylindrical sheet field running around the tube in the sense related to the direction of **E** by the right-hand rule. The source of **B**, which is $-$curl **E**, thus generates a cylindrical sheet of magnetic lines in the opposite sense. The source of **E**, c^2 curl **B**, for this **B** field contains a double cylindrical layer centered at the curved surface of the tube, directed in the initial **E** direction at the outside sheet and oppositely at the inside sheet, thus generating **E** outside the tube in the initial **E** direction and reducing the field inside the tube. The curl of **B** also contains contributions that are radially outward from the circumference of the charged disk end of the tube. (This may be seen by taking the circulation of **B** around a loop with its left arm being a small arc of the circumference and right arm a small parallel arc displaced slightly to the right.) It is these contributions that generate radially outward components of new **E** field. The continuation of these mechanisms results eventually in the electrostatic field of the uniformly charged disk, all **B** and all other contributions to **E** propagating away to infinity.

E. Creation and destruction of arbitrary electric and magnetic fields

Imagine again an initial condition with no fields, no charges, and no currents. It was shown in Sec. II C, that the current delta pulse $\mathbf{j}(\mathbf{r},t)=\sigma(\mathbf{r})\delta(t)$ generates at time $t=0^+$ the electric field $\mathbf{E}(\mathbf{r},0^+)=-\sigma(\mathbf{r})/\epsilon_0$ but does not generate any magnetic field. This result may also be stated in the following way. Any initial field with arbitrary electric field $\mathbf{E}(\mathbf{r},0^+)$ and no magnetic field, $\mathbf{B}(\mathbf{r}0^+)=0$, could be generated by the immediately preceding delta current pulse given by $\mathbf{j}(\mathbf{r},t)=\sigma(\mathbf{r})\delta(t)$ with $\sigma(\mathbf{r})=-\epsilon_0\mathbf{E}(\mathbf{r},0^+)$.

We prove at the end of this subsection that any initial field with arbitrary magnetic field $\mathbf{B}(\mathbf{r},0^+)$ and no electric field, $\mathbf{E}(\mathbf{r},0^+)=0$, could be generated by the immediately preceding "delta prime" current pulse $\mathbf{j}(\mathbf{r},t)=\Sigma(\mathbf{r})\delta'(t)$, where $\delta'(t)$ is the derivative of the delta function and $\Sigma(\mathbf{r})$ satisfies curl $\Sigma(\mathbf{r})=\epsilon_0\mathbf{B}(\mathbf{r},0^+)$ and has arbitrary divergence.

The following pulse creation theorem is then true. Any specified field at any time can be created by an immediately preceding current pulse. The fields at time t_0^+, $\mathbf{E}(\mathbf{r},t_0^+)$ and $\mathbf{B}(\mathbf{r},t_0^+)$, can be created out of zero field at time t_0^-, by the current pulse

$$\mathbf{j}(\mathbf{r},t)=\sigma(\mathbf{r})\,\delta(t-t_0)+\Sigma(\mathbf{r})\,\delta'(t-t_0), \tag{17}$$

where

$$\sigma(\mathbf{r})=-\epsilon_0\mathbf{E}(\mathbf{r},t_0^+),\qquad \text{curl }\Sigma(\mathbf{r})=\epsilon_0\mathbf{B}(\mathbf{r},t_0^+) \tag{18}$$

and div $\Sigma(\mathbf{r})$ is arbitrary. As a corollary, the following destruction theorem is true. Any given field at any time can be destroyed, canceled out, by an immediate current pulse, that pulse which would create the negative of the given field.

Consider again any given field at any time t_0^+ and ask the question, "Which past currents generated them?" The answer is that there are infinitely many past current distributions that could have generated the field. The immediately preceding pulse described by Eqs. (17) and (18) is only one of the possibilities. [There are actually an infinite number of these due to the arbitrariness of div $\Sigma(\mathbf{r})$.] By running the source-free equations (4) and (5) with $\mathbf{j}=0$, backward to an earlier time $t_0-\tau$, and creating the field at this earlier time by an immediately preceding current pulse, it is seen that an arbitrary field at any time could have been created by pulses at any earlier time, or by any number of combinations of pulses at various earlier times, or by smoothing out these pulses over time, by infinitely many continuous current distributions in the past.

