

Feshbach resonance: A one dimensional example

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We present a simple, one-dimensional example of a total spin-1/2 atom that interacts with another static atom in the presence of an external magnetic field. The interaction consists of delta potentials that act differently with each of the two components of the wavefunction. The system has two coupled channels, admits a closed solution, and features the Feshbach resonance phenomenon by proper tuning of the magnetic field. © 2013 American Association of Physics Teachers. [http://dx.doi.org/10.1119/1.4804193]

I. INTRODUCTION

The purpose of this article is to present a toy model for the interaction of two atoms at very low energy in the presence of an external static magnetic field. The model aims to mimic the collisions of alkali atoms in ultracold dilute gases and to retain important features such as the Feshbach resonance mechanism and the association of universal Feshbach molecules from such atoms, the so called halo states.^{1,2}

Resonance effects are frequent in Physics and are generically associated with large effects observed under circumstances of coincidence of two or more parameters intervening in the problem, rather than by a particular large value of any one of them. In the case of the Feshbach resonance, the coincidence is given in an atom–atom collision where the atoms can undergo a virtual transition to a state where they are bound and when the incoming energy coincides with the bound state energy. Such a coincidence is rare in nature, but a properly adjusted magnetic field fosters it in an atom–atom collision. By means of the Feshbach mechanism, the atom–atom interaction can be made attractive or repulsive, very large or very small, thus modifying the properties of the gas.

Let us consider low-energy scattering of alkali atoms, and let each atom be in a hyperfine state of low energy with zero internal orbital angular momentum. These states depend on the interaction of the nuclear spin \vec{I} and the spin \vec{S} of the single valence electron through the atom total spin $\vec{F} = \vec{I} + \vec{S}$. Moreover, if an additional external magnetic field *B* is applied both spins will interact with it and cause the splitting of the hyperfine levels (the Zeeman effect).

The atoms are well described as point-like objects and the collision depends, by means of Born-Oppenheimer potentials, on both the distance of the centers-of-mass of the separated atoms and the spins $\vec{S_1}, \vec{S_2}$ of their valence electrons (though not on the spins of their nuclei). As a consequence, these potentials produce transitions that may change the total spins of the colliding atoms.

In this article, we present a one-dimensional toy model that reduces the above many degrees of freedom and their intricacies to a minimum but still retains some distinctive features of enhancement that we would like to illustrate. It is clear that no potential depending on the *z*-coordinate alone can generate in three dimensions any outgoing spherical wave from a dispersive center, as a short-range central potential does. Nevertheless, multichannel scattering as well as the occurrence of Feshbach resonances are not restricted to three dimensions. Indeed, examples can be found where one-dimensional Feshbach resonances play an important role.^{3–5} The one-dimensional model we present below is chosen to keep the technical details as simple as possible. However, similar three-dimensional examples exist in the literature that are exactly solvable as well.^{6,7}

For the sake of simplicity, let us thus consider a total spin-1/2 atom of mass m that interacts with a much heavier, spinless atom, with their motion confined to the *z*-axis. We propose a Hamiltonian for the relative coordinate z of the form

$$H = -\mathbf{I} \frac{\hbar^2}{2m} \frac{d^2}{dz^2} + \begin{pmatrix} V_1(z) & 0\\ 0 & V_2(z) \end{pmatrix} - \mu B \sigma_x, \tag{1}$$

where μ is the atom's magnetic dipole moment, σ_x is the Pauli spin matrix, and **I** is the 2 × 2 identity matrix. This Hamiltonian consists of a kinetic energy term, a short-range attractive potential term that interacts differently with the (total) spin-up and spin-down components of the light atom, and a term representing the interaction with a constant external magnetic field pointing in the *x*-direction $\vec{B} = (B, 0, 0)$.

The total spin states $|\uparrow\rangle, |\downarrow\rangle$ mimic the hyperfine states of the atom (degenerate in this case), and the potentials $V_1(z), V_2(z)$ mimic the spin-dependent Born-Oppenheimer interatomic potentials. The time-independent Schrödinger equation for such a system reads, in terms of the spin-up $\psi(z)$ and spin-down $\varphi(z)$ components of the wavefunction,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + V_1(z)\,\psi(z) - \mu B\,\varphi(z) = E_t\,\psi(z),\qquad(2)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(z)}{dz^2} + V_2(z)\,\varphi(z) - \mu B\,\psi(z) = E_t\,\varphi(z),\qquad(3)$$

where E_t is the total energy of the system.

