(i) Owing to the idempotency of A_i and to commutation under the trace, one has $\mathrm{Tr}\, A_i\rho = \mathrm{Tr}\, A_i\rho A_i$. Hence, $A_i\rho A_i = 0$ implies $\mathrm{Tr}\, A_i\rho = 0$.

(ii) If $\mathrm{Tr}\, A_i\rho A_i = 0$, then in any complete orthonormal basis $\{|\varphi_n\rangle : n = 1,2,...\}$ we have $\Sigma_n \langle\varphi_n|A_i\rho A_i|\varphi_n\rangle = 0$. Since $\rho \geqslant 0$, each $(\langle\varphi_n|A_i)\rho(A_i|\varphi_n\rangle) \geqslant 0$ as well.

Hence,

$$\langle\varphi_n|A_i\rho A_i|\varphi_n\rangle = 0, \quad n = 1,2,... .$$

Since, e.g., $|\varphi_1\rangle$ is an arbitrary (normalized) vector, and since two linear operators have the same expectation value in all state vectors only if they coincide, we must have $A_i\rho A_i = 0$. □

APPENDIX B

To prove that the Corollary, which gives a necessary condition, does not give simultaneously also a sufficient one, we present an example for the validity of the Corollary with lack of interference.

Let $\{|n\rangle : n = 1,2,...\}$ be an orthonormal and complete basis. We define $|\Psi\rangle \equiv 2^{-1/2}|1\rangle + 2^{-1/2}|2\rangle$,

$$|\varphi\rangle \equiv 2^{-1/2}|3\rangle + 2^{-1/2}|4\rangle,$$

$$|\chi\rangle \equiv 2^{-1/2}|3\rangle + i2^{-1/2}|4\rangle.$$

Then, we take

$$B \equiv |2\rangle\langle 2| + |\chi\rangle\langle\chi|,$$

$$\rho \equiv |\Psi\rangle\langle\Psi|,$$

$$\{A_i i = 1,2,...,I\} \equiv \{P_1,P_2\},$$

with

$$P_1 \equiv |1\rangle\langle 1| + |\varphi\rangle\langle\varphi|,$$

$$P_2 \equiv |2\rangle\langle 2| + (|3\rangle\langle 3| + |4\rangle\langle 4| - |\varphi\rangle\langle\varphi|) + \sum_{n=5}^{\infty} |n\rangle\langle n|.$$

One has $[P_1,|\Psi\rangle\langle\Psi|] \neq 0$ [because $|\Psi\rangle$ is not a characteristic vector of P_1, cf (7)], and analogously for $P_2 \equiv 1 - P_1$. Further, $[B,P_i] \neq 0$, $i = 1,2$. Hence, both incompatibilities required by the necessary condition from the Corollary are valid.

One always has

$$\mathrm{Tr}\, B\rho = \mathrm{Tr}\, BP_1\rho P_1 + \mathrm{Tr}\, BP_2\rho P_2 + \mathrm{Tr}\, BP_1\rho P_2 + \mathrm{Tr}\, BP_2\rho P_1.$$

To demonstrate lack of interference [the validity of (6)], we evaluate

$$\mathrm{Tr}\, BP_2\rho P_1 = \mathrm{Tr}\, P_1BP_2\rho = \langle\Psi|P_1BP_2|\Psi\rangle = 0$$

(because $BP_1|\Psi\rangle = 0$), and analogously

$$\mathrm{Tr}\, BP_1\rho P_2 = \mathrm{Tr}\, P_2BP_1\rho = \langle\Psi|P_2BP_1|\Psi\rangle = 0. \quad \square$$

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Kaluza–Klein unification and the Fierz–Pauli weak-field limit

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An introduction to higher-dimensional unification is provided by a simple modern treatment of the original 5-D Kaluza–Klein ansatz. A demonstration is given of the way electric charge arises from momentum in the fifth dimension and an estimate of the radius of the extra dimension is obtained from simple quantum-mechanical considerations. The Fierz–Pauli weak-field approximation is applied and found to exhibit some of the essential features of Kaluza–Klein theory in a few lines without the need for a full general relativistic analysis.

I. INTRODUCTION

Two of the most important theoretical physics problems are the consistent quantum description of gravitation and its unification with the electrodynamic and nuclear interactions in a way that also unifies the fermionic description of matter with the bosonic exchange quanta of interacting fields. Important ingredients in some of the most promising theories, for example, supergravity and superstrings,[1] are the key ideas of Kaluza[2] and Klein.[3,4]

We present here a new elementary way of deriving and understanding some of the Kaluza–Klein ideas. We use SI units and retain all physical constants, in particular G, c, and $\hbar$.

Any theory unifying electrodynamics and gravitation must overcome enormous differences between the two interactions. Although Einstein's relativity theories gave both a common nonquantum space-time description, gravitation is universal whereas only particles with nonzero charge interact electrodynamically. The motion of a gravitational test particle obeys the weak equivalence principle and is thus independent of its internal characteristics, in particular its rest mass. In sharp contrast is the acceleration of a charged particle according to the Lorentz law with its dependence on the charge-to-mass ratio. This difference is intimately connected with the geometrical nature of general relativity theory and the nongeometrical nature of electrodynamics. Yet another difference is the nonlinearity of general relativistic gravitation compared to the linearity of Maxwell's theory.

We deal in this paper with the five-dimensional (5-D) theory of gravity and electromagnetism in the spirit of the early pioneers of Kaluza–Klein theory. After a brief historical overview in Sec. II, we summarize in Sec. III the generalization of general relativity to 5-D and its subsequent reduction to standard four-dimensional (4-D) general relativity plus Maxwell's electrodynamics. An inconsistency in Kaluza's paper is clarified as we build up the 5-D metric from the line element.

In Sec. IV we show how a test particle traveling along a geodesic trajectory in 5 D (interacting gravitationally only) acquires an electric charge in 4-D due to its momentum in the fifth dimension. Simple quantum mechanical arguments, originally due to Klein,[3,4] suffice to demonstrate the quantization of charge in terms of the radius of the extra dimension. This is shown to be only an order of magnitude larger than the Planck length, consistent with its nonobservation.