We now show that a delta prime current pulse creates only magnetic field. Start with zero fields at time $t=0^-$, $\mathbf{E}(\mathbf{r},0^-)=0$ and $\mathbf{B}(\mathbf{r},0^-)=0$. Let $\mathbf{j}(\mathbf{r},t)$ denote a pulse of current that acts only during the infinitesimal time interval from $t=0^-$ to $t=0^+$, and let $\mathbf{k}(\mathbf{r},t)$ be the cumulative current from the beginning of the pulse, so that

$$\mathbf{k}(\mathbf{r},t)=\int_{0^-}^{t}\mathbf{j}(\mathbf{r},t')dt',\qquad \mathbf{j}(\mathbf{r},t)=\frac{\partial\mathbf{k}(\mathbf{r},t)}{\partial t}. \tag{19}$$

Now integrate Eqs. (4) and (5) from the beginning of the pulse at time 0^- to any time t during the pulse, $0^-\leqslant t\leqslant 0^+$. The results are

$$\mathbf{E}(\mathbf{r},t)=-\frac{1}{\epsilon_0}\,\mathbf{k}(\mathbf{r},t), \tag{20}$$

$$\mathbf{B}(\mathbf{r},t) = -\text{curl}\int_{0^-}^{t} \mathbf{E}(\mathbf{r},t')dt',$$

when it is assumed that **B** remains finite during the pulse and therefore that the integral of **B** in Eq. (4) remains infinitesimal over the infinitesimal duration of the pulse. Using the formula for $\mathbf{E}(\mathbf{r},t)$ in the integral for $\mathbf{B}(\mathbf{r},t)$ leads to

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{\epsilon_0}\,\text{curl}\int_{0^-}^{t} \mathbf{k}(\mathbf{r},t')dt'. \tag{21}$$

The fields generated by the pulse, $\mathbf{E}(\mathbf{r},0^+)$ and $\mathbf{B}(\mathbf{r},0^+)$, are then

$$\mathbf{E}(\mathbf{r},0^+) = -\frac{1}{\epsilon_0}\,\mathbf{k}(\mathbf{r},0^+),$$
$$\mathbf{B}(\mathbf{r},0^+) = \frac{1}{\epsilon_0}\,\text{curl}\int_{0^-}^{0^+} \mathbf{k}(\mathbf{r},t')dt'. \tag{22}$$

Choosing **k** as the step function $\mathbf{k}(\mathbf{r},t) = \sigma(\mathbf{r})\eta(t)$, where $\eta(t)$ is the unit step function, gives the delta current pulse $\mathbf{j}(\mathbf{r},t) = \partial\mathbf{k}/\partial t = \sigma(\mathbf{r})\delta(t)$, and from Eq. (22) reproduces the results obtained earlier in Eq. (11). The delta function, $\mathbf{k}(\mathbf{r},t) = \Sigma(\mathbf{r})\delta(t)$, gives the delta prime current pulse $\mathbf{j}(\mathbf{r},t) = \Sigma(\mathbf{r})\delta'(t)$, and from Eq. (22) gives $\mathbf{E}(\mathbf{r},0^+) = 0$ and $\mathbf{B}(\mathbf{r},0^+) = \text{curl}\ \Sigma(\mathbf{r})/\epsilon_0$. Using $\mathbf{k}(\mathbf{r},t) = \Sigma(\mathbf{r})\delta(t)$ in Eq. (21) shows that **B** does remain finite through the pulse, as assumed in Eq. (20).

III. THE "QUASISTATIC" AND "FARADAY" APPROXIMATIONS

A. The quasistatic approximation

Consider an arbitrary distribution of charge and current, $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$, satisfying charge conservation (6), and ask what the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ would be in the absence of Faraday induction and Maxwell's displacement current, and what then would be the differential equations for these fields. More correctly, we ask how would we approximate the fields without taking into account Faraday induction and Maxwell's displacement current. The most reasonable answer to this question is what we call the first approximation, or the quasistatic approximation to Maxwell's equations. The fields, denoted by $\mathbf{E}_1(\mathbf{r},t)$ and $\mathbf{B}_1(\mathbf{r},t)$, may be called the "instantaneous" Coulomb and Biot–Savart fields, respectively, and are given by

$$\mathbf{E}_1(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)}{\epsilon_0}\,\frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}', \tag{23}$$

$$\mathbf{B}_1(\mathbf{r},t) = \int \mu_0\mathbf{j}(\mathbf{r}',t)\times\frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}', \tag{24}$$

for any time varying charge and current distributions. Because $\mathbf{E}_1(\mathbf{r},t)$ is determined instantaneously by the instantaneous charge distribution, $\rho(\mathbf{r},t)$, through Eq. (23), it is legitimate in this quasistatic approximation to say that electric charge is the source of electric field. Similarly, from Eq. (24) it is legitimate to say that electric current is the source of magnetic field.

