Essentials of the Casimir effect and its computation

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A historical introduction to the issues raised by the Casimir effect is provided, putting special emphasis on the experiments which attempt to test this and related phenomena. The classical and modern quantum interpretations are explained, and an account of the existing calculation methods is given. The discussion is illustrated with an example of the zeta-function regularization procedure.

I. THE CLASSICAL INTERPRETATION OF THE CASIMIR EFFECT

A. Introduction

The Casimir effect takes its name after the Dutch prominent physicist H. B. G. Casimir, who in 1948 published a paper in the Proceedings of the Royal Academy of Sciences of the Netherlands where a rather remarkable property, namely, the attraction of two neutral metallic plates, was predicted theoretically.[1] In all the research papers and reviews about the Casimir effect that have been published in the last years,[2–9] this paper by Casimir is taken as the undubious beginning of a whole branch of research, which aims nowadays at answering very profound questions about the vacuum structure of quantum field theory (QFT). However, it is very difficulty to get a clear idea from these—on the other hand, excellent review papers—of what the contribution of Casimir was precisely or of what was the specific physical context in which his paper appeared. Some authors go even further and attribute to Casimir very deep ideas about QFT that by no means could he have had at his time. The interest of the subject, which is reflected by the increasing number of papers which are dealing with it, deserves a proper clarification of several points.

To start with, it is fair to say that the 1948 paper by Casimir attracted comparatively small attention during the following two or three decades. For instance, another work by Casimir and Polder,[10] which was published in the Physical Review also in 1948, got by far much more citations from experimental and theoretical colleagues. This second paper is nowadays considered a mere addition or an initial stimulus to the first, fundamental paper about the Casimir effect. Maybe part of this puzzle can be explained by the importance and availability of the Physical Review (in comparison with the aforementioned Dutch journal). However, even in contributions where the two papers were mentioned, the one by Casimir alone deserved no special comment; i.e., in no way was it singled out with respect to the other one co-authored by Polder.

B. Connection with the van der Waals forces and the London theory

A second point to be remarked is the following. Nowadays, when dealing with the Casimir effect itself, a particular emphasis is usually put on its own spectacularity, that is, on the fact that two noncharged plates do attract themselves in the vacuum. One needs to understand that this is actually much more mysterious *today* than it was in 1948. At that time, 75 years after the celebrated dissertation of J. D. van der Waals[11] (published for the first time in 1881)—where his famous weak attractive forces between neutral molecules were introduced—and already 18 years after the formulation by F. London of his celebrated theory[12]—which gave a precise (for that time) explanation of the nature and strength of the van der Waals forces as due to the interaction of the fluctuating electric dipole moments of the neutral molecules—there was nothing specially mysterious about two neutral bodies attracting each other. van der Waals forces play a very important role in biology and medical sciences. They are in general particularly significant in surface phenomena, such as adhesion, colloidal stability, and foam formation. One could dare to say that they are the most fundamental physical forces controlling living beings and life processes. Three different classes of van der Waals forces can in principle be distinguished: orientation, induction, and dispersion forces.[13] The ones involved in the attraction of the plates correspond in this classification to the third group.

C. The specific contribution of Casimir and Polder: Retarded van der Waals forces

The works by Casimir and Casimir and Polder, addressed a rather (of course, important but) more technical point: the fact that the polarization of the neighboring molecules (or atoms) induced by a given molecule (atom) is delayed as a consequence of the finiteness of the velocity of light. So these forces could be termed long-range retarded dispersion van der Waals forces. This was clearly noted, experimentally, by the fact that when the molecules were separated far enough (always in the range of microns, so that the effect could make any sense), the power law corresponding to the attractive force between two given molecules changed to the inverse eighth power,

$$F = C_2/r^8 \tag{1}$$

(B being a constant and r the distance between the two molecules), as compared with the inverse seventh power obtained in London's theory:

$$F = C_1/r^7 , \tag{2}$$

typical of the van der Waals forces for very close molecules (which did not feel so much the retardation effect due to the finite velocity of interaction). In particular, the expression obtained by Casimir and Polder has become famous for the potential energy U corresponding to two atoms separated by a distance r and whose static polarizabilities are α_1 and α_2, respectively:

$$U = -(23\hbar c/4\pi)\,(\alpha_1\alpha_2/r^7) . \tag{3}$$

Note the appearance of $\hbar$. This energy U is the result of making a semiclassical approximation to a theory intended

for the description of quantum effects. The word *classical* is to be understood this way all along the text. It is now (and was then) a matter of an elementary exercise on surface integration to obtain the force per unit surface (that is, the pressure) which attracts two neutral, parallel, metallic, perfectly conducting plates of infinite extension in the vacuum, under the hypothesis that they are formed by a rarefied distribution of neutral, polarizable atoms. However, Casimir used a *novel* technique: that of calculating the effect due to the zero-point energy of the electromagnetic field. Simple dimensional reasons show immediately that the just-mentioned power laws give rise, respectively, to an inverse third power, if the plates are very close (distance say less or equal than 0.01 μm, i.e., 100 Å, the penetration depth of electromagnetic waves in the metal),

$$P = A/6\pi d^3 , \tag{4}$$

and to an inverse fourth power, if they are more separated (say somewhat above 0.02 μm),

$$P = B/d^4 . \tag{5}$$

This last case was the one explicitly calculated by Casimir in his seminal paper,[1] with the expression

$$P = 0.013/d^4 \ \text{dyn/cm}^2 , \tag{6}$$

where the distance in this expression has to be given in microns, the basic unit length for this kind of calculation, as already mentioned. The calculations can be easily extended to different geometrical configurations. For instance, for a hemisphere of radius R held at distance d from an infinite plane, the attractive force when they are very close was given by London's theory:

$$F = AR/6d^2 . \tag{7}$$

On the other hand, when they are a bit more separated, the retarded interaction changed this result to