We show in Sec. V that the Fierz–Pauli theory of weak-field gravity (not available to Kaluza) suffices to demonstrate many of the essential features of Kaluza–Klein theory, namely, the reduction of 5-D pure gravity to both electrodynamics and gravity in 4 D, in a few lines, without necessary recourse to a full general relativistic analysis. It does however depend on a few results from the full treatment, which is summarized first for that reason. The reader not confident in general relativity theory may take Secs. III and IV, specifically Eqs. (9), (23), (29), and (30), on faith and still find that Sec. V provides new insight into Kaluza–Klein theories.

The developments in this paper consciously follow those of Kaluza[2] and Klein[3,4]. They are however presented in a more didactic way and recast in a modern notation and context. The weak-field approximation in particular benefits from the use of the well-known Fierz–Pauli linearized gravity theory in which the dynamics are contained in explicit derivatives rather than in connection coefficients buried inside the Einstein tensor.

The detailed development of the fully relativistic Kaluza–Klein ansatz and the illuminating weak-field approximation provide an accessible yet solid first step into Kaluza–Klein theory.

All standard results, most notably the use of the $(-+++)$ metric, follow the same conventions and terminology as Misner *et al.*[5] and Doughty,[6] with 5-D analogs of 4-D quantities being denoted by a circumflex.

II. HISTORICAL OVERVIEW

In 1919, Theodor Kaluza, a junior scholar at the University of Königsberg (now Kaliningrad, USSR) proposed a generalization[2] of general relativity from 4 D to 5 D in an attempt to unify the interactions of relativistic gravitation and electrodynamics. This was achieved via a weak-field approximation of an extended 5-D metric tensor containing the electromagnetic as well as the gravitational potentials.

However, the idea of higher-dimensional unification was not new. In 1912, Finnish physicist Nordström[7] had developed a relativistic theory of gravity based on a scalar field. In 1914, before the final form of general relativity theory, he utilized an extra spatial dimension to form a flat 5-D space-time with a five-vector electromagnetic potential to extend Maxwell's electrodynamics and found[8] that the fifth component was equivalent to his scalar gravitational field. He supposed that 4-D scalar gravity was a remnant of an Abelian gauge theory in his flat 5-D space-time. However, Nordström's scalar gravity theory could not explain the bending of light near the sun[6] and was soon overtaken by the new general relativity theory.

Since general relativity was the foundation of Kaluza's work, he first sent his paper to Einstein, it being necessary at that time for papers to be recommended to journals by established scientists on behalf of the author. Einstein was amenable to Kaluza's ansatz,[9] but his few reservations[10] disheartened Kaluza, and publication was delayed until 1921, when Einstein decided to recommend publication after all.[11]

An early rival to Kaluza's theory was that of Weyl,[12] who relaxed the parallel transport property of general relativity and allowed a "gauge" scaling of the space and time dimensions in an attempt to have both gravity and electromagnetism arising from the geometry of a 4-D space-time. However, electric charge is not conserved in Weyl's theory. It was thus eliminated as a unified theory, only to be revived in a different form in 1929 as a key property of quantum electrodynamics, where the electromagnetic field couples to the electron spinor field.[13]

In 1926, Swedish physicist Oskar Klein showed that Kaluza's theory reduced rigorously to 4-D Einstein–Maxwell theory in a full relativistic analysis.[3] He also supposed that the fifth dimension, which a relativistic analysis implies (Sec. III C) must be compact, to be curled up unobservably small and found that this led naturally to the quantization of electric charge.[3,4]

By the early thirties, Einstein had begun studying Kaluza–Klein theory again. With Mayer, he attempted[14] to do away with the extra dimension by associating a vector space with a 5-D basis to each point in a 4-D space-time. Unsatisfied with this, he then followed Klein's idea[3,4] and assumed the fifth dimension to be closed (or periodic) on a small scale.[15] This was an attempt to impose a physical reality on the extra dimension after earlier work had tended to regard it as an embarrassment, only mathematically necessary. Again no great success was forthcoming and Einstein soon abandoned the effort.[16]

The extended 5-D metric tensor contains 15 components. However, only 14 are required to describe an electromagnetically sourced gravitational field in 4-D. Kaluza's ansatz involves a dimensionless scalar field as well. We shall show in Sec. III B that Kaluza's assignment of the scalar field to only one metric component is inherently inconsistent. Nearly all of the subsequent early work followed Klein[3] and assumed the constancy of this scalar field, normalized to unity. Its inclusion was left unexplored until Jordan[17] and Thiry[18] independently derived the full Kaluza–Klein metric [Eq. (9)] in the 1940s. Jordan's work[17] on this led to the Brans–Dicke–Jordan scalar-tensor theories of gravitation.[19] For further discussion on the historical background, see Ref. 20.

Generalization of the Kaluza–Klein ansatz to include the strong- and weak-nuclear forces first discovered[21] in the early 1930s had to await the non-Abelian gauge theories of Yang and Mills[22] in 1954. The local gauge symmetry of electromagnetism is completely described by the Abelian (commutative) U(1) group, which may be thought of as the group of translations around the circumference of a circle. Thus the single extra dimension in 5-D Kaluza–Klein theory suffices to include the gauge character of electrodynamics that then arises naturally as a geometric coordinate transformation in 5-D (Sec. III B). The more complex gauge symmetries of the nuclear interactions, however, require the inclusion of many more spatial dimensions if they are also to be unified under a Kaluza–Klein type procedure.

The first "modern" reference to a Kaluza–Klein theory incorporating the Yang–Mills theories is probably a problem posed by DeWitt in his lectures[23] at the 1963 Les Houches summer school on relativity, groups, and topology. Further work by Kerner[24] and Cho *et al.*[25] in 1975 laid the foundations of modern Kaluza–Klein theory, which has been incorporated into the currently popular theories of supergravity and superstrings. These developments are discussed in Refs. 1, 20, 26–29.

III. KALUZA-KLEIN ANSATZ

Following Kaluza,[2] we consider a generalization of 4-D gravitation to 5-D space-time, whose coordinates we denote by $\{x^A\} = \{x^\mu, x^5\}$, where $A,B,C,... = \{0,1,2,3,5\}$ and the 5-D line element is $d\hat{s}^2 = \hat{g}_{AB}\, dx^A\, dx^B$ in analogy with 4-D general relativity. The extra dimension must be *spacelike* to avoid causality violations due to the existence of closed timelike curves which, for example, could allow an event to cause itself.[30]

A. Transformation law of the fifth coordinate

In 4 D, general coordinate transformations of the form $x^\mu \to x^{\bar\mu} = x^{\bar\mu}(x^\nu)$, which imply that $dx^\mu \to dx^{\bar\mu} = (\partial x^{\bar\mu}/\partial x^\mu)\partial x^\mu$, leave the line element ds^2 unchanged provided the metric functions $g_{\mu\nu}(x)$ correspondingly transform according to $g_{\bar\mu\bar\nu} = (\partial x^\mu/\partial x^{\bar\mu})(\partial x^\nu/\partial x^{\bar\nu})g_{\mu\nu}$.

