The origin of the phase in the interference of Bose-Einstein condensates

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We consider the interference of two overlapping ideal Bose-Einstein condensates. The usual description of this phenomenon involves the introduction of a condensate wave function with a definite phase. We investigate the origin of this phase and the theoretical basis of treating interference. It is possible to construct a phase state for which the particle number is uncertain, but the phase is known. How such a state would be prepared before an experiment is not obvious. We show that a phase can also arise from experiments using condensates with known particle numbers. The analysis of measurements in such states also gives us a prescription for preparing phase states. The connection of this procedure to questions of spontaneously broken gauge symmetry and to hidden variables is discussed.
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I. INTRODUCTION

One of the most impressive experiments using trapped Bose gases is the interference experiment of Ketterle and co-workers.[1] Two condensates are separately prepared and allowed to overlap. An interference pattern arises showing the remarkable quantum coherence of the condensates. There have been other interesting condensate interference experiments as well.[2–5] If we assume that the separated clouds initially have a definite phase relation, then the experiments are well described by straightforward theory.[6] However, questions immediately arise. Do the separately prepared condensates have a phase relation?[7] The preparation of the sample certainly did not involve the establishment of a state with known phase. More likely the particle number in each condensate would have, or could have, been initially known ahead of time. Nevertheless, an interference pattern with a well established phase emerges when the two condensates are allowed to overlap. So how does this phase arise? The question was answered in several papers that showed how a phase appears even when the two clouds are prepared in Fock states, that is, states with well-defined known particle numbers.[8–15] In this paper we revisit the question and give a derivation of this result. This result is satisfying, because it justifies the usual simple assumption of interfering coherent systems having a well-defined, but unknown initial phase relationship.

To discuss the properties of condensates and superfluids we usually introduce an order parameter or condensate wave function $\langle \hat{\psi}(\mathbf{r}) \rangle$. The resulting wave function has a magnitude equal to the square root of the condensate density and a phase. The state in which $\langle \hat{\psi}(\mathbf{r}) \rangle$ is nonzero cannot have a fixed particle number. With one of these wave functions for each condensate, it is straightforward to discuss interference of the two, because each one has its own phase, and an interference pattern arises with a relative phase equal to the difference between the individual phases. In essence we have described each condensate by a single particle wave function so that interference is no more than the overlap and interference of two classical waves.

However, how does this single particle wave function arise? Its existence involves spontaneously broken gauge symmetry,[16–19] the necessity of which has been brought into question in recent years.[18–21] Suppose we consider a condensate described by a wave function $e^{i(\mathbf{k}\cdot\mathbf{r}+\phi)}$. We might describe the direction specified by the angle ϕ by a "spin" in a two-dimensional plane. How do we prepare such a state? What is it that selects the direction of this pseudospin from all the degenerate possible directions? There is an analogy with ferromagnetism, where there is a symmetry in the possible degenerate directions of the magnetization. A small external field in a particular direction in space will select the direction of the magnetization. In a similar way the phase angle is selected. In the ferromagnetism case we can assume that in practice there is always a small field to select a preferred direction so that the symmetry is broken. However, the local field that is used theoretically to choose a phase direction for a Bose condensate does not exist in nature.

The treatment of phase (actually relative phase) in Sec. IV, and the spontaneous appearance of a relative phase under the effect of the measurement of particle position in Fock states avoids violating particle conservation and does not require use of any symmetry-breaking field and so helps in this regard. A closely related idea is that the phase emerging from successive measurements of particle position starting with a Fock state is similar to the emergence of a hidden or additional variable in quantum mechanics.[14,23,24] Was the phase there before the experiment started, or did the experiment itself cause it to take on its final value? Hidden variables can be invoked to specify noncommuting variables. In quantum mechanics, particle number and relative phase can be considered to be conjugate variables; the knowledge of one excludes that of the other. As we measure particle position our knowledge of the particle number becomes less certain while the uncertainty of the relative phase decreases.

In the following we first discuss the kind of state that has a known phase. With this state the interference pattern emerges with just the prepared phase. Among these states are the coherent states of Glauber.[25] These can either have particle number completely unknown or have the total number of particles in the two condensates known (in which case they are called phase states), although the number in each condensate is still unknown. We then derive the interference pattern starting with Fock states and see the emergence of a relative phase even though no phase was present at the beginning of the experiment (or was at least hidden). We even find a way to prepare a state that has a known relative phase. The controversial theoretical constructs are seen to be unnecessary.

II. SIMPLE VIEW OF AN INTERFERENCE PATTERN

A gas (or liquid) undergoing Bose-Einstein condensation (BEC) is often described by a classical field known as an order parameter or condensate wave function. Such a quantity can arise in several ways. Suppose that $\hat{\psi}(\mathbf{r})$ represents a second-quantized operator that destroys a boson at position $\mathbf{r}$. The one-particle density matrix is defined as $\rho(\mathbf{r},\mathbf{r}') = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$. Penrose and Onsager[26] showed that a criterion for a Bose condensate or off-diagonal long-range order is that the density matrix has the form

$$\rho(\mathbf{r},\mathbf{r}') = \psi^*(\mathbf{r})\psi(\mathbf{r}') + f(\mathbf{r},\mathbf{r}'), \tag{1}$$

where $f(\mathbf{r},\mathbf{r}')$ vanishes when $|\mathbf{r}-\mathbf{r}'| \to \infty$. The function $\psi(\mathbf{r})$ is the condensate wave function. It is often assumed that the system is in a state such that the destruction operator has a nonzero average:

$$\langle \hat{\psi}(\mathbf{r})\rangle = \psi(\mathbf{r}) = \sqrt{n_0(\mathbf{r})}e^{i\phi(\mathbf{r})}, \tag{2}$$

where n_0 is the condensate density and ϕ its phase. Such a state is said to have spontaneously broken gauge symmetry because a particular phase (out of many possible degenerate phase states) has been chosen.[16–21]