$$F = 2\pi BR/3d^3 . \tag{8}$$

The result obtained by Casimir and Polder for the potential energy corresponding to a particle of electric polarizability α, inside a cavity of a perfectly conducting material and separated a distance r from the flat wall, was

$$U = -(3\hbar c/8\pi)(\alpha/r^4) . \tag{9}$$

D. The Lifshitz theory

E. M. Lifshitz, in a not less important paper[14] than those previously referred to (submitted to the Russian Zh. Eksp. Teor. Fiz. in 1954 and whose English translation was published in 1956), developed a different theory in order to deal with the two major difficulties of London's theory, namely, the already mentioned one, that it did not take into account the finite velocity of propagation of the electromagnetic interaction (this had been already taken care of by Casimir and Polder), and a second one, namely, the fact that the van der Waals force is not additive. This prevents us from treating the problem of extensive bodies in a proper way as composed of elementary constituents (atoms or molecules), and to derive the force between macroscopic bodies by integration of the forces which exist between the elementary constituents—unless one makes the hypothesis (advanced before) of considering a very dilute distribution of constituents in the extensive bodies, but this is a very unrealistic assumption.

Lifshitz's theory started from the opposite direction, treating matter as a continuum with a well-defined frequency-dependent dielectric susceptibility. It was a completely closed theory: It could treat any kind of material bodies, it explained in a precise and continuous way the transition from one power law to the other (due to the retardation effect) when the distance is increased, and it contained the formulas of London for the elementary constituents of matter and of Casimir and Polder for the perfectly conducting, neutral metallic plates, as limiting and particular cases, respectively, as was rigorously proven by Dzyaloshinskii, Lifshitz, and Pitaevski in 1961.[15] More precisely, the picture of the interacting bodies in Lifshitz's theory was that of two media filling half-spaces with plane-parallel boundaries separated from one another by a certain distance d. Just as for the case of the random force introduced in the theory of Brownian motion, a "random" field was introduced into the Maxwell equations of motion. In the case of large (that is, not so small) separations, Lifshitz's formula for the force per unit area betwen two parallel plates separated by a distance d reduced to the following. For plates of an infinitely conducting metal,

$$P = \hbar c\pi^2/240d^4 ; \tag{10}$$

for two identical dielectrics of dielectric constant ϵ_0,

$$P = \frac{\hbar c\pi^2}{240d^4}\left(\frac{\epsilon_0 - 1}{\epsilon_0 + 1}\right)^2 \varphi(\epsilon_0) , \tag{11}$$

where $\varphi(\epsilon_0)$ is a function defined by the theory and which has the following behavior for $\epsilon_0 \to \infty$:

$$\varphi(\epsilon_0) \simeq 1 - (1.11/\sqrt{\epsilon_0})\ln(\epsilon_0/7.6) ; \tag{12}$$

for a metal (infinitely conducting, $\epsilon = \infty$) and a dielectric of constant ϵ_0,

$$P = \frac{\hbar c\pi^2}{240d^4}\frac{\epsilon_0 - 1}{\epsilon_0 + 1}\varphi(\epsilon_0) ; \tag{13}$$

also, for two individual atoms belonging to materials of dielectric constants ϵ_{10} and ϵ_{20}, respectively, Lifshitz obtained the attractive force between them as a limiting case of the formula for continuous media, assuming that both media were sufficiently rarefied:

$$F = (23\hbar c/640\pi^2 d^4)(\epsilon_{10} - 1)(\epsilon_{20} - 1) , \tag{14}$$

from which the preceding formula of Casimir and Polder for the potential energy [Eq. (3)] follows immediately.

A most important point of this theory was the fact that the general equations derived by Lifshitz to calculate the dispersion force require only information about the dielectric properties of the bodies (in particular, the dielectric susceptibility of the bodies as a function of the frequency), and this information can, in principle, be obtained from independent spectroscopic measurements. Thus Lifshitz's theory could be applied by Parsegian and Ninham (in 1970) to a detailed study of the dispersion forces between biological membranes.[16] A very original contribution of the theory was also to consider the effect of temperature on the force of interaction. However, this part of Lifshitz's theory disagreed with subsequent (independent) calculations by Sauer,[17] Mehra,[18] and Brown and Maclay,[3] who agreed among themselves and said that Lifshitz's results concerning this point were in error.

II. EXPERIMENTAL VERIFICATION

A. The first direct experiments: Abrikosova and Deriagin, Kitchener and Prosser, and Sparnaay *et al.*

The experimental verification of the transition from one power dependence to the other when increasing the approach of the two bodies came through a work of Tabor and Winterton,[19] published in Nature in 1968. The first measurements of van der Waals forces had been always indirect, mainly based on determining the necessary force in order to break the adhesion between two surfaces. In the early 1950s, the first direct measurements for the forces between surfaces as a function of separation, carried out by I. I. Abrikosova and B. V. Deriagin, were published.[20] They used different configurations, such as a hemisphere and a flat surface of polished quartz, two flat plates of quartz, and one made of quartz, and the other of metal. The separation between them was measured by optical interference and the attractive force with the help of an elaborate feedback mechanism. They reached a smallest distance of approach of about 0.1 μm, being the overall range from 0.1 to 0.4 μm. A similar study was reported by J. A. Kitchener and A. P. Prosser in 1957, who used parallel flat surfaces of borosilicate glass.[21] They only reached about 0.7 μm. The results of further experiments by M. J. Sparnaay,[22] Sparnaay and Jochems,[23] and Black *et al.*[24] were published between 1958 and 1960. They used, in particular, aluminum, glass, and quartz plates, and the smallest distance of approach was always well above 0.1 μm. In all these experiments, it was observed that the forces agreed in magnitude with the formula corresponding to the retarded van der Waals forces, which for the distances of approach involved are ten times smaller than the ones corresponding to nonretarded forces. All of them worked with distances of maximal approach not less than the 0.1 μm, as mentioned above.