We partition the 5-D general coordinate transformation $x^A \to x^{\bar A}(x^A)$ by regarding it as a two-index array that we display in matrix form, and similarly partition the symmetric ($\hat{g}_{AB} = \hat{g}_{BA}$) metric tensor to obtain:

$$x^{\bar A}(x^A) = \begin{pmatrix} x^{\bar\mu}(x^\mu, x^5) \\ x^{\bar 5}(x^\mu, x^5) \end{pmatrix} \quad \text{and} \quad \hat{g}_{AB} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu 5} \\ \hat{g}_{5\nu} & \hat{g}_{55} \end{pmatrix}. \tag{1}$$

This already shows that they may easily include a 4-D general coordinate transformation $x^\mu \to x^{\bar\mu}(x^\mu)$. A symmetric four-tensor $g_{\mu\nu}$, appropriate to 4-D gravity, may be constructed from the symmetric quantities $\hat{g}_{\mu\nu}$ and/or $\hat{g}_{\mu 5}\hat{g}_{5\nu}$. A single four-vector field constructed from $\hat{g}_{\mu 5} = \hat{g}_{5\mu}$ is a possible electromagnetic potential. Both may include factors of the $\hat{g}_{55}$ scalar.

To account for the observed 4-D character of space-time, Kaluza[2] introduced the *cylinder condition*:

> All components of the 5-D metric and the first four coordinates x^μ must be independent of the fifth coordinate x^5, so that $\partial_5 \hat{g}_{AB}$ and $\partial_5 x^\mu$ vanish identically.

We also demand that the first four coordinates of 5-D space-time characterize 4-D space-time, which implies that $\partial x^{\bar\mu}/\partial x^5$ (and thus $\partial x^\mu/\partial x^{\bar 5}$) must vanish identically.

To determine the transformation law of x^5, we note that the symmetric 5-D metric must transform according to

$$\hat{g}_{\bar A\bar B} = \frac{\partial x^A}{\partial x^{\bar A}}\frac{\partial x^B}{\partial x^{\bar B}}\hat{g}_{AB}, \tag{2}$$

in analogy with the 4-D case, and that the cylinder condition must hold in all frames. Now, $\partial_{\bar 5}\hat{g}_{\bar\mu\bar 5}$ and $\partial_{\bar 5}\hat{g}_{\bar 5\bar 5}$ are both zero by the cylinder condition, and so the resulting coefficients of $\hat{g}_{\mu 5}$ and $\hat{g}_{55}$ must each vanish. The coefficient of $\hat{g}_{\mu 5}$ gives $(\partial x^\mu/\partial x^{\bar\mu})(\partial^2 x^5/\partial x^{\bar 5}\,\partial x^{\bar 5}) = 0$, and the invertibility of $x^{\bar\mu}(x^\mu)$ then gives $\partial^2 x^5/\partial x^{\bar 5}\,\partial x^{\bar 5} = 0$, for which the solution is $x^5 = a(x^{\bar\mu})x^{\bar 5} + b(x^{\bar\mu})$. With this condition the coefficient of $\hat{g}_{55}$ becomes $a(x^{\bar\mu})\partial a/\partial x^{\bar\mu} = 0$. Invertibility of $x^{\bar A}(x^A)$ ensures $a \neq 0$ giving $\partial a/\partial x^{\bar\mu} = 0$, and so a must therefore be independent of $x^{\bar\mu}$. The most general transformation of the fifth coordinate is thus of the form

$$x^5 \to x^{\bar 5} = x^5 + f(x^\mu), \tag{3}$$

where we have put $a = 1$ without loss of generality. The inverse transformation is given by $x^{\bar 5} \to x^5 = x^{\bar 5} + g(x^{\bar\mu})$, where $g(x^{\bar\mu}) = -f[x^\mu(x^{\bar\mu})]$.

B. Five-dimensional metric

In a stationary 4-D space-time (where $\partial_0 g_{\mu\nu} = 0$) the line element $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ between two events x and $x + dx$ may be split[31] into $ds^2 = g_{00}\, d\lambda^2 + dl^2$, where the temporal and spatial separations $d\lambda$ and dl are given by

$$d\lambda = dx^0 + \frac{g_{k0}}{g_{00}}\, dx^k \quad \text{and} \quad dl^2 = \left(g_{kl} - \frac{g_{k0}g_{l0}}{g_{00}}\right)dx^k\, dx^l. \tag{4}$$

These quantities are invariant with respect to the transformation $x^0 \to x^0 + f(x^k)$ in the time direction, and arbitrary coordinate transformations $x^k \to x^{\bar k}(x^k)$, on each constant x^0 spacelike cross section.

Klein[3] notes that, analogously to $d\lambda$ and dl^2 of Eq. (4), the line element $d\hat{s}^2$ between two events in 5-D space-time may be similarly split, as a result of the cylinder condition, into the invariant quantities

$$d\hat\lambda = dx^5 + \frac{\hat{g}_{\mu 5}}{\hat{g}_{55}}\, dx^\mu$$

and

$$d\hat{l}^2 = \left(\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu 5}\hat{g}_{5\nu}}{\hat{g}_{55}}\right)dx^\mu\, dx^\nu. \tag{5}$$

Putting $\hat{g}_{55} = \phi$ (a dimensionless scalar field) suggests we also put $\hat{g}_{\mu 5} = \kappa\phi A_\mu$ with κ a constant, so that Eqs. (5) become

$$d\hat\lambda = dx^5 + \kappa A_\mu\, dx^\mu$$

and

$$d\hat{l}^2 = (\hat{g}_{\mu\nu} - \kappa^2\phi A_\mu A_\nu)dx^\mu\, dx^\nu. \tag{6}$$