To describe the interference pattern in the experiment of Ref. 1 we must consider the overlap of Bose clouds released from harmonic oscillator traps.[6] This overlap leads to some interesting features such as fringes whose separation changes with time. In our analysis here we will consider only plane waves and ignore any changes in time. Suppose we have an order parameter that involves two condensate wave functions, with condensate densities n_a and n_b in momentum states $\mathbf{k}_a$ and $\mathbf{k}_b$. This dual order parameter has the form

$$\psi(\mathbf{r}) = \sqrt{n_a}e^{i\mathbf{k}_a\mathbf{r}}e^{i\phi_a} + \sqrt{n_b}e^{i\mathbf{k}_b\mathbf{r}}e^{i\phi_b}. \tag{3}$$

The density of the combined system is then

$$\begin{aligned} n(\mathbf{r}) &= |\sqrt{n_a}e^{i\mathbf{k}_a\mathbf{r}}e^{i\phi_a} + \sqrt{n_b}e^{i\mathbf{k}_b\mathbf{r}}e^{i\phi_b}|^2 \\ &= n[1 + x\cos(\mathbf{k}\cdot\mathbf{r} + \phi)], \end{aligned} \tag{4}$$

where $n=n_a+n_b$, $\mathbf{k}=\mathbf{k}_a-\mathbf{k}_b$, $\phi=\phi_b-\phi_a$, and $x=2\sqrt{n_a n_b}/n$. We have an interference pattern with *relative* phase $\mathbf{k}\cdot\mathbf{r}+\phi$. The phase shift ϕ is measurable, although the individual phases ϕ_a and ϕ_b are not.

This analysis is simple, but it requires the preparation of the system in a state with known individual phases. How can we do that? What is the nature of such a state? The expectation value of Eq. (2) cannot be in a state of definite particle number or the expectation value would vanish. We next investigate this question more deeply.

III. PHASE STATES

As noted by Johnston[21] the coherent states introduced by Glauber[25] for photons are appropriate for superfluids.[22,27] They are also called "classical states" and are the minimum uncertainty states of the harmonic oscillator.[22] Here we will not use them in full generality, but rather use a subset of them known as phase states. (A full treatment of coherent states in a treatment of a condensate wave function is given in Appendix A.) Phase states describe two condensates (in states $\mathbf{k}_a$ and $\mathbf{k}_b$) with variable particle numbers, N_a and N_b (both macroscopic), but fixed total number $N=N_a+N_b$. No other momentum states are occupied. If particle creation operators $a^\dagger$ and $b^\dagger$ (obeying Bose commutation relations) for the two states act on the vacuum to put particles into these two states, then we define the (properly normalized) state as

$$\begin{aligned} |\alpha_a\alpha_b;N\rangle &= \frac{1}{\sqrt{g^N}}\sum_{N_a=0}^{N}\sqrt{\frac{N!}{N_a!\,(N-N_a)!}} \\ &\quad\times \alpha_a^{N_a}\alpha_b^{N-N_a}|N_a,N-N_a\rangle \end{aligned} \tag{5a}$$

$$= \frac{1}{\sqrt{g^N}}\sum_{N_a}\frac{\sqrt{N!}}{N_a!\,(N-N_a)!}(\alpha_a a^\dagger)^{N_a}(\alpha_b b^\dagger)^{N-N_a}|0\rangle \tag{5b}$$

$$= \sqrt{\frac{1}{g^N N!}}(\alpha_a a^\dagger + \alpha_b b^\dagger)^N|0\rangle, \tag{5c}$$

where the quantities α_i are complex. We separate them into magnitudes γ_i and phases ϕ_i according to

$$\alpha_i = \gamma_i e^{i\phi_i}. \tag{6}$$

Also $g=\gamma_a^2+\gamma_b^2$.

We can easily calculate the average number of particles $\bar{N}_a$ in this state. We use Eq. (5a) to give

$$\begin{aligned} a|\alpha_a\alpha_b;N\rangle &= \frac{1}{\sqrt{g^N}}\sum_{N_a}\sqrt{\frac{N!}{N_a!\,(N-N_a)!}} \\ &\quad\times \alpha_a^{N_a}\alpha_b^{N-N_a}\sqrt{N_a}|N_a-1,N-N_a\rangle \end{aligned} \tag{7a}$$

$$\begin{aligned} &= \alpha_a\frac{\sqrt{N}}{\sqrt{g^N}}\sum_{N_a'}\sqrt{\frac{(N-1)!}{N_a'!\,(N-1-N_a')!}} \\ &\quad\times \alpha_a^{N_a'}\alpha_b^{N-1-N_a'}|N_a',N-N_a\rangle \end{aligned} \tag{7b}$$

$$= \alpha_a\sqrt{\frac{N}{g}}|\alpha_a\alpha_b;N-1\rangle, \tag{7c}$$

where $N_a'=N_a-1$. Thus

$$\bar{N}_a = \langle\alpha_a\alpha_b;N|a^\dagger a|\alpha_a\alpha_b;N\rangle = \gamma_a^2\frac{N}{\gamma_a^2+\gamma_b^2}. \tag{8}$$

Similarly we find $\bar{N}_b=\gamma_b^2 N/(\gamma_a^2+\gamma_b^2)$, so that $\gamma_i=|\alpha_i|=\sqrt{\bar{N}_i}$ and $g=N_a+N_b=N$.

The fact that N is known in our phase state does not affect the results for the interference patterns which depend just on the relative phase. Such states have been often used to discuss the interference of two condensates.[11,12,15,28] Our state can be used to discuss how the relative phase can be conjugate to particle number. We write it in a form that makes the phases explicit:

$$|\alpha_a\alpha_b;N\rangle = \sqrt{\frac{N!}{g^N}}\sum_{(N_a+N_b=N)}\frac{\gamma_a^{N_a}\gamma_b^{N_b}e^{iN_a\phi_a}e^{iN_b\phi_b}}{\sqrt{N_a!\,N_b!}}|N_a,N_b\rangle. \tag{9}$$

Now express the phases in terms of the relative phase $\phi = \phi_b - \phi_a$ and the total phase $\Phi = (1/2)(\phi_b + \phi_a)$ and take the derivative with respect to ϕ:

$$\begin{aligned}-2i\frac{\partial}{\partial\phi}|\alpha_a\alpha_b;N\rangle &= \sqrt{\frac{N!}{g^N}}\sum_{(N_a+N_b=N)}(N_a-N_b)\\ &\quad\times\frac{\gamma_a^{N_a}\gamma_b^{N_b}e^{iN_a(\Phi+\phi/2)}e^{iN_b(\Phi-\phi/2)}}{\sqrt{N_a!\,N_b!}}\\ &\quad\times|N_a,N_b\rangle = (a^\dagger a - b^\dagger b)|\alpha_a\alpha_b;N\rangle.\end{aligned} \tag{10}$$

The phase derivative operator gives the same result as the number difference operator so that ϕ and $(N_a - N_b)$ are conjugate variables.[19,28] Note that the total phase appears only as an external factor $\exp[iN\Phi]$ and so has no physical significance. Thus the individual phases have no physical significance; only the relative phase ϕ is a meaningful quantity.