In contrast with $\mathbf{E}_1(\mathbf{r},t)$ and $\mathbf{B}_1(\mathbf{r},t)$, the equilibrium fields, $\mathbf{E}_C(\mathbf{r})$ and $\mathbf{B}_{BS}(\mathbf{r})$, of Eqs. (8) and (9 are defined only for time independent current and charge distributions $\mathbf{j}(\mathbf{r})$ and $\rho(\mathbf{r})$, or equivalently when **j** is stationary and $\text{div}\,\mathbf{j}=0$. This difference leads to a significant change in the differential equations for the fields. For the equilibrium fields, the differential equations are given in Eq. (7). For the quasistatic approximation, they are obtained by evaluating the divergence and curl of Eqs. (23) and (24), and are[8]

$$\text{div}\ \mathbf{E}_1 = \frac{\rho}{\epsilon_0}, \qquad \text{curl}\ \mathbf{E}_1 = 0, \tag{25}$$

$$\text{div}\ \mathbf{B}_1 = 0, \qquad \text{curl}\ \mathbf{B}_1 = \mu_0(\mathbf{j} + \epsilon_0\dot{\mathbf{E}}_1). \tag{26}$$

The change is the presence of $\dot{\mathbf{E}}_1$ in the equation for curl $\mathbf{B}_1$. It arises[8] because the curl of Eq. (24), or of Eq. (9), has a term involving an integral containing div **j** which vanishes in the equilibrium case, leading to curl $\mathbf{B}_{BS} = \mu_0\mathbf{j}$. Here, div **j** need not vanish, and the integral turns out to be $\dot{\mathbf{E}}_1$, after replacing div **j** by $-\dot{\rho}$. Because div curl $\mathbf{B}_1$ vanishes identically, the divergence of the right side of the curl $\mathbf{B}_1$ equation must also vanish, thus requiring the $\dot{\mathbf{E}}_1$ term when $\text{div}\,\mathbf{j}\neq 0$.

In spite of the appearance of the $\dot{\mathbf{E}}_1$ term in Eq. (26), the field $\mathbf{B}_1$ is still given by the Biot–Savart integral of Eq. (24), for the following reason. The Helmholtz theorem applied to Eq. (26) gives for $\mathbf{B}_1$ the integral in Eq. (24) with **j** replaced by $(\mathbf{j}+\epsilon_0\dot{\mathbf{E}}_1)$. But the contribution of $\dot{\mathbf{E}}_1$ to the integral vanishes[9] because curl $\dot{\mathbf{E}}_1 = 0$. Although $\dot{\mathbf{E}}_1$ does not contribute to the Biot–Savart integral, it must be included in the Ampere circuital form of the curl $\mathbf{B}_1$ equation

$$\int \mathbf{B}_1\cdot d\mathbf{l} = \int \mu_0(\mathbf{j}+\epsilon_0\dot{\mathbf{E}}_1)\cdot d\mathbf{A} \tag{27}$$

which is a direct consequence of the curl $\mathbf{B}_1$ equation in Eq. (26).

The differential equations in the quasistatic approximation Eqs. (25) and (26), differ from Maxwell's in two respects. They do not contain a Faraday induction term in the equation for curl $\mathbf{E}_1$, and they do not contain the full Maxwell displacement current in the curl $\mathbf{B}_1$ equation. The full Maxwell displacement current, $\dot{\mathbf{E}}$, generally has a non-vanishing curl and does contribute to the Biot–Savart integral for the magnetic field.