In scattering theory, the concept of a *channel* is introduced as the quantum state of the colliding atoms before or after the collision takes place. In our case, when the incoming atom is sufficiently far from the target so that it does not feel the potential, the magnetic field lifts the degeneracy of the light atom ground state and gives rise to two Zeeman levels with energies $\pm \mu B$ and a level splitting of $\Delta_Z = 2\mu B$. The corresponding Zeeman states are the eigenstates of σ_x : $|\pm\rangle_Z = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$.⁸ (The heavy atom does not interact with the magnetic field.) Therefore, we are faced with a twochannel scattering problem, one channel for each possible Zeeman state of the incoming (or outgoing) light atom.

Let us express the Schrödinger equation in the basis of Zeeman states—in terms of the components $u = (\psi + \varphi) / \sqrt{2}$ and $v = (\psi - \varphi) / \sqrt{2}$ —and label the channels as the u

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and the *v* channels, accordingly. In this case, we form linear combinations of Eqs. (2) and (3). As is customary, we shift the origin of energies upwards by μB and define $E = E_t + \mu B$. An incoming atom in the *u* channel thus has an energy equal to E > 0 that can be fixed experimentally. In terms of the *u* and *v* channels, we have

$$-\frac{\hbar^2}{2m}\frac{d^2u(z)}{dz^2} + V_u(z)\,u(z) + W(z)\,v(z) = E\,u(z),\tag{4}$$

$$-\frac{\hbar^2}{2m}\frac{d^2v(z)}{dz^2} + V_v(z,B)\,v(z) + W(z)\,u(z) = E\,v(z),\qquad(5)$$

with

$$V_{u}(z) \equiv \frac{V_{1} + V_{2}}{2}, \quad V_{v}(z, B) \equiv 2\mu B + \frac{V_{1} + V_{2}}{2},$$
$$W(z) \equiv \frac{V_{1} - V_{2}}{2}.$$
(6)

Here, V_u and V_v are the *bare* potentials of the *u* and *v* channels; the two channels are coupled by the off-diagonal potential *W* and they only decouple if $V_1 = V_2$.

We consider attractive delta-function potentials

$$V_1(z) = -g_1 \,\delta(z), \quad V_2(z) = -g_2 \,\delta(z),$$
 (7)

with g_1 and g_2 positive constants and $g_1 \neq g_2$. One can think of these delta functions as effective substitutes for short range potentials in a low energy scattering experiment where the range is much smaller than the wavelength of the scattered atom. Using these definitions, the potentials in Eq. (6) become

$$V_u(z) = -\frac{g_1 + g_2}{2} \,\delta(z), \quad V_v(z, B) = 2\mu B - \frac{g_1 + g_2}{2} \,\delta(z),$$
$$W(z) = \frac{g_2 - g_1}{2} \,\delta(z). \tag{8}$$

The magnetic field has the effect of unbalancing the dissociation threshold of the potentials V_u and V_v .⁹ We see from Eq. (8) that the threshold for V_v is lifted by $2\mu B$ with respect to V_u (see Fig. 1).

If *E* lies below the dissociation threshold of a given potential the corresponding channel is said to be *closed*; otherwise, it is said to be *open*. There are several possibilities:

- (i) if E < 0, both channels are closed;
- (ii) if $0 < E < 2\mu B$, *u* is open and *v* is closed;
- (iii) if $E > 2\mu B$, both channels are open.

We first consider an open u channel and a closed v channel; later we will look for bound states with both channels closed.

II. ONE OPEN AND ONE CLOSED CHANNEL $(0 < E < 2\mu B)$

In the absence of coupling between the channels (W = 0) Eqs. (4) and (5) become independent equations, each driven by a delta-function potential. Let us call these the *bare* equations and write the solutions as $u^{(b)}(z)$ and $v^{(b)}(z)$. As shown in the Appendix, $u^{(b)}(z)$ has scattering solutions for any value of E > 0, whereas $v^{(b)}(z)$ admits only a single bound state with $E < 2\mu B$, given by



Fig. 1. Schematic representation of $V_u(z)$ and $V_v(z)$ in Eq. (6). The solid horizontal line is the energy *E* and the dashed segment is the bound state energy of $V_v(z)$. Note that by increasing or decreasing the magnetic field, the $V_v(z)$ dissociation threshold at $2\mu B$ (horizontal dotted line) is shifted up and down. In the figure, $\Delta = \hbar^2 \alpha^2 / 2m$.

$$\varepsilon^{(b)}(B) = 2\mu B - \frac{\hbar^2}{2m} \alpha^2, \qquad (9)$$

with $\alpha = m(g_1 + g_2)/2\hbar^2$.

If the channels are coupled ($W \neq 0$) the situation is completely different. Now u(z) and v(z) are no longer independent and Eqs. (4) and (5) now admit v(z) (scattering) solutions for any continuous value of $0 < E < 2\mu B$. It is interesting to investigate the cause of this rather drastic change.