Now $d\hat{l}^2$ is independent of x^5 and will be the line element $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ of 4-D space-time if we make the identification

$$g_{\mu\nu} = \hat{g}_{\mu\nu} - \kappa^2\phi A_\mu A_\nu. \tag{7}$$

Thus the 5-D line element $d\hat{s}^2$ may be written (cf. the 4-D case above) as

$$d\hat{s}^2 = ds^2 + \hat{g}_{55}\, d\hat{\lambda}^2$$

or

$$d\hat{s}^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + \phi(\kappa A_\mu\, dx^\mu + dx^5)^2, \tag{8}$$

which yields the partitioned 5-D metric

$$\hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \kappa^2\phi A_\mu A_\nu & \kappa\phi A_\mu \\ \kappa\phi A_\nu & \phi \end{pmatrix}, \tag{9}$$

now expressed totally in terms of the 4-D quantities $g_{\mu\nu}$, A_μ, and ϕ. The inverse metric $\hat{g}^{AB}$ is easily obtained from the standard prescription $\hat{g}^{AB}\hat{g}_{BC} = \hat{\delta}^A_C$. This analysis demonstrates that if ϕ is included in $\hat{g}_{55}$, then it ought also to appear, for consistency, in the other three components. A field redefinition of A_μ, for example $A^{\text{new}}_\mu = \phi A_\mu$, may permit the removal of ϕ from some but not all of these three components. Kaluza's identifications, $\hat{g}_{\mu 5} = \kappa A_\mu$ and $\hat{g}_{55} = \phi$, lead to a metric that is inconsistent with those same identifications.

In general relativity the first-order gravitational potentials $\Gamma^\mu_{\nu\lambda}$, called the *affine connections* or *connection coefficients*,[5,32] symmetric in their last two indices, are related to the metric by

$$2\Gamma^\mu_{\nu\lambda} = g^{\mu\pi}(\partial_\lambda g_{\pi\nu} + \partial_\nu g_{\pi\lambda} - \partial_\pi g_{\nu\lambda}). \tag{10}$$

Instead of the above approach, Kaluza considered the 5-D generalization of Eq. (10) and found, using the cylinder condition, that only 64 of the $5^2(5+1)/2 = 75$ independent components of $\hat{\Gamma}_{ABC} = \hat{\Gamma}_{A(BC)} = \hat{g}_{AD}\hat{\Gamma}^D_{BC}$ are nonvanishing after partitioning $A = \{\mu,5\}$:

$$2\hat{\Gamma}_{\mu\nu\lambda} = \partial_\lambda\hat{g}_{\mu\nu} + \partial_\nu\hat{g}_{\lambda\mu} - \partial_\mu\hat{g}_{\nu\lambda}, \tag{11}$$

$$2\hat{\Gamma}_{\mu\nu 5} = 2\hat{\Gamma}_{\mu 5\nu} = \partial_\nu\hat{g}_{\mu 5} - \partial_\mu\hat{g}_{\nu 5}, \tag{12}$$

$$2\hat{\Gamma}_{5\mu\nu} = 2\hat{\Gamma}_{5\nu\mu} = \partial_\nu\hat{g}_{\mu 5} + \partial_\mu\hat{g}_{\nu 5}, \tag{13}$$

$$2\hat{\Gamma}_{55\mu} = -2\hat{\Gamma}_{\mu 55} = \partial_\mu\hat{g}_{55}, \tag{14}$$

with $\hat{\Gamma}_{555}$ and 10 others vanishing. The form of Eq. (12) led Kaluza to the identification $\hat{g}_{\mu 5} = \kappa A_\mu$, so that the electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is given by $2\hat{\Gamma}_{\mu\nu 5} = 2\hat{\Gamma}_{\mu 5\nu} = \kappa F_{\mu\nu}$ in which the SI dimensions of the proportionality constant κ must be $[\kappa] = [q]/[p]$, namely, those of electric charge q over momentum p. The combination

$$2\hat{\Gamma}_{5\mu\nu} = \kappa(\partial_\mu A_\nu + \partial_\nu A_\mu) \equiv \kappa\Sigma_{\mu\nu}, \tag{15}$$

does not appear in the equations of electrodynamics and must not therefore appear in the physical equations that result if the Kaluza idea is to be successful. Our analysis above shows that these 5-D connection coefficients are correct only in the special case of ϕ constant, here normalized to $\phi = \hat{g}_{55} = 1$.

Whether or not the scalar field is constrained to be constant, the electromagnetic Bianchi identity,[6] namely $\partial_\nu\tilde{F}^{\mu\nu} = 0$, follows immediately from the following 5-D general relativistic identity[2] arising from the symmetries of the Riemann tensor:

$$\partial_D(\hat{\Gamma}_{ABC} + \hat{\Gamma}_{BCA} + \hat{\Gamma}_{CAB}) = \partial_A\hat{\Gamma}_{CDB} + \partial_B\hat{\Gamma}_{ADC} + \partial_C\hat{\Gamma}_{BDA}, \tag{16}$$

which is easily verified by writing out the components. Taking $(ABCD) = (\mu\nu\lambda 5)$ and substituting $2\hat{\Gamma}_{\mu 5\nu} = \kappa(\phi F_{\nu\mu} + A_\mu\partial_\nu\phi - A_\nu\partial_\mu\phi)$ we find

$$\partial_\mu F_{\lambda\nu} + \partial_\nu F_{\mu\lambda} + \partial_\lambda F_{\nu\mu} = 0 \Leftrightarrow \partial_\nu\tilde{F}^{\mu\nu} = 0. \tag{17}$$

Under the coordinate change of Eq. (3) for the fifth coordinate, $\hat{g}_{\mu 5} = \kappa\phi A_\mu$ transforms by

$$\hat{g}_{\bar{\mu}\bar{5}} = \hat{g}_{\mu 5} + \hat{g}_{55}\partial_{\bar{\mu}} g \;\Rightarrow\; A_{\bar{\mu}} = A_\mu - \kappa^{-1}\partial_\mu f, \tag{18}$$

which may be recognized as a gauge transformation of the vector field A_μ. This demonstrates one of the most powerful results of Kaluza's ansatz, namely, that the *gauge* freedom of A_μ, previously considered "internal" in some sense, arises naturally here as a *geometric* freedom in the extra dimension. More recent attempts[26] to incorporate the strong- and weak-nuclear forces into the Kaluza–Klein framework require more dimensions to account for their more complicated non-Abelian gauge symmetries.