To emphasize this last point and put the phase state in a more compact form to treat interference, we rename and rewrite it as

$$\begin{aligned}|\phi,N\rangle &= \frac{1}{\sqrt{g^N N!}}(a^\dagger + \gamma e^{i\phi}b^\dagger)^N|0\rangle\\ &= \frac{1}{\sqrt{g^N}}\sum_{n=1}^{N}\sqrt{\frac{N!}{n!\,(N-n)!}}(\gamma e^{i\phi})^{N-n}|n,N-n\rangle,\end{aligned} \tag{11}$$

where now $\gamma = \sqrt{\bar{N}_a/\bar{N}_b}$ and ϕ is the relative phase. Also now $g = (1+\gamma^2)$. We have dropped a meaningless factor of unit magnitude.

Because we have just two occupied states, the terms in the Fourier transform of $\hat{\psi}(\mathbf{r})$ [see, for example, Eq. (A7)] not referring to states k_a and k_b never contribute, and we can more simply write

$$\hat{\psi}(\mathbf{r}) \to c_{\mathbf{r}} \equiv \sqrt{\frac{1}{V}}(ae^{i\mathbf{k}_a\cdot\mathbf{r}} + be^{i\mathbf{k}_b\cdot\mathbf{r}}). \tag{12}$$

We can also make Eq. (12) more compact by writing

$$c_{\mathbf{r}} = \sqrt{\frac{1}{V}}(a + be^{i\mathbf{k}\cdot\mathbf{r}}), \tag{13}$$

with $\mathbf{k} = \mathbf{k}_b - \mathbf{k}_a$. The quantity $e^{i\mathbf{k}_a\cdot\mathbf{r}}$ can again be dropped as a meaningless factor of unit magnitude.

We want to consider how $c_{\mathbf{r}}$ acts on the phase state. We have

$$\begin{aligned}c_{\mathbf{r}}|\phi,N\rangle &= \frac{1}{\sqrt{g^N V}}\sum_{n=1}^{N}\sqrt{\frac{N!}{n!\,(N-n)!}}(\gamma e^{i\phi})^{N-n}\\ &\quad\times(\sqrt{n}|n-1,N-n\rangle + e^{i\mathbf{k}\cdot\mathbf{r}}\sqrt{N-n}|n,N-n-1\rangle).\end{aligned} \tag{14}$$

If we change variables in the first state to $n' = n-1$, we can express both terms in the same form and obtain

$$c_{\mathbf{r}}|\phi,N\rangle = A(\mathbf{r},\phi)|\phi,N-1\rangle, \tag{15}$$

where

$$A(\mathbf{r},\phi) = \sqrt{\frac{N}{gV}}(1 + \gamma e^{i\phi}e^{i\mathbf{k}\cdot\mathbf{r}}). \tag{16}$$

The average density follows immediately as

$$\begin{aligned}\langle\phi,N|\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})|\phi,N\rangle &= \langle\phi,N|c_{\mathbf{r}}^\dagger c_{\mathbf{r}}|\phi,N\rangle = |A(\mathbf{r},\phi)|^2\\ &= \bar{n}(1 + \bar{x}\cos(\mathbf{k}\cdot\mathbf{r} + \phi)),\end{aligned} \tag{17}$$

just like Eq. (4), where now $\bar{n}$ and $\bar{x}$ have definitions in terms of averages. Thus a phase state provides a rigorous context for a discussion of condensate wave functions and for the simplified form of treating interference between the two condensates. How we might actually prepare one *before* an experiment is a separate difficult question, which we treat in the following.

We will find it useful and necessary in Sec. IV to consider more general cases in which we make measurements of many particle positions essentially simultaneously. This measurement process allows interference fringes to emerge where they would otherwise not occur. For our phase state such calculations are straightforward and add no additional information because the multiparticle densities all factor in the phase states. For example, consider the expectation value of

$$\hat{\psi}^\dagger(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_2)\hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_1) = c_{\mathbf{r}_2}^\dagger c_{\mathbf{r}_2}c_{\mathbf{r}_1}^\dagger c_{\mathbf{r}_1} \approx c_{\mathbf{r}_2}^\dagger c_{\mathbf{r}_1}^\dagger c_{\mathbf{r}_2}c_{\mathbf{r}_1}. \tag{18}$$

Although $c_{\mathbf{r}_2}$ and $c_{\mathbf{r}_1}^\dagger$ do not commute, we drop a term of order N compared to one of order N^2 in the approximation. The last form is more convenient to use. By the behavior of the phase state, we easily obtain

$$\begin{aligned}&\langle\phi,N|\hat{\psi}^\dagger(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_2)\hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_1)|\phi,N\rangle\\ &\quad\approx\langle\phi,N|c_{\mathbf{r}_2}^\dagger c_{\mathbf{r}_1}^\dagger c_{\mathbf{r}_2}c_{\mathbf{r}_1}|\phi,N\rangle = |A(\mathbf{r}_2)|^2|A(\mathbf{r}_1)|^2\\ &\quad= \prod_{i=1}^{2}\bar{n}(1 + \bar{x}\cos(\mathbf{k}\cdot\mathbf{r}_i + \phi)).\end{aligned} \tag{19}$$

This result generalizes to

$$\begin{aligned}&\langle\phi,N|\hat{\psi}^\dagger(\mathbf{r}_m)\hat{\psi}(\mathbf{r}_m)\cdots\hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_1)|\phi,N\rangle\\ &\quad\approx\langle\phi,N|c_{\mathbf{r}_m}^\dagger\cdots c_{\mathbf{r}_1}^\dagger c_{\mathbf{r}_m}\cdots c_{\mathbf{r}_1}|\phi,N\rangle\\ &\quad= \prod_{i=1}^{m}\bar{n}(1 + \bar{x}\cos(\mathbf{k}\cdot\mathbf{r}_i + \phi)).\end{aligned} \tag{20}$$

A state in the form $c_{\mathbf{r}_m}\cdots c_{\mathbf{r}_1}|\Psi\rangle$ is useful for interpreting experiments. We can consider our experiment as detecting particle 1 and then 2 shortly thereafter, and so on. After m detections the wave function evolves to a state missing several particles. What is the nature of the state to which it has evolved? For a phase state it is $c_{\mathbf{r}_m}\cdots c_{\mathbf{r}_1}|\phi,N\rangle \sim |\phi,N-m\rangle$. However, it is more interesting to consider the case of $|\Psi\rangle$, a Fock state as we do next.