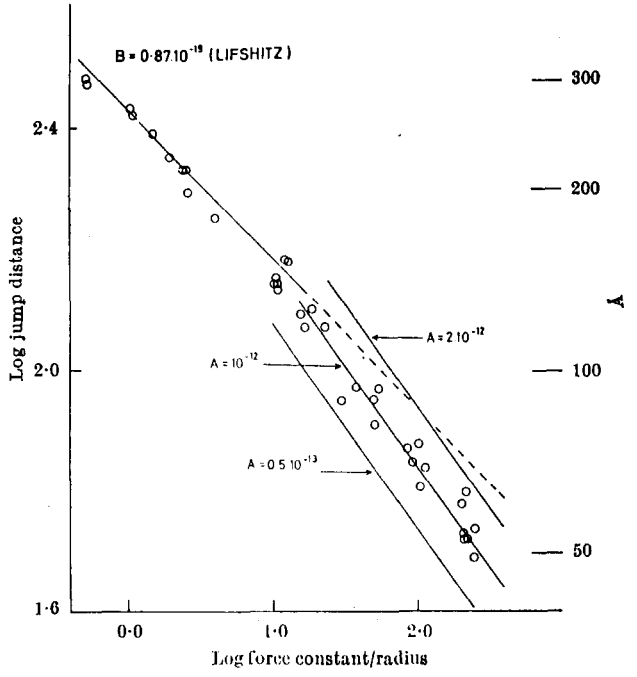

Fig. 1. Logarithmic plot of critical jump distance h against the parameter stiffness of beam/radius of cylinder (c/R). The results show a transition from nonretarded to retarded van der Waals forces at a separation of the order of 150 Å. For a separation above 200 Å, the results agree well with those calculated from Lifshitz's theory for retarded van der Waals forces using a theoretical value of $B = 0.87 \times 10^{-19}$. For separations below 100 Å, the results agree with those calculated for nonretarded van der Waals forces using a value of the Hamaker constant $A = 10^{-12}$ erg. (Reprinted by permission from D. Tabor and R. H. S. Winterton, Nature **219**, 1120 (Copyright (c) 1968 Macmillan Magazines Ltd.).

B. The experiment of Tabor and Winterton: Transition from normal to retarded van der Waals forces

D. Tabor and R. H. S. Winterton[19] worked with very smooth surfaces—which they achieved by using the cleavage face of muscovite mica—bent them in partially cylindrical form, and arranged them perpendicularly. Multiple-beam interferometry allowed these authors to determine the separation between the sheets of mica to an accuracy of $\pm$ 4 Å. From previous experiments, they had realized that in order to investigate the transition from normal to retarded van der Waals forces it was necessary to attain separations much less than the 0.1 μm (see Fig. 1).

They managed to work in the range between 50 and 300 Å and clearly observed the transition from one power dependence with distance to the other: The normal van der Waals forces were seen to predominate for separations less than 100 Å and the retarded forces for separations greater than 200 Å. As seen in Fig. 1, for separations greater than 150 Å the experimental points approached very well the solid line corresponding to the predictions of the Lifshitz theory.

C. The experiment of Sabiski and Anderson

A further experimental verification of the Lifshitz theory came in 1973 with the paper by E. S. Sabiski and C. H. Anderson.[25] They presented accurate measurements of the properties of helium films absorbed on cleaved surfaces of alkaline–earth fluoride crystals at 1.38 K. The thickness of the films were measured using an acoustic interferometry technique, and their value ranged between 10 and 250 Å. The experimental results were in excellent agreement with the corresponding calculations based on the Lifshitz theory of the van der Waals forces (see Fig. 2).

This is an example of a paper which makes no reference to the work of Casimir, although it mentions the one by Casimir and Polder on the retarded van der Waals forces. The reason for this seems clear, if we consider that Casimir's paper deals with the attraction of metals in the vacuum, while the idea and specific device used by Sabiski and Anderson makes only sense with dielectrics. After noting the extreme difficulty in performing more accurate experiments on the traditional line which had been followed until then, they started from an old observation by Schiff[26] (made in 1941) that the relatively thick liquid-helium films observed in the walls of containers are formed by the van der Waals force. Although there was some controversy concerning the importance of additional forces which could come into play, the remarkable agreement obtained between the experimental results and Lifshitz theory—for different materials and as a function of the thickness of the helium film—provided a strong confirmation of the validity of the latter. As admitted by Sabisky and Anderson, the results cannot be extrapolated to smaller distances, comparable with the surface roughness, only a few atoms thick. In this case stronger van der Waals forces have been reported corresponding to the adsorption–isoterm measurements of unsaturated films in cases where the surface roughness