C. Reduction of the 5-D action

To simplify subsequent calculations we now set $\phi = 1$ in all components of the metric (9), bearing in mind that they are therefore not the most general solutions. The reduction from the Einstein–Hilbert action in 5 D to the sum of the Maxwell and Einstein–Hilbert actions in 4 D is clearer and more enlightening without the extra terms due to the scalar field, which will be restored in Sec. V. It plays no important part in this introductory look at Kaluza–Klein theory. Its interpretation as a Goldstone boson for example would be considered in the context of the more realistic higher-dimensional theories that are naturally field theoretic and involve the weak- and strong-nuclear forces.

Einstein's field equations $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (8\pi G/c^4)T_{\mu\nu}$ of general relativity may be obtained from the variational principle $\delta S/\delta g_{\mu\nu} = 0$, in which S is the Einstein–Hilbert action[6,32]

$$S_{(\text{grt})} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R, \tag{19}$$

where $g = \|g_{\mu\nu}\|$, and $d^4x\sqrt{-g}$ is the invariant four-volume element in curved space-time.

By analogy with the 4-D variational principle, we consider a generalization of the Einstein–Hilbert action (19) and the variation of the 5-D Ricci scalar $\hat{R}$, which may be expressed[33] in terms of 4-D quantities as $\hat{R} = R - \frac{1}{4}\kappa^2F^{\mu\nu}F_{\mu\nu}$, therefore being independent of x^5. The Einstein–Hilbert action in 5 D is

$$\hat{S} = \frac{c^3}{16\pi\hat{G}}\int d^5x\sqrt{-\hat{g}}\hat{R}, \tag{20}$$

where $\hat{g} = \|\hat{g}_{AB}\| = \|g_{\mu\nu}\| = g$ and the dimensions of $\hat{G}$ must be those of (G/length). We may now separate the integral into a 4-D part and an integration over x^5, namely,

$$\hat{S} = \frac{c^3}{16\pi\hat{G}}\int dx^5\int d^4x\sqrt{-g}\hat{R}$$

and immediately carry out the integration over x^5. To avoid meaningless infinities the fifth dimension must be *compact*,[34] although this does not require it to be small relative to macroscopic scales. We consider a constant radius r_5 for the fifth dimension and denote its circumference by $C_5 = \int dx^5 = 2\pi r_5$.

For Eq. (20) to eventually reduce to the Einstein–Maxwell action:

$$S_{(\mathrm{grt+em})} = S_{(\mathrm{grt})} + S_{(\mathrm{em})} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{4\mu_0 c}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{21}$$

we make the standard association $\hat{G} = 2\pi r_5 G$. Hence, Eq. (20) becomes, on substitution of $\hat{R}$,

$$\hat{S} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\,(R - \tfrac{1}{4}\kappa^2 F^{\mu\nu}F_{\mu\nu}), \tag{22}$$

which implies that

$$\kappa = (16\pi G/\mu_0 c^4)^{1/2}. \tag{23}$$

In evaluating this constant, it is instructive to rewrite it in the form

$$\kappa = \left(\frac{2\alpha_g}{\alpha_e}\right)^{1/2}\frac{e}{m_p c} = \sqrt{2}\,\frac{g_g}{g_e}\frac{e}{m_p c} = 5.75\times 10^{-19}\,\mathrm{A\,s^2\,m^{-1}\,kg^{-1}}, \tag{24}$$

where $\alpha_e = e^2/4\pi\epsilon_0\hbar c = 7.297\times 10^{-3}$ and $\alpha_g = 2Gm_p^2/\hbar c = 1.18\times 10^{-38}$ are the dimensionless electromagnetic fine structure constant and a gravitational analog, the latter expressed by convention in terms of the proton mass m_p. The dimensionless Lagrangian coupling constants g_e and g_g are given by $\alpha_e = g_e^2/4\pi$ and $\alpha_g = g_g^2/4\pi$.

With our 5-D action $\hat{S}$ now indistinguishable from $S_{(\mathrm{grt+em})}$, standard variational procedures yield Einstein's equations with an electromagnetic source and the vacuum Maxwell equation. The variation of $\hat{S}$ with respect to $g_{\mu\nu}$ becomes:

$$\delta\hat{S} = \frac{c^3}{16\pi G}\int d^4x\,\delta(\sqrt{-g}R) - \frac{1}{4\mu_0 c}\int d^4x\,\delta(\sqrt{-g}F^{\mu\nu}F_{\mu\nu}), \tag{25}$$

where $\delta(\sqrt{-g}R) = \sqrt{-g}(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R)\delta g_{\mu\nu}$ is given by a standard general relativistic analysis.[32] The variation of the second term in (25) is

$$\delta(\sqrt{-g}F^{\mu\nu}F_{\mu\nu}) = 2\sqrt{-g}\,(F^{\mu\lambda}F^{\nu}{}_{\lambda} + \tfrac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho})\delta g_{\mu\nu},$$

giving

$$\delta\hat{S} = \frac{-c^3}{16\pi G}D\int d^4x\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \frac{8\pi G}{\mu_0 c^4}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}\right)\right]\delta g_{\mu\nu}. \tag{26}$$

Since the $\delta g_{\mu\nu}$'s are arbitrary, the condition $\delta\hat{S} = 0$ yields:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{\mu_0 c^4}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}\right)$$

or (27)

$$G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}_{(\mathrm{em})},$$

where $T^{\mu\nu}_{(\mathrm{em})}$ is the electromagnetic energy-momentum tensor:

$$T^{\mu\nu}_{(\mathrm{em})} = (1/\mu_0)(F^{\mu\lambda}F^{\nu}_{\lambda} - \tfrac{1}{4}g^{\mu\nu}F_{\lambda\pi}F^{\lambda\pi}). \tag{28}$$

Variation of $\hat{S}$ with respect to the rest of $\hat{g}_{AB}$, namely A_μ, leads to the vacuum Maxwell equation $\partial_\mu F^{\mu\nu} = 0$.

This extension to 5 D in order to unify electromagnetism and gravity closely parallels the Minkowskian extension of 3-D space to 4-D space-time in special relativity to unify $\mathbf{E} = \{cF^{k0}\}$ and $\mathbf{B} = \{\frac{1}{2}\epsilon^{klm}F_{lm}\}$ in the Faraday tensor $F_{\mu\nu}$ (with $F_{kl} = \epsilon_{klm}B^m$).