IV. INTERFERENCE IN FOCK STATES

It is not evident that experimentalists can prepare a phase state as we have described. It is more likely that they are working with Fock states, that is, states in which the particles numbers N_a and N_b in the two condensates are known rather well. In any case it is likely that there is a greater probability of initially preparing such a state. However, as several workers[8–15] have realized in recent years, and as we will show, an interference pattern with some phase can still arise in a Fock state. If we denote the state sharp in particle number as $|N_a,N_b\rangle$ and use the Bose annihilation relation

$$c_{\mathbf{r}}|N_a,N_b\rangle=\sqrt{\frac{1}{V}}(\sqrt{N_a}|N_a-1,N_b\rangle+e^{i\mathbf{k}\cdot\mathbf{r}}\sqrt{N_b}|N_a,N_b-1\rangle, \tag{21}$$

the one-body density in a Fock state is

$$D_1=\langle N_a,N_b|c_{\mathbf{r}}^{\dagger}c_{\mathbf{r}}|N_a,N_b\rangle=n, \tag{22}$$

and there is no interference. Phase and particle number are conjugate, and the particle number is initially known.

However, if we consider measuring the position of two particles simultaneously, some correlation should arise. We have

$$\begin{aligned}c_{\mathbf{r}_2}c_{\mathbf{r}_1}|N_a,N_b\rangle=\frac{1}{V}[&\sqrt{N_a(N_a-1)}|N_a-2,N_b\rangle\\&+e^{i\mathbf{k}\cdot(\mathbf{r}_1+\mathbf{r}_2)}\sqrt{N_b(N_b-1)}|N_a,N_b-2\rangle\\&+\sqrt{N_aN_b}(e^{i(\mathbf{k}\cdot\mathbf{r}_1)}+e^{i(\mathbf{k}\cdot\mathbf{r}_2)})|(N_a-1)\\&\times(N_b-1)\rangle],\end{aligned} \tag{23}$$

so the particle number is now slightly less certain. The two-body Fock correlation function is

$$D_2=\langle N_a,N_b|c_{\mathbf{r}_1}^{\dagger}c_{\mathbf{r}_2}^{\dagger}c_{\mathbf{r}_2}c_{\mathbf{r}_1}|N_a,N_b\rangle \tag{24a}$$

$$=\frac{1}{V^2}[N_a(N_a-1)+N_b(N_b-1)+N_aN_b|e^{i(\mathbf{k}\cdot\mathbf{r}_1)}+e^{i(\mathbf{k}\cdot\mathbf{r}_2)}|^2] \tag{24b}$$

$$=n^2[1+x\cos\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)]. \tag{24c}$$

For two particles there is indeed a position correlation. As we make more and more measurements the state becomes more and more mixed among states with various numbers of particles.

We can rewrite Eq. (24c) in a somewhat different and useful way. A simple integration shows that

$$D_2=n^2\int_0^{2\pi}\frac{d\phi}{2\pi}\prod_{i=1}^{2}[1+x\cos(\mathbf{k}\cdot\mathbf{r}_i+\phi)]. \tag{25}$$

If we compare Eqs. (25) and (19), we see that the results are similar except that we now integrate over all relative phases. Remarkably Eq. (25) can be extended to higher order correlation functions. In Ref. 14 it is shown that

$$D_m=n^m\int_0^{2\pi}\frac{d\phi}{2\pi}\prod_{i=1}^{m}[1+x\cos(\mathbf{k}\cdot\mathbf{r}_i+\phi)], \tag{26}$$

where it is assumed that $m\ll N$.

To derive Eq. (26), we invert Eq. (11), multiply both sides by $e^{-i\phi(N-n)}=e^{-i\phi N_b}$, and integrate over ϕ to give

$$|N_a,N_b\rangle=\frac{g^{N/2}}{\gamma^{N_b}}\sqrt{\frac{N_a!\,N_b!}{N!}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{-i\phi N_b}\,|\phi,N\rangle. \tag{27}$$

Thus, by Eq. (15) we find

$$c_{\mathbf{r}_m}\cdots\ c_{\mathbf{r}_1}|N_a,N_b\rangle\sim\int_0^{2\pi}\frac{d\phi}{2\pi}e^{-i\phi N_b}\prod_{i=1}^{m}A(\mathbf{r}_i,\phi)\ |\phi,N-m\rangle. \tag{28}$$

We will return to analyze this interesting state later. First consider the m-body Fock correlation function:

$$D_m=\langle N_a,N_b|c_{\mathbf{r}_m}^{\dagger}\cdots c_{\mathbf{r}_1}^{\dagger}c_{\mathbf{r}_m}\cdots c_{\mathbf{r}_1}|N_a,N_b\rangle \tag{29a}$$

$$\sim\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{-i(\phi-\phi')N_b}\prod_{i=1}^{m}A^*(\mathbf{r}_i,\phi')\times A(\mathbf{r}_i,\phi)\langle\phi',N-m||\phi,N-m\rangle. \tag{29b}$$

Phase states are not actually orthogonal, but for large N they are essentially as we show in Appendix A. So if $m\ll N$, we can write $\langle\phi',N-m||\phi,N-m\rangle\sim\delta(\phi-\phi')$, and

$$D_m\sim\int_0^{2\pi}\frac{d\phi}{2\pi}\prod_{i=1}^{m}|A(\mathbf{r}_i,\phi)|^2, \tag{30}$$

just as we claimed in Eq. (26). We will show how to obtain this result by direct calculation in Appendix B as done in Ref. 14.