B. The Faraday approximation

We now describe a second approximation that takes into account Faraday induction, but not Maxwell's full displacement current. We call this the "Faraday approximation," and denote the fields by $\mathbf{E}_2(\mathbf{r},t)$ and $\mathbf{B}_2(\mathbf{r},t)$. The fields may be described as follows. The magnetic field, $\mathbf{B}_2(\mathbf{r},t)$, is simply the instantaneous Biot–Savart field, $\mathbf{B}_1$ of Eq. (24), and the electric field, $\mathbf{E}_2(\mathbf{r},t)$, is the sum of the instantaneous Coulomb field, $\mathbf{E}_1$ of Eq. (23), and a non-Coulombic Faraday field denoted by $\mathbf{E}_F$. This Faraday field is regarded as the field induced by a time varying $\mathbf{B}_1$, and is defined by

$$\text{div}\ \mathbf{E}_F = 0, \qquad \text{curl}\ \mathbf{E}_F = -\dot{\mathbf{B}}_1. \tag{28}$$

The Faraday approximation is then described by Eq. (28) and

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_F, \qquad \mathbf{B} = \mathbf{B}_1. \tag{29}$$

The differential equations for the fields defined by Eqs. (28) and (29) are

$$\text{div}\ \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \text{curl}\ \mathbf{E} = -\dot{\mathbf{B}}, \tag{30}$$

$$\text{div}\ \mathbf{B} = 0, \qquad \text{curl}\ \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0\dot{\mathbf{E}}_1) \tag{31}$$

and differ from Maxwell's, Eqs. (1) and (2), by having $\dot{\mathbf{E}}_1$ rather than $\dot{\mathbf{E}}$ in the curl **B** equation.

There is no field propagation in the Faraday approximation, and thus no radiation. The fields at all points of space are instantaneously determined by the instantaneous values of ρ, **j**, and $\dot{\mathbf{j}}$; $\rho(\mathbf{r},t)$ determines $\mathbf{E}_1(\mathbf{r},t)$ by Eq. (23), $\mathbf{j}(\mathbf{r},t)$ determines $\mathbf{B}(\mathbf{r},t)=\mathbf{B}_1(\mathbf{r},t)$ by Eq. (24), and $\dot{\mathbf{j}}$ determines $\dot{\mathbf{B}}_1(\mathbf{r},t)$, which in turn determines $\mathbf{E}_F(\mathbf{r},t)$ by Eq. (28). When ρ and **j** vanish everywhere during any time interval, then **E** and **B** also vanish in that time interval. When only **j** vanishes, and ρ does not, there remains only the stationary Coulomb field $\mathbf{E}_1$ associated with the necessary stationary charge distribution.

In this approximation, it is both natural and legitimate to say that magnetic field is generated by currents because $\mathbf{B}=\mathbf{B}_1$ is determined solely and instantaneously by **j**. It is also legitimate to say that the $\mathbf{E}_1$ part of $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_F$ is generated by charges and that Faraday's induced part, $\mathbf{E}_F$ is generated by the time-varying magnetic field, $\dot{\mathbf{B}}=\dot{\mathbf{B}}_1$.

It is difficult to characterize generally the range of validity of the approximation. From one point of view,[10] it can be said that the error stems from neglecting time derivatives of **j** higher than the first. For a moving charge in uniform translational motion with speed v, this is equivalent to saying that the Faraday approximation gives the fields correctly to first order in v/c.

From a more fundamental viewpoint, the basic error is the absence of field propagation, which requires mutual generation between electric and magnetic field. Although there is Faraday generation of **E** by $\dot{\mathbf{B}}$, there is no generation of **B** by any property of the electric field. Mutual generation and propagation appear only when Maxwell replaces $\dot{\mathbf{E}}_1$ in Eq. (31) by the full displacement current $\dot{\mathbf{E}}$, in which event the equations can be dynamically interpreted as described in Sec. II.

IV. AN INDIRECT DYNAMIC INTERPRETATION: RESOLVING THE FIELDS

We describe an indirect interpretation of Maxwell's equations that is of some value in spite of its artificial nature as a mathematical trick. We resolve the electric field into its Coulomb part $\mathbf{E}_1$ and its remainder $\mathbf{E}^*$, and the magnetic field into its Biot–Savart part $\mathbf{B}_1$ and its remainder $\mathbf{B}^*$,

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}^*, \qquad \mathbf{B}=\mathbf{B}_1+\mathbf{B}^*. \tag{32}$$