The potential W that couples the channels endows the incoming atom in the $|+\rangle_Z$ state with a probability amplitude for flipping its (total) spin to the $|-\rangle_{Z}$ state, and vice versa. Therefore, the two colliding atoms with energy E can make a virtual transition to a bound state of energy $\varepsilon^{(b)}(B)$. According to the Heisenberg uncertainty relation, this is only allowed if the duration of the process lasts at most $\hbar/|E - \varepsilon^{(b)}(B)|$, after which the atom's state is restored to the original $|+\rangle_{7}$. There is no restriction whatsoever on the values of E with an atom's presence in the closed channel in an energy range otherwise forbidden. The existence of the coupled closed channel modifies the interaction felt by the atom and the influence may be conspicuous for certain values of E. If $E \approx \varepsilon^{(b)}(B)$, the duration of the virtual transition can last a very long time, thus causing a large effect. This is the so-called Feshbach effect.^{10,11} There arises an interference between two alternative mechanisms that contribute to scattering: a direct transition through the potential V_u that leads to the final state in a single step, and an indirect transition that proceeds through a virtual transition to the other channel. The interference can be constructive or destructive, thus enhancing or suppressing the scattering. Note that for a given value of the energy E, the amount of the detuning $|E - \varepsilon^{(b)}(B)|$ can be controlled externally by changing B.¹²

In order to find the explicit solutions, we rewrite Eqs. (4) and (5) as

$$\left(\frac{d^2}{dz^2} + k^2\right)u(z) = -S\,\delta(z),\tag{10}$$

$$\left(\frac{d^2}{dz^2} - \beta^2\right)v(z) = -S'\,\delta(z),\tag{11}$$

where we have defined

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$$k^{2} = \frac{2mE}{\hbar^{2}}, \quad K_{B}^{2} = \frac{2m}{\hbar^{2}} (2\mu B), \quad \beta'^{2} = K_{B}^{2} - k^{2}, \quad (12)$$

and

$$\frac{S}{2} = \alpha u(0) + w v(0), \quad \frac{S'}{2} = \alpha v(0) + w u(0), \quad (13)$$

with

$$\alpha = \frac{m}{2\hbar^2}(g_1 + g_2)$$
 and $w = \frac{m}{2\hbar^2}(g_1 - g_2).$ (14)

The scattering solution—the incoming wave e^{ikz} at the open channel entrance plus outgoing scattered waves—can be written as (see Appendix)

$$u(z) = e^{ikz} - S \frac{e^{ik|z|}}{2ik} = e^{ikz} - \frac{1}{ik} [\alpha u(0) + w v(0)] e^{ik|z|},$$
(15)

$$v(z) = S' \frac{e^{-\beta'|z|}}{2\beta'} = \frac{1}{\beta'} \left[\alpha \, v(0) + w \, u(0) \right] e^{-\beta'|z|}.$$
 (16)

The sources in the right-hand-sides of these equations are proportional to the values that the wavefunctions take at the origin, which can be obtained self-consistently by setting z=0 in Eqs. (15) and (16) and solving the resulting linear system. From Eq. (16), we find

$$v(0) = \frac{w}{\beta' - \alpha} u(0), \tag{17}$$

allowing us to write Eq. (15) as

$$u(z) = e^{ikz} - \alpha_{\rm eff}(\beta') \, u(0) \, \frac{e^{ik|z|}}{ik}, \tag{18}$$

where

$$\alpha_{\rm eff}(\beta') = \alpha + \frac{w^2}{\beta' - \alpha}.$$
(19)

Equations (18) and (19) summarize the remarkable result we want to emphasize: the net influence of the closed channel is to provide the open channel with an effective interaction $V_{\rm eff}$, in this case, an effective delta-function potential given by (see Appendix)^{13–17}

$$V_{\rm eff}(z) = -g_{\rm eff}(\beta')\,\delta(z), \quad \text{with} \quad g_{\rm eff}(\beta') \equiv \frac{\hbar^2}{m}\alpha_{\rm eff}(\beta').$$
(20)

Therefore, the entire range $-\infty < \alpha_{\rm eff}(\beta') < +\infty$ is available for the effective coupling. The effective coupling constant $\alpha_{\rm eff}$ strongly depends on the incident energy as well as on the magnetic field through the combination $\beta'(k, K_B)$ [see Eq. (12)].

We have found that by varying B at fixed energy, both the strength and the sign of the interaction are under external control. This is the Feshbach phenomenon.