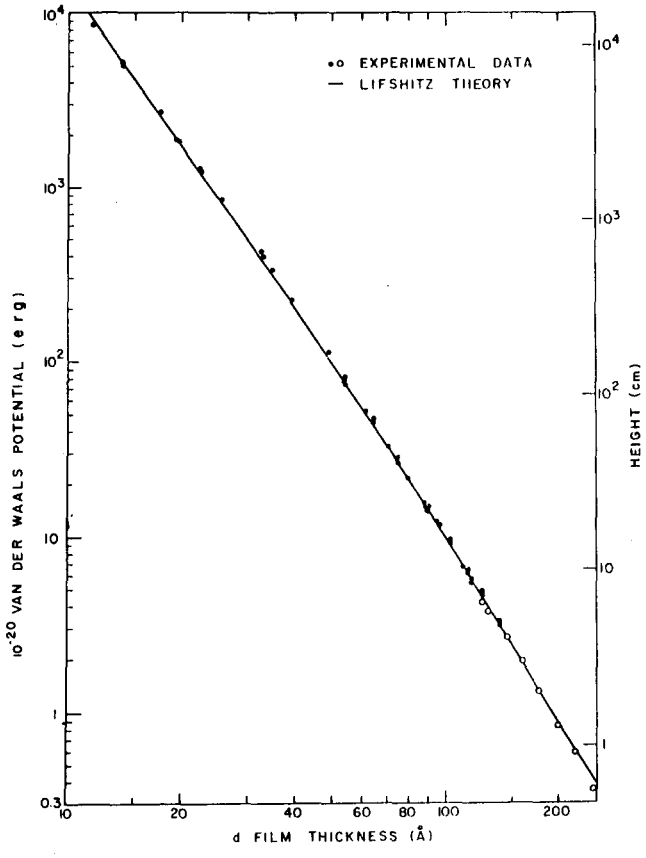

Fig. 2. van der Waals potential of a helium atom on the surface of the helium film which is absorbed on a SrF_2 cleaved surface as a function of the film thickness at 1.38 K. The data represented by closed and open points are for unsaturated and saturated films, respectively. The solid line represents the theory of Lifshitz [taken from E. S. Sabiski and C. H. Anderson, Phys. Rev. A 7, 790, 799 (1973)].

was certainly significant. It is fair to mention also that these techniques are extremely difficult to control properly and that the reported values of the very many experiments which have been carried out (in this, on the other hand, basic field in solid-state and condensed-matter research) give values for the film thickness which vary considerably.

III. THE CASIMIR EFFECT IN QUANTUM FIELD THEORY

A. The local formualtion of the Casimir effect

The paper by L. S. Brown and G. J. Maclay,[3] published 21 years after the work of Casimir, was specially significant from the theoretical point of view, a kind of milestone on the road leading to the modern QFT interpretation of the Casimir effect. For the first time, it contains the local formulation of the Casimir effect, in terms of the vacuum energy density and vacuum pressure. These authors derived the following explicit expression for the regularized energy-momentum (or stress-energy) tensor:

$$\Theta^{\mu\nu} = -(\pi^2/180d^4)(\tfrac{1}{4}g^{\mu\nu} + z^\mu z^\nu), \qquad (15)$$

which was computed with the aid of an image-source construction of the corresponding Green's function. Here, z^μ denotes a spacelike four-vector orthogonal to the parallel plates and d (as before) its separation.

Also, for the first time, and although not explicitly stated there, the calculations in Ref. 3 involved the zeta-function procedure, in particular, Riemann and Epstein zeta functions. This method has evolved, starting from the seminal works of J. S. Dowker and R. Critchley[27] (in 1976) and of S. Hawking[28] (in 1977)—and incorporating a rather long list of contributions from very different authors[29]—into the most elegant, simple, and mathematically rigorous way of defining regularized vacuum energy densities in situations that nowadays very much generalize the original case considered by Casimir. This method is termed the *zeta-function regularization procedure* and pervades different aspects of QFT.

B. The mystery of the Casimir effect

Before introducing the concrete expressions which lead to the calculation of the Casimir effect from the point of view of the modern QFT, let us say a few words about an intriguing question that was posed before, namely, why is the Casimir effect less understood now than it was 40 years ago? Why did it become a subject of more and more interest as decades went by? Though the (tentative) answer to these questions will become more clear after the discussion below and once the actual calculations are performed, we can already point out in advance to the basic problem: Unlike the van der Waals forces, which are always attractive, the ones appearing in the Casimir effect can be either attractive or repulsive. In the most simple case, the modern calculations about the Casimir effect doubtlessly indicate that if instead of two plates we considered the configuration formed by the two halves of a hollow cube of a neutral perfectly conducting metallic material, if we would bring the two halves together in order to form a closed cube, they would experience a *repulsive* pressure. The same would happen when bringing together the two halves of a metallic sphere. The sign of the Casimir force (and that of the vacuum energy density) is positive or negative depending crucially on the nature of the field (electromagnetic, i.e., the only one considered until now, scalar, etc.) and, for a given field, on the dimension of the space-time (always four, until now, but can be arbitrarily generalized), and with the dimension also fixed, on the particular geometry of the boundary. Here, we are always talking about a flat space-time, but of course, curved manifolds can also come into play.

One speaks nowadays of different (generalized) Casimir effects, as due to (1) the existence of a background field in the vacuum; (2) the geometry of the boundary; (3) the dimension of the space-time; and (4) the possible curvature of the space-time. Summing up, just by closing up in a trivial way the configuration considered by Casimir, we obtain a repulsive pressure, which can in no way be explained as a kind of van der Waals-like force. On the other hand, for the multiple generalizations of the Casimir effect (to different fields, boundaries, dimensions, and space-times), the dependence of the force sign on them is anything but trivial. So is the *mystery* of the Casimir force born.

C. The concept of the vacuum energy

It has been this generalization of the concept of the Casimir effect to incorporate all kinds of contributions to the vacuum energy density in the situations above described that has rendered this concept so popular in QFT. For an excellent review, much more detailed than the present summary, the reader is addressed to the paper by Plunien *et al.*,[8] which contains a very detailed exposition of all the

present developments in this field and 156 basic references. In particular, the Casimir effect can give rise to contributions to the surface tension of a curved conductor, can have cosmological consequences due to deviations from the Minkowskian geometry of space-time, and can lead to corrections to the self-energy for a scalar field confined to a cavity or even to calculations in the bag model as a confining mechanism for quarks and gluons in QCD. It also gives the response of the vacuum to the presence of external fields. In this context, Ambjørn and Wolfram[7] have discussed the vacuum energy of a charged scalar field in the presence of an external electrical field. Finally, deviations from ideal conditions have also been the object of investigation, namely, nonzero temperature, noninfinitely conducting metals, plates of nonzero depth, etc.