Then $\mathbf{E}^*=\mathbf{E}-\mathbf{E}_1$ and $\mathbf{B}^*=\mathbf{B}-\mathbf{B}_1$ may be regarded as the displacements or departures of **E** and **B** from their instantaneous Coulomb and Biot–Savart values $\mathbf{E}_1$ and $\mathbf{B}_1$. If we now regard $\mathbf{E}_1(\mathbf{r},t)$ and $\mathbf{B}_1(\mathbf{r},t)$ of Eqs. (23) and (24) as the fields that are instantaneously in equilibrium with the charge and current distributions, $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$, then $\mathbf{E}^*$ and $\mathbf{B}^*$ are the displacements of the fields from their instantaneously equilibrium values. The differential equations for $\mathbf{E}^*$ and $\mathbf{B}^*$ then describe the vibrations of the fields around these instantaneously equilibrium values. This is analogous to the example of a vertical mass spring system in a g (gravity) field, where it is of value to measure the displacement of the mass not from the unstressed position of the spring, but from the shifted equilibrium position. This shifted equilibrium position would be only an instantaneous equilibrium position when g varies with time.

The differential equations for $\mathbf{E}^*$ and $\mathbf{B}^*$ may be obtained by subtracting Eqs. (25) and (26) from Eqs. (1) and (2) and are

$$\operatorname{div} \mathbf{E}^*=0, \qquad \operatorname{curl} \mathbf{E}^*=-\dot{\mathbf{B}}^*-\dot{\mathbf{B}}_1, \tag{33}$$

$$\operatorname{div} \mathbf{B}^*=0, \qquad \operatorname{curl} \mathbf{B}^*=\frac{1}{c^2}\dot{\mathbf{E}}^*. \tag{34}$$

Taking the curl of the curl equations in Eqs. (33) and (34) results in

$$\nabla^2\mathbf{E}^*-\frac{1}{c^2}\ddot{\mathbf{E}}^*=\operatorname{curl}\dot{\mathbf{B}}_1, \qquad \nabla^2\mathbf{B}^*-\frac{1}{c^2}\ddot{\mathbf{B}}^*=\frac{1}{c^2}\ddot{\mathbf{B}}_1, \tag{35}$$

which show that $\mathbf{E}^*$ and $\mathbf{B}^*$ propagate at speed c.

If at some time **j** becomes and remains zero, then $\mathbf{B}_1$ becomes and remains zero, and ρ becomes and remains stationary or zero, and therefore $\mathbf{E}_1$ becomes and remains stationary or zero. It follows from Eqs. (33) and (34), with $\mathbf{B}_1=0$, that the departure fields, $\mathbf{E}^*$ and $\mathbf{B}^*$ then propagate away to infinity, leaving behind only the static equilibrium field, $\mathbf{E}_1$, associated with the remaining static charge distribution, if any. The examples in Sec. II D may be interpreted in this manner. At the moment that **j** becomes and remains zero, $\mathbf{E}^*$ and $\mathbf{B}^*$ may be considered to be "detached" from the current sources that initially generated them,[11] and then to propagate freely on their own.

The $\dot{\mathbf{B}}_1$ term of Eq. (33) vanishes not only when **j** is zero but also when **j** is stationary, $\mathbf{j}=\mathbf{j}_0(\mathbf{r})$. Thus when **j** becomes stationary and remains so, the departure fields, $\mathbf{E}^*$ and $\mathbf{B}^*$, again simply propagate away to infinity, and the fields settle down to their instantaneous equilibrium values $\mathbf{E}_1$ and $\mathbf{B}_1$. The magnetic field $\mathbf{B}_1$ is the stationary Biot–Savart field associated with $\mathbf{j}_0$, and the electric field $\mathbf{E}_1$ is the instantaneous Coulomb field associated with ρ. When $\operatorname{div}\mathbf{j}_0$ vanishes, both ρ and $\mathbf{E}_1$ are stationary. When $\operatorname{div}\mathbf{j}_0\neq 0$, both ρ and $\mathbf{E}_1$ increase linearly with time. Another situation with a simple outcome[12] arises when **j** becomes and remains linear in time for a long time, and consequently $\dot{\mathbf{B}}_1$ becomes constant in time. .

V. COMPARISON WITH OTHER AUTHORS

Reading Maxwell's equations directly as a set of local dynamic field equations leads to the conclusions that an external current generates **E** locally, and that all further action involves the internal dynamics of the field, operating by the dual mechanism of $-$curl **E** generating **B** and c^2 curl **B** generating **E**. In contrast, a number of past and recent writings in this journal have downplayed the electromagnetic field as the agent that propagates signals across space.