IV. GEODESIC EQUATION IN 5 D

A. Classical

We consider now the motion in 5 D of a free particle of mass m, namely a geodesic trajectory, with ϕ still equal to 1. A massive test particle in 4 D has an action $S[z(\tau)]$ and an energy-momentum tensor $T^{\mu\nu}(x)$ given[6] by

$$S[z(\tau)] = -mc\int_{-\infty}^{\infty} d\tau(-g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu)^{1/2} \tag{29}$$

and

$$T^{\mu\nu}(x) = mc\int_{-\infty}^{\infty} d\tau\,\delta^4[x - z(\tau)]\dot{z}^\mu\dot{z}^\nu, \tag{30}$$

leading to a trajectory $z(\tau)$ which obeys the geodesic equation $\ddot{z}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{z}^\mu\dot{z}^\nu = 0$, where $\dot{z}^\mu = dz^\mu/d\tau$ and τ is the proper time.

The geodesic equation in 5 D is, correspondingly,

$$\ddot{z}^A + \hat{\Gamma}^A_{BC}\dot{z}^B\dot{z}^C = 0, \tag{31}$$

where the connection coefficients are given in the Appendix. We partition Eq. (31) by $A = \{\mu,5\}$. With $A = \mu$ we obtain

$$\ddot{z}^\mu + \hat{\Gamma}^\mu_{\nu\lambda}\dot{z}^\nu\dot{z}^\lambda + 2\hat{\Gamma}^\mu_{\nu 5}\dot{z}^\nu\dot{z}^5 = 0,$$

which becomes, using connection coefficients (A1) and (A2),

$$\ddot{z}^\mu + \Gamma^\mu_{\nu\lambda}\dot{z}^\nu\dot{z}^\lambda = \kappa(\kappa A_\lambda\dot{z}^\lambda + \dot{z}^5)F^\mu{}_\nu\dot{z}^\nu. \tag{32}$$

The $A = 5$ part of (31) becomes, using Eqs. (A3), (A4), and (15),

$$\ddot{z}^5 - (\kappa A^\rho\Gamma_{\rho\nu\lambda} - \tfrac{1}{2}\kappa\Sigma_{\lambda\nu})\dot{z}^\nu\dot{z}^\lambda + \kappa^2 A^\rho F_{\rho\nu}(\kappa A_\lambda\dot{z}^\lambda + \dot{z}^5)\dot{z}^\nu = 0, \tag{33}$$

and it can be verified using Eqs. (32) and (33) that $C = \kappa A_\lambda\dot{z}^\lambda + \dot{z}^5$ is a constant, that is $dC/d\tau = 0$. Hence, Eq. (32) becomes the Lorentz equation

$$\ddot{z}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{z}^\mu\dot{z}^\nu = (q/m)F^\mu{}_\nu\dot{z}^\nu \tag{34}$$

for the motion of a particle with mass m and charge q in curved space-time, provided we demand that $\kappa C = q/m$, which determines the dimensions of C to be those of a velocity, $(C) = (\mathrm{m\,s^{-1}})$. We see also that $\Sigma_{\mu\nu}$ is decoupled from the 4-D dynamics as it must be.

Once again a consideration of the purely gravitational effects in 5-D Kaluza–Klein space-time produces, in a fully relativistic context, motions governed by both gravitational and electromagnetic interactions in 4 D.

Now consider a generalization to 5 D of the free particle geometric Lagrangian[6] contained in the action (29), namely

$$\hat{L} = -mc(-\hat{g}_{AB}\dot{z}^A\dot{z}^B)^{1/2}. \quad (35)$$

From the usual prescription[6] and the relation $c^2 = -\hat{g}_{AB}\dot{z}^A\dot{z}^B$ we get $\hat{p}_A = \partial\hat{L}/\partial\dot{z}^A = m\hat{g}_{AB}\dot{z}^B$. The fifth component is

$$\hat{p}_5 = m\kappa A_\mu \dot{z}^\mu + m\dot{z}^5 = mC, \quad (36)$$

which is consistent with the dimensions of C. Thus the charge q in the Lorentz equation is related to $\hat{p}_5$ by

$$q = \kappa\hat{p}_5 \Rightarrow \frac{q}{e} = \left(\frac{2\alpha_g}{\alpha_e}\right)^{1/2} \frac{\hat{p}_5}{m_p c}. \quad (37)$$

In classical Kaluza–Klein theory, the charge (in units of the proton charge) is a manifestation of momentum in the fifth dimension in units of $m_p c$. Electric charge is bipolar and, in contrast to the total mass energy of a system, may have a zero value. The Kaluza–Klein ansatz associates these properties with similar properties of $\hat{p}_5$.

Using Eq. (36) in the μ component of $\hat{p}_A$, namely

$$\hat{p}_\mu = m\dot{z}_\mu + m\kappa^2 A_\mu A_\nu \dot{z}^\nu + m\kappa A_\mu \dot{z}^5, \quad (38)$$

we have $\hat{p}_\mu = m\dot{z}_\mu + \kappa A_\mu \hat{p}_5$ which with the first part of Eq. (37) agrees with the generalized momenta $\pi_\mu = m\dot{z}_\mu + qA_\mu$ of a particle of charge q interacting with the electromagnetic field. Thus we can identify $\hat{p}_\mu$ with π_μ. The $A = \mu$ component of the momentum of a free particle in 5 D becomes the momentum of a charged particle in 4 D interacting with the electromagnetic field via its electric charge q which arises from the $A = 5$ component.