Let us now describe the *conceptual revolution* brought about by Casimir's experiment. As already remarked, at the beginning there was no difficulty in explaining the Casimir effect—and classical generalizations of it—by means of the Lifshitz's theory of the van der Waals forces. However, modern QFT offers an alternative, very general, much more fundamental explanation from first priciples of the Casimir force: It is due to a change of the vacuum energy, i.e., to a deviation of the zero-point energy caused by the presence of external constraints. In other words, Casimir's work stimulated investigations about the zero-point energy problem in QFT, which resulted in what is now commonly called "Casimir's concept of the vacuum energy": The physical vacuum energy of a quantized field must necessarily be calculated with respect to its interaction with external constraints and is thus defined as the difference between the zero-point energy corresponding to the vacuum configuration with constraints and the one corresponding to the free-vacuum configuration, respectively. This formal definition must be supplemented, in general, with a regularization prescription in order to obtain a finite final expression. In this way a precise field quantization scheme starting from first principles is constructed (at least theoretically; in practice, things are not so easy).

A first application of this concept of vacuum energy is an alternative calculation of the Casimir effect, which gives exactly the same result as the one obtained through its interpretation as due to retarded van der Waals forces. Actually, it was Casimir himself who showed for the first time[30] that the zero-point energy of the electromagnetic field could successfully explain van der Waals attraction. But the concept of vacuum (or Casimir) energy goes much further than his simple example and leads to results which in no way can be put down to van der Waals forces, such as the cases of repulsive pressure already mentioned. In the modern Casimir theory, these forces arise naturally and cannot be understood in the framework of conventional field theory, in which the zero-point energy is simply neglected. Two general methods for evaluation of the Casimir energy can be distinguished: (i) summation of the series of energy eigenvalues corresponding to the zero-point field modes and (ii) determination of the vacuum stress-energy tensor in terms of local Green's functions, obtained by considering the constrained propagation of virtual field quanta. Both methods should lead to the same result, but this is problematic because of the infinites involved and the different regularization schemes that one may choose to use. Aside from this, specific (technical) difficulties appear in both cases. In the first, one is led (in principle) to calculate the whole energy spectrum for the free and for the constrained field modes, and this can only be achieved easily for simple geometries. In the second case, one has to determine the exact Green's functions describing propagation in the presence of external boundaries. This is done by the usual image-source construction, which is again easy only for simple geometries, or perturbatively by multiple scattering expansions in the case of general constraints.

D. The explicit definition of the Casimir energy as a mode sum

From what has been said, it is now clear that a fundamental question connected with the Casimir energy is the determination of the "true" field Hamiltonian. In a model without boundary conditions, the Hamiltonian eigenvalue associated with the ground state or vacuum (the zero-point energy) is always discarded because, in spite of being infinite, it can be reabsorbed in a suitable redefinition of the energy origin. The most popular way of putting such an adjustment into practice is normal ordering. Now, a most important implication of the concept of vacuum energy stemming from the work of Casimir is the fact that the vacuum energy in QFT *cannot* be defined by means of normal ordering, because this procedure cannot possibly take into account the presence of arbitrary boundaries. The canonical formalism of QFT tells us that, for a scalar field of mass m, the Hamiltonian operator takes the form

$$\begin{aligned}\hat{H} &= \frac{1}{2}\sum_k \omega_k (a_k^\dagger a_k + a_k a_k^\dagger) \\ &= \sum_k \omega_k \left(n_k + \frac{1}{2}\right) \\ &= \sum_k \omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right), \end{aligned} \tag{16}$$

where $\omega_k^2 = k^2 + m^2$ are the eigenvalues of the Klein–Gordon operator, $a_k^\dagger$ and a_k satisfy the canonical commutation relations for bosonic fields

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0, \tag{17}$$

and $n_k = a_k^\dagger a_k$ is the number operator, whose eigenvalues are non-negative integers.

Since the vacuum state $|0\rangle$ is defined by

$$a_k |0\rangle = 0, \tag{18}$$

when computing the vacuum expectation value

$$E_0 \equiv \langle 0|\hat{H}|0\rangle, \tag{19}$$

one gets a half of the *sum of all the eigenfrequencies*:

$$E_0 = \frac{1}{2}\sum_k \omega_k, \tag{20}$$

which is in general a divergent quantity.

When instead of a bosonic field one takes a Dirac fermion, the levels above and below the Fermi energy have to be considered separately. Apart from the different roles of particles and antiparticles, the result is quite the same: Only some signs are reversed, because of the replacement of commutators with anticommutators. After requiring charge-conjugation invariance, the zero-point energy of the symmetrized noninteracting Dirac vacuum is

$$E_0 = -\frac{1}{2}\left(\sum_{k>F} E_k + \sum_{k<F} E_k\right), \tag{21}$$

where the E_k's are the eigenenergies, up to a sign change for values below the Fermi level F.

E. About the relation between the "old" and the "new" Casimir effects

While the dependence of the Casimir effect on the plate separation can be easily inferred on dimensional grounds, the numerical factor can only be derived by a more or less involved calculation. The relevance of this point is that it shows that such a result cannot reflect a general situation. What is more, the sign of the vacuum energy, which determines the attractive or repulsive character of the zero-point pressure, depends, in a rather intricate way, on the particular geometry of the configuration.

An early classical model for the electron, based on a negative charge distribution, was long ago proposed by Casimir himself.[31] For such an object to be stable, additional *Poincaré* stresses were needed. Boyer's work[2] on the Casimir effect for a perfectly conducting spherical shell was aimed at the justification of those stresses as coming from the Casimir energy. However, after explicitly computing the vacuum energy, Boyer found that its sign was the opposite to the one expected; i.e., the configuration was repulsive, thus putting an end to the hope of stabilizing the classical electron model by the Casimir energy. Nevertheless, for solid balls with a given dielectricity and permeability, the energy turns out to be negative again, but its contribution is too small to account for the observed charge quantization.