Equation (26) has the form of Eq. (20) but with an integration over the unknown phase. This result makes sense in that our initial Fock state did not have a phase defined and can be expressed as a sum over phase states as in Eq. (27).

Suppose we have started with a Fock state and have made $m-1$ particle measurements and have found particles at positions $\mathbf{R}_1,\ldots,\mathbf{R}_{m-1}$. Then the probability of finding the mth particle at position $\mathbf{r}_m$ is

$$P_m=n^m\int_0^{2\pi}\frac{d\phi}{2\pi}g_m(\phi)[1+x\cos(\mathbf{k}\cdot\mathbf{r}_m+\phi)], \tag{31}$$

where

$$g_m(\phi)=\prod_{i=1}^{m-1}[1+x\cos(\mathbf{k}\cdot\mathbf{R}_i+\phi)]. \tag{32}$$

As we will show by simulation, $g(\phi)$ develops a sharp peak at some *a priori* unpredictable phase. If we make measurements from the first to the mth particle by this prescription, the peak becomes narrower as we proceed. As more measurements are made, the particle number in each condensate becomes less certain [as, for example, in Eq. (23)], so the phase can be more sharply defined.

Now look back at Eq. (28). After a fair number m of measurements, the real part of $\Pi_{i=1}^{m}A(\mathbf{r}_i,\phi)$ peaks sharply at some ϕ value, which we denote by ϕ_0. This peak means that the measurements have converted the Fock wave function into a narrow sum of phase states around $|\phi_0,N-m\rangle$. The more measurements that are made, the better the definition of the phase state. Measurements in a Fock state provide a way to prepare a phase state. We can understand the MIT experiments[1] in this way. The starting state was prepared as two separate condensates, whose particle numbers could have been known. Many subsequent particle measurements sharpened the phase to some random value and the final overall observation showed that phase.

V. NUMERICAL SIMULATION

We chose the initial position $\mathbf{r}_1$ randomly and then the next particle is chosen from the probability distribution P_2 given by Eq. (31) and so on. We will find that if m is large enough, g_m in Eq. (32) peaks at some *a priori* unpredictable phase angle ϕ_0 which may fluctuate as m changes, but gradually settles down. Starting a new experiment from the Fock state will lead to a randomly different phase angle. We will consider only the case in which the initial Fock state has $N_a=N_b=N/2$, that is, $x=1$. It is convenient to Fourier expand g_m. We write

$$g_m(\phi) = a_0 + \sum_{q=1}^{\infty} [a_q \cos q\phi + b_q \sin q\phi]. \tag{33}$$

In the integration of Eq. (31) only a_0, a_1, and b_1 will contribute. The integrals gives

$$P_m(\mathbf{r}_m) \sim 1 + \frac{a_1}{2a_0}\cos(\mathbf{k}\cdot\mathbf{r}_m) - \frac{b_1}{2a_0}\sin(\mathbf{k}\cdot\mathbf{r}_m). \tag{34}$$

If we define $\cos(\phi_m) \equiv a_1/\sqrt{a_1^2+b_1^2}$, $\sin(\phi_m) \equiv b_1/\sqrt{a_1^2+b_1^2}$, and $A_m \equiv \sqrt{a_1^2+b_1^2}/2a_0$, we can write

$$P_m = K[1 + A_m(\cos \mathbf{k}\cdot\mathbf{r}_m \cos \phi_m - \sin \mathbf{k}\cdot\mathbf{r}_m \sin \phi_m)] \tag{35a}$$

$$= K(1 + A_m \cos(\mathbf{k}\cdot\mathbf{r}_m + \phi_m)), \tag{35b}$$

where K is a normalization factor, and

$$\tan(\phi_m) = \frac{b_1}{a_1} \tag{36}$$

gives the value of the angle in the mth experiment. Because P_m is a probability, we must have $A_m < 1$, so that P_m is always positive. Because A_m has this property, we can write $A_m \equiv \sin(\alpha_m)$, where $0 < \alpha_m < \pi$ behaves like a polar angle. Then the emerging phase actually has a space angle designation (α_m, ϕ_m). We will find numerically that $A_m \to 1$ rapidly as we make measurements. In that case the probability of Eq. (35) looks just like the density prediction of Eq. (4). Moreover because $g_m(\phi)$ is a narrow function with a peak at ϕ_0, the phase defined by the Fourier coefficients is the same as that defined by the peak of g_m, as seen using Eq. (31).

We work in one dimension for simplicity. At iteration m we form a g_m according to Eq. (32), whose Fourier transform gives the parameters a_0, a_1, and b_1. From these we find ϕ_m and A_m. To simulate a corresponding particle position measurement x, we must choose from the probability of Eq. (34); to do so we form the cumulative probability $C_m(x) = \int_0^x dx' \, P_m(x')$ and then solve the equation $r=C_m(x)$ for x, where r is a random number uniformly distributed in the interval $[0,1]$. We take the box size $L=1$ and choose a k value such that kL is an integer times 2π to provide periodic boundary conditions. The normalization of the probability in Eq. (35) is just the factor $K=1/L=1$.

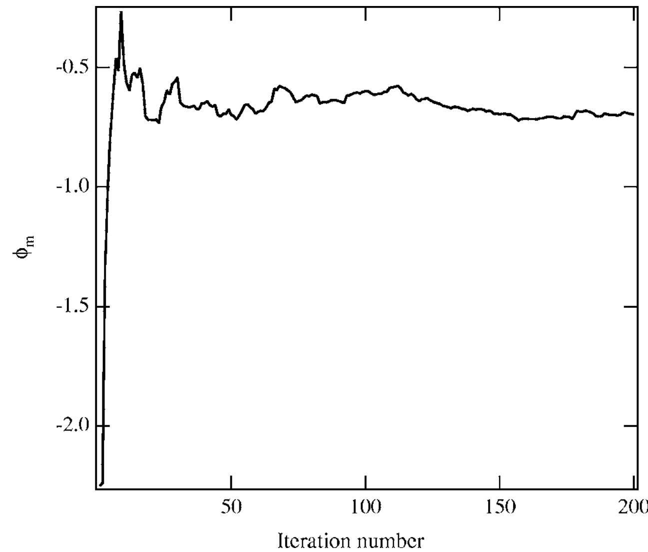

Fig. 1. The phase angle ϕ_m as a function of the number of iterations.