B. Quantum mechanical

With the fifth dimension now considered compact we apply some elementary ideas from quantum theory to get an estimate of r_5. We know from experiment that the electric charge of free particles is quantized into integral multiples of the electronic charge e. Hence, from Eq. (37) we may write

$$\hat{p}_5 = e/\kappa, \quad (39)$$

for the proton. Assuming an integral number N of de Broglie wavelengths fit around C_5, we have $N\lambda_5 = 2\pi N\hbar/\hat{p}_5 = 2\pi r_5$. With Eqs. (39) and (24) we find

$$r_5 = \frac{N\hbar\kappa}{e} = \left(\frac{2\alpha_g}{\alpha_e}\right)^{1/2} \frac{N\hbar}{m_p c}. \quad (40)$$

We rewrite this as ($N = 1$)

$$r_5 = \frac{2}{\sqrt{\alpha_e}} l_P \approx 2\sqrt{137} l_P \approx 23.41 l_P = 3.784 \times 10^{-34}\ \text{m}, \quad (41)$$

where the Planck length $l_P \equiv (G\hbar/c^3)^{1/2} = 1.616 \times 10^{-35}$ m. This shows that the radius of the fifth dimension differs from l_P by a factor dependent only on the electromagnetic fine structure constant. The numerical value of r_5 is consistent with the fact that the fifth dimension has not been observed in high-energy experiments. To probe length scales of the order of 10^{-34} m would require energies $\sim 10^{18}$ GeV—far beyond the direct capabilities (≈ 1000 GeV) of modern accelerators.

Equation (40) describes the quantization of charge in terms of the radius of the fifth dimension,

$$q = N\kappa\hbar/r_5. \quad (42)$$

This illustrates the geometric origins of electric charge and its quantization under this ansatz. A goal of many modern Kaluza–Klein theories is to somehow determine the size of the extra dimension(s) from other arguments, thus "explaining" the quantization and value of electric charge.

V. WEAK-FIELD APPROXIMATION

The Fierz–Pauli field is a massless spin-2 tensor field,[6] the equations for which may also be formed by considering the weak-field limit of general relativity. In an asymptotically flat space-time, one can form the Fierz–Pauli tensor field $h_{\mu\nu}(x)$ from the metric tensor of the geometry by

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \quad (43a)$$

and

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}h, \quad (43b)$$

where $h = h^\mu_\mu = -\bar{h}$ is the trace of $h^{\mu\nu}$, and re-express Einstein's field equations in linearized form as[5,6]

$$-\Box\bar{h}_{\mu\nu} + 2\partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \eta_{\mu\nu}\partial^\lambda\partial^\rho\bar{h}_{\lambda\rho} = (16\pi G/c^4)T_{\mu\nu}. \quad (44)$$

The Brans–Dicke–Jordan theory of gravity[17,19] is a variation of general relativity theory which explicitly incorporates Mach's principle,[5,35] namely that it is the distribution of matter elsewhere in the universe that determines the local properties of inertia and hence mass. The left side of the Brans–Dicke–Jordan field equation is also the Einstein tensor of general relativity theory, but the right side involves a scalar gravitational field Φ_{BDJ} which plays a similar role to the *inverse* of the universal gravitational constant G. In the weak-field approximation, the Einstein tensor reduces to the left side of Eq. (44). We also approximate the scalar field by $\Phi_{BDJ} = \phi_0 + \xi$, where ϕ_0 is constant and ξ contains the small variations about ϕ_0. The Brans–Dicke–Jordan equation in the weak-field approximation is then[19]

$$-\Box\bar{h}_{\mu\nu} + 2\partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \eta_{\mu\nu}\partial^\lambda\partial^\rho\bar{h}_{\lambda\rho} = (16\pi\phi_0^{-1}/c^4)T_{\mu\nu} + (\partial_\mu\partial_\nu\xi - \eta_{\mu\nu}\Box\xi)\phi_0^{-1}. \quad (45)$$

In a straight generalization of the 4-D result $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ to 5 D we take the full metric of Eq. (9) and form

$$\hat{h}_{AB} = \hat{g}_{AB} - \hat{\eta}_{AB}, \quad (46)$$

where $\|\hat{h}_{AB}\| \ll 1$, enabling us to ignore second-order terms and obtain the 5-D generalization of the weak-field equation (44) as

$$-\Box\bar{h}_{AB} + 2\partial^C\partial_{(B}\bar{h}_{A)C} - \hat{\eta}_{AB}\partial^C\partial^D\bar{h}_{CD} = (16\pi\hat{G}/c^4)\hat{T}_{AB}. \quad (47)$$

Consider again the case of a *free particle* in the 5-D Kaluza–Klein space-time. The energy-momentum tensor $\hat{T}_{AB}$ is therefore given[6] by

$$\hat{T}_{AB}(x) = mc\int_{-\infty}^{\infty} d\hat{\tau}\,\delta^5[\hat{x} - \hat{z}(\hat{\tau})]\dot{z}_A\dot{z}_B, \quad (48)$$

the 5-D generalization of Eq. (30). We now consider Eq. (47) with the three index choices $(AB) = (\mu\nu)$, $(\mu 5)$, and (55).

Taking $(AB) = (\mu\nu)$ we obtain

$$-\Box\bar{h}_{\mu\nu} + 2\partial^{\lambda}\partial_{(\nu}\bar{h}_{\mu)\lambda} - \hat{\eta}_{\mu\nu}\partial^{\lambda}\partial^{\rho}\bar{h}_{\lambda\rho} = (16\pi\hat{G}/c^4)\hat{T}_{\mu\nu}. \quad (49)$$

Now $\Box\bar{h}_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{h}$, where $\hat{h}_{\mu\nu} = h_{\mu\nu} + \kappa^2\phi A_\mu A_\nu \approx h_{\mu\nu}$ since $\kappa^2\phi A_\mu A_\nu = h_{\mu 5}h_{5\nu}$ which is neglected in our linear approximation. The 5-D trace $\hat{h}$ in the second term is $\hat{h}^A_A = \hat{h}^\mu_\mu + \hat{h}^5_5 \approx h + \phi$, where we have again dropped a term in h^2. Thus from Eq. (43b), $\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi$ and Eq. (49) becomes

$$-\Box\bar{h}_{\mu\nu} + 2\partial^{\lambda}\partial_{(\nu}\bar{h}_{\mu)\lambda} - \eta_{\mu\nu}\partial^{\lambda}\partial^{\rho}\bar{h}_{\lambda\rho} = (16\pi\hat{G}/c^4)\hat{T}_{\mu\nu} + (\partial_\mu\partial_\nu\phi - \eta_{\mu\nu}\Box\phi), \quad (50)$$

which is the weak-field equation (45) of the Brans–Dicke–Jordan theory of gravity if we identify our dimensionless scalar field ϕ with the dimensionless ratio ξ/ϕ_0, and demand $\hat{G}\hat{T}_{\mu\nu} = \phi_0^{-1}T_{\mu\nu}$. If the scalar field ϕ is constant, then Eq. (50) reverts to the standard weak-field Fierz–Pauli equation. Indeed, it was consideration of this scalar field in the context of 5-D Kaluza–Klein theory that led Jordan to his work[17] on the scalar-tensor theory of gravity in the 1940s.