Going back to perfect conductors, the cylindrical configuration, which is an intermediate case between the plates and spherical surface, gives an attractive pressure, and not a null one, as the alert reader might have expected.

The study of the Casimir effect for the e.m. field in regions bounded with thin arbitrarily shaped conducting shells has led to interpret the change in the vacuum energy as a contribution to the surface tension.[32] The vacuum energy contribution to the surface tension can be defined as the difference between the zero-point energy for a certain surface deformation and that for flat boundaries. In gravitational theories, one can use the Casimir energy to investigate cosmological consequences of changes in vacuum energy due to deviations from Minkowski geometry.

For objects with dielectric constant and magnetic permeability, the contribution to the vacuum energy does in general depend on the frequency cutoff, which specifies the degree of transparency of the material to the e.m. waves. By making suitable choices of these cutoffs, reasonable values for surface tensions have been found for liquid helium and for many metals.

Casimir's papers prodded many physicists into making further investigations about the zero-point energy problem in field theory. The numerous questions raised on the issue of the physical reality of the zero-point energy can be adequately answered only by showing observable effects that fully display the influence of the vacuum. The reason why the e.m. field is so often taken as the first example is our wide knowledge of its interaction. Thus the Casimir energy has provided alternative calculations of van der Waals forces. Nevertheless, unlike those interactions, the Casimir effect can bring about not just attractive forces, but repulsive ones as well, as was dramatically observed for the conducting spherical shell.

This new aspect pointed out that there were at least two implications of the vacuum energy: a different interpretation of known phenomena and the exhibition for the first time of effects not just classically incomprehensible, but not even understandable in field theories which disregard the zero-point energy.

In theory, in order to make the mode sum in the preceding subsection applicable, one should first introduce a space cutoff or quantization box Σ with the same shape as the real boundaries $\mathscr{B}$, but much larger, so that the real situations may be described by the limit when the size of Σ tends to infinity. The free-field zero-point energy inside the volume bounded by Σ will be

$$E_0[\Sigma,\phi] = \frac{1}{2}\sum_k \omega_k[\Sigma,\phi]\,. \tag{22}$$

Let $\mathscr{B}$ be the surface on which the field is constrained, e.g., by imposing Dirichlet or Neumann boundary conditions, and let $\omega_k[\Sigma,\mathscr{B}]$ denote the eigenfrequencies found when, inside the region limited by Σ, the boundary $\mathscr{B}$ with its corresponding conditions is present. The *difference in zero-point energy* between this and the previous configuration is given by

$$\Delta E\omega_k[\Sigma,\mathscr{B}] = \frac{1}{2}\sum_k \omega_k[\Sigma,\mathscr{B}] - \frac{1}{2}\sum_k \omega_k[\Sigma,\phi]\,. \tag{23}$$

This difference is meaningless because, as it stands, both terms are infinite. However, if one prescribes the introduction of a regulating mechanism—such as a frequency cutoff—the Casimir energy will emerge as the regularized limit of this difference after the space cutoff has been removed by sending Σ to infinity. If the regulator is taken to be a frequency cutoff function, the limit will have meant the removal of two cutoffs.

F. An example of calculation by means of the zeta-function regularization procedure

Let us obtain the Casimir effect for the original perfectly conducting parallel-plate configuration. The electromagnetic field in three-dimensional space can be described as if it was made of two scalar massless fields, one satisfying Dirichlet boundary conditions and another subject to Neumann ones. They correspond to the transverse electric and transverse magnetic modes, and the conditions come from requiring the standard perfectly conducting behavior for the electric and magnetic fields, namely, $\mathbf{n}\cdot\mathbf{B}=0$ and $\mathbf{n}\times\mathbf{E}=\mathbf{0}$ on the surface of the conductor. For each of them, the eigenmodes have to satisfy the free Klein–Gordon equation

$$\Box\varphi(\mathbf{x},t) = 0\,. \tag{24}$$

The Dirichlet boundary conditions are enforced by demanding

$$\varphi^{\mathrm{D}}(\mathbf{x},t) = 0, \quad \text{on the boundaries}\,; \tag{25}$$

i.e., the field has to vanish on the plates. If we assume that they are orthogonal to the x_1 axis and held at a distance L from each other, then, after renaming (x_1,x_2,x_3) into $(x,\mathbf{x}_T)$, the solutions may be expressed as

$$\varphi(x,\mathbf{x}_T,t) \sim \sin(\pi n x/L)\,e^{i\mathbf{k}_T\cdot\mathbf{x}_T}\,e^{-i\omega^{\mathrm{D}}_{kn}(L)t}\,,$$
$$\omega^{\mathrm{D}}_{kn}(L) = [(\pi n/L)^2 + \mathbf{k}_T^2]^{1/2}, \quad n = 1,2,3,\ldots\,. \tag{26}$$

As for the eigenvalues obeying Neumann boundary conditions, one has to bear in mind the constraint

$$\partial_{\mathbf{n}}\varphi^{\mathrm{N}}(\mathbf{x},t)=0, \quad \text{on the boundaries}, \tag{27}$$

$\mathbf{n}$ being the normal vector to the surface, i.e., $\partial_{\mathbf{n}}\varphi^{\mathrm{N}}\sim\partial_x\varphi^{\mathrm{N}}$. The field eigenvalues satisfying this restriction are

$$\varphi(x,\mathbf{x}_T,t)\sim\cos(\pi nx/L)e^{i\mathbf{k}_T\cdot\mathbf{x}_T}e^{-i\omega_{kn}^{\mathrm{N}}(L)t},$$
$$\omega_{kn}^{\mathrm{N}}(L)=[(\pi n/L)^2+\mathbf{k}_T^2]^{1/2}, \quad n=0,1,2,3,\ldots. \tag{28}$$