Papers by Rosser, Jefimenko, and Griffiths and Heald effectively deny dynamic action within the electromagnetic field. They argue against a local dynamic reading of Maxwell's curl equations, Eqs. (1) and (2). In discussing Maxwell's curl **B** equation, Rosser,[13] writes "It is not necessary to state that it is the displacement "current" which gives rise to the magnetic field, it is sufficient to say that one effect is accompanied by the other. They have a common *cause*, namely, moving charges, at some earlier time. For a full discussion of this viewpoint the reader is referred to O'Rahilly." Rosser goes on to make the analogous statement about the curl **E** equation. We agree with Rosser only in saying that $\dot{\mathbf{E}}$ is not a source of **B**, and $\dot{\mathbf{B}}$ is not a source of **E**. We differ greatly when we say it is the other way around;

that $-\text{curl}\,\mathbf{E}$ generates $\mathbf{B}$, and c^2 curl $\mathbf{B}$ generates $\mathbf{E}$, and this describes the dual mechanism by which field is propagated. We object to the underlying ideas in Rosser's statement which denies any dynamic action within the electromagnetic field. The book by O'Rahilly,[14] to which Rosser refers, is an absolutely thorough rejection of the entire field concept and of Maxwell's theory.

The papers by Jefimenko and by Griffiths and Heald base their denial of field dynamics on two equations derived by Jefimenko which give $\mathbf{E}$ and $\mathbf{B}$ directly in terms of retarded values of ρ, $\dot{\rho}$, $\mathbf{j}$, and $\dot{\mathbf{j}}$. Jefimenko[15] writes, "These equations indicate that the sources of a time-dependent electric field are electric charges together with conduction and convection currents, while those of a time-dependent magnetic field are only the conduction and convection currents but not the displacement currents. This means that although a displacement current is associated with a magnetic field, this does not constitute a cause and effect relationship." Griffiths and Heald,[16] writing about Maxwell's curl $\mathbf{B}$ equation say "The Maxwell term $\partial\mathbf{E}/\partial t$ is necessary in the *local* equation (2) (the curl $\mathbf{B}$ equation) as a surrogate for the source currents *at other places...* . In principle, $\partial\mathbf{E}/\partial t$ is no more a source of $\nabla\times\mathbf{B}$, than $\nabla\times\mathbf{B}$ is a source of $\partial\mathbf{E}/\partial t$." These authors state that the "true source" of $\mathbf{B}$ is the current distribution $\mathbf{j}$.

In our opinion the primary nature of the field, on an equal footing with matter, needs no defense other than the force of history from Maxwell to Einstein and beyond. We also refer to the quotations from Einstein and Infeld[3] and Feynman.[11]

However, we comment on Griffiths and Heald's description of $\dot{\mathbf{E}}$ as merely a "surrogate" for the "true" source currents at other places. It was demonstrated earlier that any given field distribution could have been created by infinitely many past current distributions. In our view, any field that could be a surrogate for an infinite number of possible past current distributions, must be worthy of more consideration, and in fact should be regarded as an independent entity. (Additionally, although it may be unfair in this discussion to invoke quantum phenomena, we do so, and point out the process of pair creation by photons, in which currents and charges are created out of pure electromagnetic field.)

[1]See, for example, A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic, New York, 1950), pp. 147–149, where the Helmholtz theorem is called "a fundamental theorem of vector analysis."

[2]Pieter B. Visscher, *Fields and Electrodynamics* (Wiley, New York, 1988). This book was brought to our attention during the review process. Visscher's perspective on fields starts by discretizing all field equations with respect to space and time.

[3]The dynamics of field propagation has been similarly described by Einstein and Infeld in A. Einstein and L. Infeld, *The Evolution of Physics* (Simon and Shuster, New York, 1938), where they write that Maxwell's equations

> do not...connect two widely separated events: they do not connect the happenings *here* with the conditions *there*. The field *here* and *now* depends on the field *in the immediate neighborhood* at a time *just past*. The equations allow us to predict what will happen a little further in space and a little later in time, if we know what happens here and now... . We can deduce what happens here from that which happened far away by the summation of these very small steps.

[4]A fascinating introduction to the nineteenth century models and interpretations of Maxwell's electromagnetism is given by Bruce J. Hunt in an article entitled " 'How My Model Was Right': G. F. FitzGerald and the Reform of Maxwell's Theory" which appears in *Kelvin's Baltimore Lectures and Modern Theoretical Physics Historical and Philosophical Perspectives,* edited by Robert Kargon and Peter Achinstein (MIT, Cambridge, Massachusetts, 1987).