The index choice $(AB) = (\mu 5)$ yields $-\Box\hat{h}_{\mu 5} + \partial^{\lambda}\partial_\mu\hat{h}_{5\lambda} = 16\pi\hat{G}\hat{T}_{\mu 5}/c^4$, where we have used the cylinder condition and $\hat{\eta}_{\mu 5} = 0$. On substitution of the metric components (9), using Eq. (23), and constraining $\phi = 1$ we obtain

$$\Box A_\mu - \partial_\mu\partial\cdot A = -(16\pi\hat{G}/\kappa c^4)\hat{T}_{\mu 5} = -\mu_0\kappa C_5\hat{T}_{\mu 5}. \quad (51)$$

Now the energy-momentum tensor on the right side is the $(\mu 5)$ part of Eq. (48). Using

$$J_\mu = qc\int d\tau\,\delta^4[x - z(\tau)]\dot{z}_\mu$$

and letting $\hat{p}_5 \approx m\dot{z}_5$ in the weak-field approximation since $\kappa \ll 1$, we find

$$\hat{T}_{\mu 5} = (1/\kappa C_5)J_\mu, \quad (52)$$

where we have made use of the relation $\hat{p}_5 = q/\kappa$ from Sec. IV. With this result, Eq. (51) becomes Maxwell's second-order equation $\Box A_\mu - \partial_\mu\partial\cdot A = -\mu_0 J_\mu$. However, this result did require the use of stronger approximations, namely that the scalar field ϕ be constant (normalized to unity) and that $\hat{p}_5 \approx m\dot{z}_5$, which involves ignoring the first-order term $\kappa A_\lambda\dot{z}^\lambda$ in $\hat{p}_5$. With these conditions in place, the motion of a test particle in 5 D becomes that of a charged particle in 4 D interacting with the electromagnetic field in flat space-time.

With $(AB) = (55)$ in Eq. (47) we obtain

$$\Box\bar{h}_{55} - \hat{\eta}_{55}\partial^{\lambda}\partial^{\rho}\bar{h}_{\lambda\rho} = 16\pi\hat{G}\hat{T}_{55}/c^4.$$

Since $\hat{\eta}_{55} = 1$, we find $\bar{h}_{55} = \frac{1}{2}\phi - \frac{1}{2}h$ and $\bar{h}_{\lambda\rho} \approx h_{\lambda\rho} - \frac{1}{2}\eta_{\lambda\rho}(h + \phi)$, and using Eq. (43b) we obtain

$$\Box\bar{h} = -(16\pi\hat{G}/c^4)\hat{T}_{55}. \quad (53)$$

Equating this with the trace of the wave equation $\Box\bar{h}_{\mu\nu} = -16\pi GT_{\mu\nu}/c^4$ we get the following relation between $\hat{T}_{55}$ and the trace of the 4-D energy-momentum tensor:

$$\hat{T}_{55} = (1/C_5)T, \quad (54)$$

where we have used $\hat{G} = C_5 G$. This is consistent with our earlier results since the $\hat{g}_{55}$ metric component may represent a scalar gravitational field ($\phi = \xi/\phi_0$ in this section) and it is known[6] that scalar gravitational fields couple only to $T = T^\mu_\mu$ rather than all components of $T_{\mu\nu}$.

VI. CONCLUSION

We have presented a simple account of 5-D Kaluza–Klein theory and shown that construction of its metric requires the incorporation of the scalar field in all components, clarifying the inconsistency of Kaluza's original associations.

The Kaluza–Klein ansatz[2,3] was presented in detail in a fully relativistic way. The Einstein–Hilbert Lagrangian $c^3R/16\pi G$, which yields Einstein's vacuum equations in the 4-D variation, is generalized in the Kaluza–Klein technique to 5 D. Upon dimensional reduction it yields Einstein's equations with an electromagnetic source composed of pure radiation (since $J_\nu = 0$).

We also considered a free particle traveling along a 5-D geodesic as an illustration of the Kaluza–Klein process, both classically and in the context of simple quantum mechanics. This demonstrates how electric charge arises as a consequence of momentum in the fifth dimension. An estimate of the radius of the compact fifth dimension was obtained and its simple relationship with the Planck length via the electromagnetic fine structure constant explicitly determined. Consistent with its nonobservation, its numerical value was found to be only an order of magnitude larger than the Planck length ($l_P \sim 10^{-35}$ m).

We demonstrated the simple reduction to 4-D gravitational and electromagnetic results under a Fierz–Pauli weak-field approximation of the 5-D gravitational field equations sourced by a test particle. This encapsulates the essential elements of (5-D) Kaluza–Klein theory without the need for a full general relativistic treatment.

APPENDIX

The connection coefficients derived from the metric of Eq. (9) with $\phi = 1$ are

$$\hat{\Gamma}^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - \tfrac{1}{2}\kappa^2(A_\nu F^\mu{}_\lambda + A_\lambda F^\mu{}_\nu), \quad (A1)$$

$$\hat{\Gamma}^\mu_{\nu 5} = -\tfrac{1}{2}\kappa F^\mu{}_\nu, \quad (A2)$$

$$\hat{\Gamma}^5_{\nu\lambda} = -\kappa A^\rho\Gamma_{\rho\nu\lambda} - \kappa^3 A^\rho A_{(\nu}F_{\lambda)\rho} + \kappa\partial_{(\lambda}A_{\nu)}, \quad (A3)$$

$$\hat{\Gamma}^5_{\nu 5} = \tfrac{1}{2}\kappa^2 A^\rho F_{\rho\nu}, \quad (A4)$$

$$\hat{\Gamma}^\mu_{55} = \hat{\Gamma}^5_{55} = 0. \quad (A5)$$

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The elastic constant of a rubber tube

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A method is described for the evaluation of the elastic constant of a rubber automobile inner tube. The relevant theory is presented, and no expensive apparatus need be used.

I. INTRODUCTION

A number of authors have considered the stress/strain relations in rubber structures such as balloons and circular membranes.[1] We extend the analysis to the case of an automobile inner tube. A simple global stress analysis is possible if the shape is considered as a toroidal surface of revolution. The deformation of a volume element due to the local