Hence, the sum of all the eigenvalues will be

$$\begin{aligned}\mathscr{E}_0(L)&=\frac{1}{2}\left(\frac{1}{L}\sum_{n=1}^{\infty}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}\omega_{kn}^{\mathrm{D}}(L)+\frac{1}{L}\sum_{n=0}^{\infty}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\omega_{kn}^{\mathrm{N}}(L)\right)\\&=\frac{1}{2L}\left(\sum_{n=1}^{\infty}+\sum_{n=0}^{\infty}\right)\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}\left[\left(\frac{\pi n}{L}\right)^2+\mathbf{k}_T^2\right]^{1/2}\\&=\frac{1}{2L}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}(\mathbf{k}_T^2)^{1/2}+\frac{1}{L}\sum_{n=1}^{\infty}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}\left[\left(\frac{\pi n}{L}\right)^2+\mathbf{k}_T^2\right]^{1/2}.\end{aligned} \tag{29}$$

The zeta-function regularization method starts by replacing the power $\frac{1}{2}$ in the integrand with $-s/2$, where s is a complex variable, Thus,

$$\begin{aligned}\mathscr{E}_0(s;L)&=\frac{1}{2L}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}(\mathbf{k}_T^2)^{-s/2}\\&\quad+\frac{1}{L}\sum_{n=1}^{\infty}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}\left[\left(\frac{\pi n}{L}\right)^2+\mathbf{k}_T^2\right]^{-s/2}\\&\equiv\mathscr{E}_0^{(1)}(s;L)+\mathscr{E}_0^{(2)}(s;L).\end{aligned} \tag{30}$$

From now on, we will focus on the second integral. After assuming Re s to be large enough for the integral to converge, we obtain

$$\mathscr{E}_0^{(2)}(s;L)=\frac{\pi^{1-s}}{4L^{3-s}}\frac{\Gamma[(s-2)/2]}{\Gamma(s/2)}\sum_{n=1}^{\infty}(n^2)^{(2-s)/2}. \tag{31}$$

This result may be analytically continued to other values of s by noting that, if initially Re s is sufficiently large, the sum on the right-hand side is the power series defining the *Riemann zeta function*

$$\zeta(z)=\sum_{n=1}^{\infty}n^{-z}, \quad \mathrm{Re}\,z>1. \tag{32}$$

We will sum the series, get a zeta function, and continue back in its argument by using the analytic extension of ζ itself to arguments such that Re $z<1$. Then,

$$\begin{aligned}\mathscr{E}_0^{(2)}(L)&\equiv\lim_{s\to-1}\mathscr{E}_0^{(2)}(s;L)\\&=-(\pi^{3/2}/8L^4)\Gamma(-\tfrac{3}{2})\zeta(-3).\end{aligned} \tag{33}$$

The value of $\zeta(z)$ at $z=-3$ cannot be found by the series (32), but it may be obtained by analytic continuation using a variety of techniques. One that happens to be particularly easy to apply is the legitimate exploitation of the Riemann zeta-function reflection formula in terms of Euler's gamma function, which reads

$$\Gamma(z/2)\zeta(z)=\pi^{z-1/2}\Gamma[(1-z)/2]\zeta(1-z). \tag{34}$$

This is a way of performing analytic continuation in the sense that it allows us to shift from one region of the complex plane to another and in such a manner that we go from the domain where (32) is valid to a region where it is not. As a result, we obtain finite and fully computable values, i.e.,

$$\begin{aligned}\mathscr{E}_0^{(2)}(L)&=-(1/8\pi^2L^4)\zeta(4)\\&=-\pi^2/720L^4.\end{aligned} \tag{35}$$

Next, we have to evaluate again the density (energy per unit *volume*) $\mathscr{E}_0$ for the configuration where the perfectly conducting plates are at a distance S instead of L, and take the limit for $S\to\infty$ of the difference (23) afterward. This limit is the realization of the unconstrained vacuum which is to be subtracted. Since $\mathscr{E}_0^{(1)}(s;L)$ was independent of L, we will get the same first term when L be replaced with S, and their contribution will cancel each other. Formally, this will be so even before replacing $\frac{1}{2}$ with $-s/2$. The cancellation of the part independent of the separation is therefore independent of the type of device used as second regulator. As for $\mathscr{E}_0^{(2)}(s;L)$, one immediately realizes that, because of its form, $\sim 1/S^4$ will vanish for $S\to\infty$. So one is left with

$$\mathscr{E}(L)=-\pi^2/720L^4, \tag{36}$$

which is the density of Casimir energy, i.e., the Casimir energy per unit volume, for this configuration. Although the dependence on $1/L^4$ was to be expected on dimensional grounds, the coefficient and the sign are, as a rule, difficult to obtain. The pressure $\mathscr{P}$ (force per unit surface) arising from this energy density will be obtained as follows. Let $E(L)$ be the energy contained in the volume in question and $F(L)$ the Casimir force, which results from $F(L)=(-\partial/\partial L)E(L)$. Since $E(L)=$ plate surface $\times L\times\mathscr{E}(L)$, the pressure $\mathscr{P}(L)\equiv F(L)/($plate surface$)$ reads now $\mathscr{P}(L)=(-\partial/\partial L)[L\mathscr{E}(L)]$ and amounts to

$$\mathscr{P}(L)=-\pi^2/240L^4, \tag{37}$$

i.e., the negative sign of this pressure marks out the force coming from the Casimir energy as attractive. This is the result that was originally derived in Casimir's paper by application of a Euler–MacLaurin sum rule, a technique whose underlying mathematical basis is linked to the zeta-function method itself through the Bernoulli numbers.

G. Conclusions and prospects for further experiments

The Casimir energy results from the modification of the modes of quantized fields in connection with the introduction of boundaries. It may also be said that the modes can be modified by the presence of *polarizable* particles as well. These particles alter the eigenfrequencies of the e.m. field, leading to the appearance of van der Waals forces, which, for positive polarizabilities, are always attractive. The van

der Waals forces admit two other interpretations: two-photon exchange and zero-point fluctuation as explained in Sec. III C. The photon exchange would be described by Feynman diagrams containing a closed photon loop, standing for the vacuum fluctuations in the e.m. field, perturbed by two insertions representing two particles. The integration would be over all the possible momenta of the virtual intermediate photons.