[5]Integrating $(\partial/\partial t)\text{div}\,\mathbf{B}=0$ results in $\text{div}\,\mathbf{B}=f(\mathbf{r})$, where $f(\mathbf{r})$ is a stationary function of position. If now a static field $\mathbf{B}_0$ is defined by $\text{div}\,\mathbf{B}_0=f(\mathbf{r})$ and $\text{curl}\,\mathbf{B}_0=0$, then the difference field, $\mathbf{B}-\mathbf{B}_0$, satisfies Eqs. (4) and (5) and also $\text{div}\,(\mathbf{B}-\mathbf{B}_0)=0$. But $\mathbf{B}-\mathbf{B}_0$ is simply the magnetic field that arises from the source term, $-\text{curl}\,\mathbf{E}$. Integrating $\partial/\partial t[\text{div}\,\mathbf{E}-(\rho/\epsilon_0)]=0$ results in $\text{div}\,\mathbf{E}-(\rho/\epsilon_0)=g(\mathbf{r})$. If a static field $\mathbf{E}_0$ is defined by $\text{div}\,\mathbf{E}_0=g(\mathbf{r})$ and $\text{curl}\,\mathbf{E}_0=0$, then the difference $\mathbf{E}-\mathbf{E}_0$ satisfies Eqs. (4) and (5) and also $\text{div}\,(\mathbf{E}-\mathbf{E}_0)=\rho/\epsilon_0$.

[6]Set $\mathbf{E}=E_y\hat{\mathbf{y}}$ and $\mathbf{B}=B_x\hat{\mathbf{x}}$. Then, after the pulse, $E_y(z,t)$ and $B_x(z,t)$ both satisfy the source free (scalar) wave equation in one dimension, with initial conditions $E_y(z,0^+)=(\lambda/\epsilon_0)\delta(z)$ and $\dot{E}_y(z,0^+)=0$ for E_y, and $B_x(z,0^+)=0$ and $\dot{B}_x(z,0^+)=(\lambda/\epsilon_0)\delta'(z)$ for B_x. The plane wave solutions for these initial conditions are given in Eq. (16). They clearly satisfy the source-free wave equation and also satisfy the initial conditions immediately after the pulse.

[7]It may be seen from Fig. 2 that the circulation of $\mathbf{E}$ is clockwise on the positive z side of the x,y plane, and thus that curl $\mathbf{E}$ is in the positive x direction. Analogously, curl $\mathbf{E}$ is in the negative x direction on the negative z side. It is clear that curl $\mathbf{E}$ is confined to a double layer surrounding the x,y plane because $\mathbf{E}$ and therefore curl $\mathbf{E}$ vanish off the plane. Quantitatively, the curl of the initially generated $\mathbf{E}$, of Eq. (13), is $\text{curl}\,\mathbf{E}=-(\lambda/\epsilon_0)\delta'(z)$; the double layer is mathematically described as a delta prime function.

[8]Evaluating the divergence and curl of both Eq. (23) and Eq. (24) gives $\text{div}\,\mathbf{E}_1=\rho/\epsilon_0$, $\text{curl}\;\mathbf{E}_1=0$, $\text{div}\,\mathbf{B}_1=0$, and $\text{curl}\,\mathbf{B}_1=\mu_0(\mathbf{j}-\int\text{div}\,\mathbf{j}(\mathbf{r}',t)[(\mathbf{r}-\mathbf{r}')/4\pi|\mathbf{r}-\mathbf{r}'|^3]d\mathbf{r}'$ which leads to Eq. (26) after replacing div $\mathbf{j}$ by $-\dot{\rho}$, and realizing that the resulting integral containing $\dot{\rho}$ is equal to $\epsilon_0\dot{\mathbf{E}}_1$.