Figures 1 and 2 are plots of ϕ_m and A_m versus iteration number in a particular run of 200 iterations. There is no reason why A_m should be unity from the outset. However, A_m proceeds to unity after a small number of iterations. The result is that ϕ_m approaches a sharply defined random phase angle as predicted. Of course, for small m the fluctuations are relatively large and settle down only after many measurements, corresponding to an initially wide distribution, $g_m(\phi)$, which progressively narrows as more information is gathered. In Fig. 3 we show plots of the final angular distribution $g_{200}(\phi)$; it is sharply peaked at the same value found from the iteration limit. Figure 4 shows the final probability distribution, Eq. (35), versus the position x and also shows a histogram of the positions found in the 200 iterations. We see that these positions fall in the given distribution with the expected oscillations and with the same phase as found in the two other ways: from the asymptote of Fig. 1, and from the position of the peak in $g_m(\phi)$.

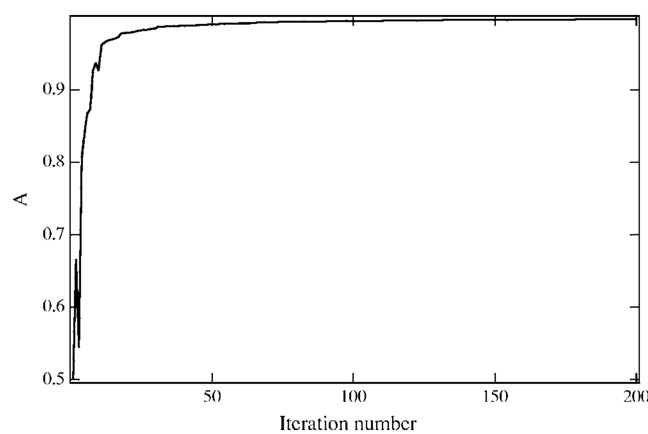

Fig. 2. The amplitude $A=\sin\alpha$ as a function of the number of iterations. The amplitude converges to unity.

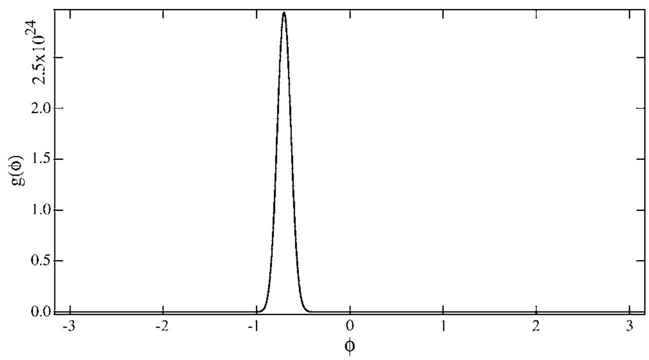

Fig. 3. The angular distribution $g(\phi)$ as a function of angle; $g(\phi)$ peaks at the same phase angle as given in Fig. 1.

VI. DISCUSSION

We have shown the interference of two Bose condensates can be treated rigorously. The usual assumption of two condensates with individually known phases becomes associated with questions about whether we can usefully define the phase of a single condensate. Using such states leads directly to the usual relations for interference patterns based on very simple assumptions. This procedure remains unsatisfying because it is not very obvious how to prepare such phase states before looking at the interference. Experimentally it seems to make no difference, because without special preparation, even with Fock states, we have seen how an interference pattern arises using Bose condensates.

The discussion of Sec. IV shows explicitly why such preparation was not necessary. Even if we start with a state where the particle number in each condensate is precisely known and many particles are involved in the measurement, we find a perfect interference pattern emerging, with a well-defined relative phase. Starting from a state with a definite number of particles, the experiment will end up with a state with a definite value of the relative phase. Thus this procedure provides a method for preparing the phase state discussed in Sec. III. Starting from a Fock state, make, say, 200 position measurements to reach a narrow $g_{200}(\phi)$; the phase of the wave function of the remaining state of $N-100$ total particles will be well defined. The final result is likely a phase state with a known *total* number of particles such as that discussed at Eq. (11), but an unknown number in each condensate.

In superfluid calculations it is simpler to treat the problem with definite phases than to use a Fock state. However, the actual existence of such a broken symmetry state is subject to question.[18–21] In a ferromagnet the presence of a small external field breaks the symmetry of the various directions of the magnetization. The existence of a field that would make $\langle\hat{\psi}\rangle$ nonzero is not as clear, because such states do not conserve particle number. The existence of a well-defined relative phase established by measuring particle positions can be established without being concerned about broken symmetry.

If the reader is uncomfortable with the idea of a phase *emerging* from a series of measurements on particle position, then the reader might assume, with no change in theoretical prediction, that the relative phase pre-existed within the two condensate clouds of particles. That is, the individual condensates had some relative phase (a hidden variable) before they overlapped, and the experiments bring out this previously hidden phase. In the next realization of the experiment, starting again from a Fock state, the phase will surely emerge with a randomly different value, in accordance with conventional quantum mechanics, which expresses the Fock state as a sum over all phase states as given in Eq. (27).