In view of the van der Waals forces between pairs of particles, a naive scheme would picture the Casimir forces between surfaces to be mere superpositions of interactions among its constituents, as conducting surfaces may be taken to consist of collections of polarizable particles. Following that line of thought, the Casimir forces between conducting surfaces would always be attractive, but we have seen that they are often repulsive. This ingenuous scheme is unsuccessful because it fails to consider two important points. One of them is the modification of the zero-point spectrum. The presence of the boundaries modifies the field modes, alters the photon propagator, and consequently, the spectrum itself. On the other hand, for connected surfaces it is not possible to separate the energy of the boundary itself from the true Casimir energy of the field.

This apparent failure of the superposition of van der Waals forces should be experimentally tested. If the experiment by Sparnaay, which involved parallel plates, was by no means easy to carry out, it is hardly surprising that probing more complicated configurations involves prohibitive difficulties. Perhaps the most interesting thing to look for would be the observation of the sign change when going from a disconnected to a connected boundary, e.g., by measuring forces before and after joining two hemispherical conducting shells. However, these surfaces would be difficult to make with metallic materials, and their regularity—at the scale required by the forces to be manifest—would be extremely hard to control. As in other experiments, one would have to make as much as possible in order to prevent accidental differences of potential between the two parts and the presence of external obstacles—dust, vibrations, etc. Although Sparnaay's measurements were so complicated for such a simple system as a pair of plates, the technology of stress calibration must be today much more advanced than in 1958. Nevertheless, it seems that all these situations have not been the object of critical experimental attention yet.

Another possibility suggested in Ref. 6 would be the detection of the repulsive nature of Casimir forces in spherical cavities by analyzing the behavior of liquids capable of forming small bubbles, such as liquid ^{4}He.

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A toy version of the interacting boson model

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For pedagogical purposes, and also to provide a simple schema wherein ideas and methods may quickly be tried out, a toy version of the interacting boson model (IBM) is developed in two dimensions.

I. INTRODUCTION

The shell model point of view, that the nucleons in the atomic nucleus move independently of each other to a first approximation, and their mutual interactions build up the detailed nuclear level schemes, has been dramatically successful throughout the periodic table. Nuclei, however, also exhibit phenomena that require for their explanation an apparently antithetical position, namely, the invocation of collective modes involving the nucleus as a whole implying the coherent or tightly correlated motion of the nucleons. Thus the photonuclear giant electric-dipole resonance (that by itself almost exhausts the dipole sum rule) is visualized as the collective oscillations of the protons of the nucleus with respect to the neutrons. Again, the occurrence of rotational bands in the spectra of many nuclei, particularly among the lanthanides and actinides, are best understood in terms of the quantized rotations of a deformed nucleus spinning as a whole. A similar case is that of observed spectra describable as being vibrational in nature. The collectivity of these states manifests itself not only through the nature of the spectrum but also by the telltale enhancement of their (electric quadrupole) transitions within the bands. The relationship between the independent particle description and the collective picture is a subject of continuing interest.

About 16 years ago, Arima and Iachello[1] initiated a new approach, providing thereby a tractable description of nuclei throughout the periodic table, but particularly in the regions of strongly developed collective motion. In its simplest form, the model consists essentially in pairing off nucleons (similar to the Cooper pairs in the theory of superconductivity) into effective bosons, limiting the boson spins to zero and two, in the first instance, and then permitting boson–boson interactions. This is the interacting boson model (IBM).[2] It is dramatically successful in describing complex nuclei with relatively few parameters. A key concept in this approach is that of a dynamical symmetry, a notion that has found application in almost all branches of physics. The Hamiltonian H, having a group structure G [which is here U(6) corresponding to the six states: a scalar ($J=0$) boson called s, and the five-component quadrupole ($J=2$) boson called d], can be written in terms of the Casimir invariants of a complete chain of groups, corresponding to successive subgroups starting from G, providing labels for the classification of the states and permitting analytic (algebraic) solutions to the eigenvalue problem. The objective of the present study is to present a toy version of the IBM in which the group structure of the Hamiltonian is U(3) and the group chain involves SU(2), O(3), and O(2) and thus requires familiarity only with angular momentum algebra. It may thus be of pedagogic value and may also provide a simple schema wherein ideas and methods could quickly be worked out and checked.

II. THE MODEL

Consider a two-dimensional nucleus wherein the nucleons are imagined to be paired off into bosons. Let us retain only bosons with angular momentum $m=0,\pm 2$ to which correspond the annihilation (creation) operators $a_0(a_0^\dagger)$ and $a_\pm$ $(a_\pm^\dagger)$, satisfying the commutation relations

$$[a_i,a_j^\dagger]=\delta_{ij}, \tag{1a}$$

$$[a_i,a_j]=0=[a_i^\dagger,a_j^\dagger]. \tag{1b}$$

Introduce the set of nine bilinears

$$\Lambda_{ij}\equiv a_i^\dagger a_j, \tag{2a}$$

which by virtue of the basic commutation relations [of Eq. (1)] satisfy the Lie algebra:

$$[\Lambda_{ij},\Lambda_{kl}]=\delta_{jk}\Lambda_{il}-\delta_{il}\Lambda_{jk}, \tag{2b}$$

whereby seeing that $\Sigma\Lambda_{ii}=a_0^\dagger a_0+a_+^\dagger a_+ +a_-^\dagger a_-$ is fixed at N (the total number of bosons), it is clear that the underlying group structure of the model is SU(3). The most general Hamiltonian for a system of N such bosons is