[9]As in the previous note, the divergence and curl of $\mathbf{F}(\mathbf{r})=\int\mathbf{f}(\mathbf{r}',t)\times[(\mathbf{r}-\mathbf{r}')/(4\pi|\mathbf{r}-\mathbf{r}'|^3)]d\mathbf{r}'$ are $\text{div}\,\mathbf{F}=0$ and $\text{curl}\,\mathbf{F}=\mathbf{f}-\int\text{div}\,\mathbf{f}(\mathbf{r}',t)[(\mathbf{r}-\mathbf{r}')/(4\pi|\mathbf{r}-\mathbf{r}'|^3)]d\mathbf{r}'$. When $\text{curl}\,\mathbf{f}=0$, Helmholtz's theorem for $\mathbf{f}$ gives $\mathbf{f}=\int\text{div}\,\mathbf{f}(\mathbf{r}',t)[(\mathbf{r}-\mathbf{r}')/(4\pi|\mathbf{r}-\mathbf{r}'|^3)]d\mathbf{r}'$ from which it follows that $\text{curl}\,\mathbf{F}=0$. Then $\mathbf{F}=0$ when $\text{curl}\,\mathbf{f}=0$ because div $\mathbf{F}$ and curl $\mathbf{F}$ are then both zero. It even follows that an irrotatinal current distribution, one with $\text{curl}\,\mathbf{j}=0$, generates no Biot–Savart magnetic field.

[10]When $\dot{\mathbf{E}}_1=\dot{\mathbf{E}}$, Eqs. (30) and (31) agree with Maxwell's equations. They then give correct results when $\dot{\mathbf{E}}_\mathrm{F}$ vanishes, because by Eq. (29), $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_\mathrm{F}$. When all derivatives of $\mathbf{j}$ higher than the first are zero, then $\dot{\mathbf{j}}$ is stationary, and then $\dot{\mathbf{B}}=\dot{\mathbf{B}}_1$ is stationary, from Eq. (24), and then $\mathbf{E}_\mathrm{F}$ is stationary from Eq. (28), and so $\dot{\mathbf{E}}_\mathrm{F}$ vanishes in such cases.

[11]More picturesquely, in describing the propagating fields that remain after all charges and currents vanish, Feynman, in R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading), p. 18–8, wrote

> The fields have "taken off," they are propagating freely through space, no longer connected in any way with the source. The caterpillar has turned into a butterfly.

[12]When $\dot{\mathbf{j}}$ is stationary, so is $\dot{\mathbf{B}}_1$, from Eq. (24), and so is the Faraday induced field, $\mathbf{E}_\mathrm{F}$, from Ref. 10. Substituting $\mathbf{E}^*=\mathbf{E}_\mathrm{F}+\mathbf{E}^{**}$ in Eqs. (33) and (34), and using $\dot{\mathbf{E}}_\mathrm{F}=0$ and Eq. (28) then shows that $\mathbf{E}^{**}$ and $\mathbf{B}^*$ satisfy the source-free field equations and thus propagate away to infinity. The fields therefore settle down to $\mathbf{B}=\mathbf{B}_1$ and $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_\mathrm{F}$. The "settled" magnetic field, $\mathbf{B}_1$, is linear with time, and the settled electric field, $\mathbf{E}_1$ is quadratic in time because ρ is quadratic when $\mathbf{j}$ is linear (and $\text{div}\,\mathbf{j}\neq 0$). The simplest example where $\dot{\mathbf{j}}$ becomes stationary is that of a ramp function starting at time zero, $\mathbf{j}=\mathbf{j}'(\mathbf{r})t\,\eta(t)$, $\eta(t)$ being the unit step function. Here, $\dot{\mathbf{j}}=0$ for $t<0$, and $\dot{\mathbf{j}}=\mathbf{j}'(\mathbf{r})$ for $t>0$, and so $\dot{\mathbf{B}}_1$ jumps from 0 to another stationary function of position during the startup. Because $\dot{\mathbf{B}}_1$ varies with time during the startup at time zero, it generates departure fields in Eqs. (33) and (34), which then propagate away to infinity.

[13]W. G. V. Rosser, "Interpretation of the displacement current," Am. J. Phys. **31**, 807–808 (1963).

[14]A. O'Rahilly, *Electromagnetics* (Longman Green, London, 1938).

[15]The quotation is from, Oleg D. Jefimenko, *Electricity and Magnetism*, 2nd ed. (Electret Scientific, Star City, 1989). See also, Oleg D. Jefimenko, "Comment on 'On the equivalence of the laws of Biot–Savart and Ampere'," Am. J. Phys. **58**, 505 (1990).

[16]David J. Griffiths and Mark A. Heald, "Time-dependent generalizations of the Biot–Savart and Coulomb laws," Am. J. Phys. **59**, 112 (1991).