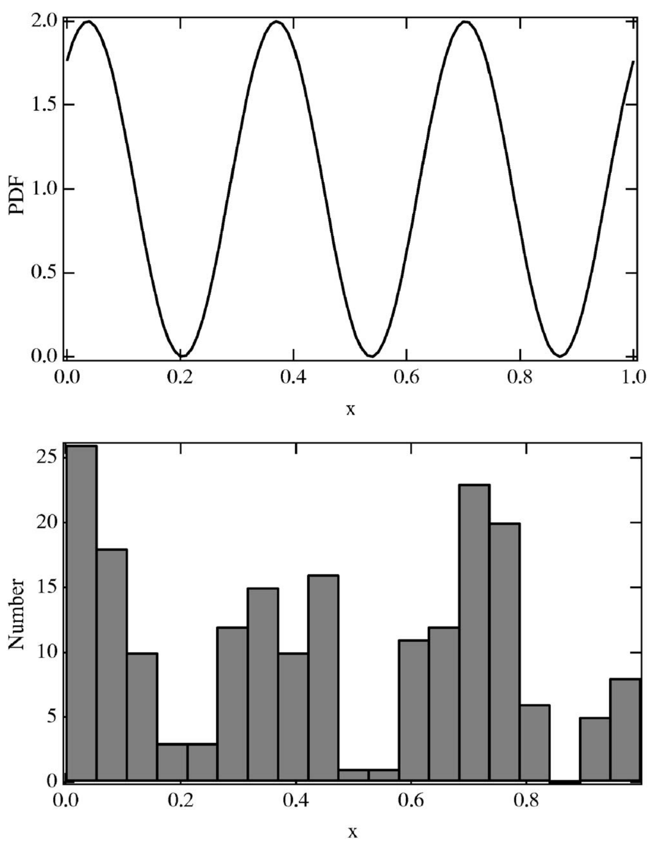

Fig. 4. The probability distribution function for the position in the interference pattern as calculated and the histogram as found in the simulated experiments. The phase here is the same as found by the asymptote of Fig. 1 or by the maximum of g_m in Fig. 3.

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APPENDIX A: COHERENT STATES

Consider the following normalized wave function which describes a single momentum state k, with a mixture of states of known particle number N_k,

$$|\alpha_k\rangle = e^{-(1/2)\gamma_k^2}\sum_{N_k} \frac{\alpha_k^{N_k}}{\sqrt{N_k!}} |N_k\rangle. \tag{A1}$$

The parameter α_k is complex, and we write it in terms of its magnitude γ_k and phase ϕ_k:

$$\alpha_k = \gamma_k e^{i\phi_k}. \tag{A2}$$

We can calculate the average number of particles $\bar{N}_k$ in this state. Let a_k be the destruction operator for particles in state $|N_k\rangle$. Then

$$\bar{N}_k = \langle \alpha_k | a_k^\dagger a_k | \alpha_k \rangle = e^{-\gamma_k^2} \sum_{N_k} \frac{\gamma_k^{2N_k}}{N_k!} N_k = \gamma_k^2, \tag{A3}$$

and $\gamma_k = |\alpha_k| = \sqrt{\bar{N}_k}$. The state $|\alpha_k\rangle$ has the property that it is an eigenstate of the lowering operator a_k:

$$a_k|\alpha_k\rangle = e^{-\gamma_k^2} \sum_{N_k} \frac{\alpha_k^{N_k}}{\sqrt{N_k!}} a_k |N_k\rangle = \alpha_k e^{-\gamma_k^2} \sum_{N_k} \frac{\alpha_k^{N_k-1}}{\sqrt{(N_k-1)!}} |N_k - 1\rangle$$
$$= \alpha_k | \alpha_k\rangle. \tag{A4}$$

Thus a_k has a nonzero expectation value in this state:

$$\langle \alpha_k | a_k | \alpha_k \rangle = \alpha_k = \sqrt{\bar{N}_k} e^{i\phi_k}. \tag{A5}$$

Clearly the states $|\alpha_k\rangle$ provide a definite phase and are not eigenstates of the number operator.

Next construct a multilevel many-body state with many possible k values. This state takes the form $|\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3}, \ldots\rangle = |\alpha_{k_1}\rangle |\alpha_{k_2}\rangle |\alpha_{k_3}\rangle \cdots$

$$|\{\alpha_k\}\rangle \equiv |\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3}, \ldots\rangle$$
$$= \sum_{\{N_k\}} \prod_k \left[e^{-(1/2)\gamma_k^2} \frac{\alpha_k^{N_k}}{\sqrt{N_k!}} \right] | N_{k_1}, N_{k_2}, N_{k_3}, \ldots \rangle, \tag{A6}$$

where $\{N_k\}$ means sum over all possible numbers of particles N_{k_i} in all the k states.

With such a state, we can consider the expectation value of the full field operator $\hat{\psi}(\mathbf{r})$. Expand the field operator in plane wave states as

$$\hat{\psi}(\mathbf{r}) = \sqrt{\frac{1}{V}} \sum_k e^{i\mathbf{k}\cdot\mathbf{r}} a_k, \tag{A7}$$

where V is the volume of the system. We have

$$\langle \{\alpha_k\} | \hat{\psi}(\mathbf{r}) | \{\alpha_k\} \rangle = \sum_k \sqrt{\frac{\bar{N}_k}{V}} e^{i\phi_k} e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{A8}$$

If one of the k states is macroscopically occupied, say, the momentum state $k = k_a$, we can write

$$\langle \hat{\psi}(\mathbf{r}) \rangle = \sqrt{\bar{n}_a} e^{i\mathbf{k}_a\cdot\mathbf{r}} e^{i\phi_a} + \xi, \tag{A9}$$

where $n_a = N_a/V$ and ξ is the total contribution of the noncondensed states. The leading term $\psi_a(\mathbf{r}) = \sqrt{\bar{n}_a} e^{i\mathbf{k}_a\cdot\mathbf{r}} e^{i\phi_a}$ represents a condensate wave function with a definite phase ϕ_a but indefinite number of particles.

Consider the case of the interference of a double condensate in momentum states k_a and k_b. For coherent states with only these two momentum states occupied we can write

$$|\alpha_a \alpha_b\rangle = e^{-(1/2)(\gamma_a^2 + \gamma_b^2)} \sum_{N_a, N_b} \frac{\alpha_a^{N_a} \alpha_b^{N_b}}{\sqrt{N_a! \, N_b!}} |N_a, N_b\rangle, \tag{A10}$$

where the averages $\bar{N}_a = \gamma_a^2$ and $\bar{N}_b = \gamma_b^2$ are macroscopic quantities. We manipulate the sums slightly in terms of particle creation operators $a^\dagger$ and $b^\dagger$ for the two states. If $N = N_a + N_b$, we have

$$|\alpha_a \alpha_b\rangle = e^{-(1/2)(\gamma_a^2 + \gamma_b^2)} \sum_{N, N_a} \frac{\alpha_a^{N_a} \alpha_b^{N-N_a}}{\sqrt{N_a! \, (N-N_a)!}} |N_a, N - N_a\rangle \tag{A11a}$$

$$= e^{-(1/2)(\gamma_a^2 + \gamma_b^2)} \sum_{N, N_a} \frac{1}{N_a! \, (N-N_a)!}$$
$$\times (\alpha_a a^\dagger)^{N_a} (\alpha_b b^\dagger)^{N-N_a} |0\rangle \tag{A11b}$$

$$= e^{-(1/2)(\gamma_a^2 + \gamma_b^2)} \sum_N \frac{1}{N!} (a a^\dagger + \alpha_b b^\dagger)^N |0\rangle \tag{A11c}$$

$$= e^{-(1/2)(\gamma_a^2 + \gamma_b^2)} e^{(\alpha_a a^\dagger + \alpha_b b^\dagger)} |0\rangle. \tag{A11d}$$

Equation (5b) is the phase state used in Sec. III, $|\alpha_a \alpha_b; N\rangle \sim (\alpha_a a^\dagger + \alpha_b b^\dagger)^N |0\rangle$. We see that it is a substate of the more general coherent state.

APPENDIX B: NEAR ORTHOGONALITY OF PHASE STATES

We calculate the inner product of two phase states to show that they are nearly orthogonal for large particle number. From Eq. (11) we find

$$\langle \phi', N | \phi, N \rangle = \frac{1}{g^N} \sum_{n=1}^{N} \frac{N!}{n! \, (N-n)!} [\gamma^2 e^{i(\phi - \phi')}]^{N-n}$$
$$= \frac{1}{g^N} [1 + \gamma^2 e^{i(\phi - \phi')}]^N. \tag{B1}$$

This is a very sharply peaked function of $(\phi - \phi')$ as can be seen by Taylor expanding the logarithm of Eq. (B1) in powers of $(\phi - \phi')$ and then exponentiating the result, keeping only terms to $(\phi - \phi')^2$. The result is

$$\langle \phi', N | \phi, N \rangle = \exp\left[-\frac{N}{2(1+\gamma^2)^2} (\phi' - \phi)^2 \right]$$
$$\times \exp\left[-i \frac{N}{(1+\gamma^2)} (\phi' - \phi) \right]. \tag{B2}$$

In the limit of very large N, $\langle \phi', N | \phi, N \rangle$ is proportional to a delta function of $(\phi - \phi')$ as we assumed in the discussion of Sec. IV.

APPENDIX C: ALTERNATIVE DERIVATION OF THE D_M EQUATION

We derive the general expression of Eq. (26) for the correlation function D_m. Consider this quantity in its original form for a Fock state:

$$D_m = \langle N_a, N_b | c_{\mathbf{r}_m}^\dagger \cdots c_{\mathbf{r}_1}^\dagger \cdots c_{\mathbf{r}_m} \cdots c_{\mathbf{r}_1} | N_a, N_b \rangle \tag{C1a}$$

$$= \frac{1}{V^m} \langle N_a, N_b | (a^\dagger + e^{-i\mathbf{k}\cdot\mathbf{r}_m} b^\dagger)(a^\dagger + e^{-i\mathbf{k}\cdot\mathbf{r}_1} b^\dagger) \cdots$$
$$\times (a + e^{i\mathbf{k}\cdot\mathbf{r}_m} b) \cdots (a + e^{i\mathbf{k}\cdot\mathbf{r}_1} b) | N_a, N_b \rangle. \tag{C1b}$$

Because this quantity is diagonal in Fock space, each time an a occurs, there must be a matching $a^\dagger$. The b operators are similar. We are assuming $m \ll N_a$ or N_b, so that we can always write $a|N_a - p, N_b - l\rangle \approx \sqrt{N_a} |N_a - p - 1, N_b - l\rangle$, etc. Thus each $a^\dagger a$ gives N_a, and each $b^\dagger b$ gives N_b. Consider a particular combination product:

$$(a^\dagger + e^{-i\mathbf{k}\cdot\mathbf{r}_j} b^\dagger)(a + e^{i\mathbf{k}\cdot\mathbf{r}_l} b) = a^\dagger a + b^\dagger b + a^\dagger b e^{i\mathbf{k}\cdot\mathbf{r}_l} + b^\dagger a e^{-i\mathbf{k}\cdot\mathbf{r}_j} \tag{C2a}$$

$$\to N_a + N_b + \sqrt{N_a N_b} e^{i\mathbf{k}\cdot\mathbf{r}_l} + \sqrt{N_a N_b} e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \tag{C2b}$$

with the restriction that every time an $e^{i\mathbf{k}\cdot\mathbf{r}_l}$-type term occurs, there must be a corresponding $e^{-i\mathbf{k}\cdot\mathbf{r}_j}$-type term somewhere in the overall product to give the proper balance of creation and destruction operators. Thus we obtain a series of terms of the form $F_{q_m}(\mathbf{r}_m)F_{q_{m-1}}(\mathbf{r}_{m-1})\cdots F_{q_2}(\mathbf{r}_2)F_{q_1}(\mathbf{r}_1)$, where

$$F_0(\mathbf{r}_i) = N_a + N_b, \tag{C3a}$$

$$F_{\pm1}(\mathbf{r}_i) = \sqrt{N_a N_b} e^{\pm i\mathbf{k}\cdot\mathbf{r}_l}, \tag{C3b}$$

and the sum of all the q_j vanishes. That is, we have

$$D_m = \frac{1}{V^m} \sum_{\{q\}} F_{q_m} \cdots F_{q_2} F_{q_1}, \tag{C4}$$

where $\{q\}$ means sum on all q_i with the restriction that $\Sigma_i q_i = 0$.

The restriction on the q values can be lifted if we substitute the integral

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi \Sigma q_i} = \delta_{\Sigma q_i, 0}, \tag{C5}$$

which allows us to write

$$D_m = \frac{1}{V^m} \int_0^{2\pi} \frac{d\phi}{2\pi} \prod_{i=1}^{m} [F_0(\mathbf{r}_i) + e^{i\phi} F_1(\mathbf{r}_i) + e^{-i\phi} F_{-1}(\mathbf{r}_i)] \tag{C6a}$$

$$= n^m \int_0^{2\pi} \frac{d\phi}{2\pi} \prod_{i=1}^{m} [1 + x \cos(\mathbf{k}\cdot\mathbf{r}_i + \phi)], \tag{C6b}$$

as we wished to prove.

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