There are two values of β' around which α_{eff} flips its sign, changing the character of the interaction from attractive to repulsive and vice-versa. These values are given by

$$\beta_c' = \alpha - \frac{w^2}{\alpha},\tag{21}$$

where the effective coupling vanishes $[\alpha_{eff}(\beta_c') = 0]$, and

$$\beta_0' = \alpha, \tag{22}$$

where the effective coupling diverges (see Fig. 2). Given a magnetic field *B*, the value of β'_0 corresponds to an incident energy coinciding to the bare closed-channel bound-state energy $E_0(B) = \varepsilon^{(b)}(B)$ [see Eq. (9) and Fig. 2]. The energy corresponding to β'_c is given by

$$E_c(B) = 2\mu B - \frac{\hbar^2}{2m} \left(\alpha - \frac{w^2}{\alpha}\right)^2,$$
(23)

which gets a contribution from the coupling w to the closed channel.¹⁸

Finally, we find for Eqs. (15) and (16)

$$u(z) = e^{ikz} + r(k,B) e^{ik|z|},$$
(24)

$$v(z) = \frac{ikw}{(ik+\alpha)(\beta'-\alpha)+w^2} e^{-\beta'|z|},$$
(25)

where

$$r(k,B) = -\frac{1}{1 + i[k/\alpha_{\rm eff}(\beta')]}$$
(26)

is the reflection amplitude. The spatial extent of the wavefunction in the v channel is $1/\beta' = \hbar/\sqrt{2m(2\mu B - E)}$, which bears no relation to its counterpart $1/\alpha$ of $v^{(b)}(z)$ (see Appendix).¹⁹

III. LOW-ENERGY SCATTERING

Three dimensional low-energy scattering with a shortrange, spherically symmetric potential can be described to a good approximation by a single parameter, the *scattering length* a_{3d} . At such low energy, the wavelength of the



Fig. 2. Plot of the effective interaction $\alpha_{\rm eff}(\beta')$ in Eq. (19). By varying $0 < \beta'(k,B) < \infty$, $\alpha_{\rm eff}$ can take any value in the range $-\infty < \alpha_{\rm eff} < +\infty$. This plot corresponds to the choice $w = 0.3 \alpha$ in Eq. (14).

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incoming atom is much larger than the range of the potential—the potential effectively acts as a point-like dispersive center and the scattering amplitude is spherically symmetric too with a behavior in terms a_{3d} given by²

$$f(k) = -\frac{a_{3d}}{1 + ik \, a_{3d} + O(k^2)}.$$
(27)

We take this expression as a reference to define a(B), the one-dimensional analogue of a_{3d} , by writing the reflection amplitude as

$$r(k,B) = -\frac{1}{1 + ik \, a(B) + O(k^3)},\tag{28}$$

where the difference stems from the different units of f(k), which has units of length, and r(k,B), which is dimensionless. Upon Taylor expanding in powers of k and using Eqs. (18), (24), and (25), we find

$$a(B) \equiv \frac{1}{\alpha_{\rm eff}(k=0,B)} = \frac{1}{\alpha + w^2/(K_B - \alpha)} = \frac{1}{\alpha} - \frac{(w/\alpha)^2}{K_B - \beta'_c},$$
(29)

where K_B , α , w, and β'_c are defined in Eqs. (12), (14), and (21). Note that in the last equality, we have separated the open *u*-channel bare piece $1/\alpha$ from the contribution due to channel coupling, the latter of which has a pole at $K_B = \beta'_c$; that is, it diverges at a field B_c given by [see Eq. (12)]

$$B_{c} = \frac{1}{2\mu} \frac{\hbar^{2}}{2m} \beta_{c}^{\prime 2} = \frac{1}{2\mu} \frac{\hbar^{2}}{2m} \left(\alpha - \frac{w^{2}}{\alpha} \right)^{2}.$$
 (30)

Retaining the pole contribution and the constant term only we find that, for values of B close to B_c ,

$$a(B) = a_0 \left[c - \frac{\Delta B}{B - B_c} + O(B - B_c) \right], \tag{31}$$

where

$$a_{0} = 1/\alpha, \quad c = 1 - \frac{1}{2} \frac{w^{2}}{\alpha^{2} - w^{2}},$$

$$\Delta B = \frac{1}{\mu} \frac{\hbar^{2}}{2m} w^{2} \left[1 - \left(\frac{w}{\alpha}\right)^{2} \right].$$
(32)

Notice that a(B) changes its sign across B_c : a(B) > 0 for $B < B_c$ and a(B) < 0 for $B > B_c$. We also note that the value B_0 for which $a(B_0) = 0$ is found from Eq. (29) to be $\hbar^2 \alpha^2 / (4\mu m)$.

A word of caution is in order at this point. In spite of the similarities with the three-dimensional case, the implications of a vanishing or a divergent scattering length are very different in one and three dimensions. While vanishing a_{3d} means vanishing cross-section, or complete transparency, in the one-dimensional case, it means quite the opposite—a reflection coefficient equal to unity coincides with complete opacity; the converse being true when the scattering length diverges.

What remains true in both the three- and one-dimensional cases is that the divergence of the scattering length appears in the presence of a zero-energy bound state in the spectrum.^{20,21}

IV. BOUND STATES AND HALO STATES

Finally, let us look for the discrete bound states (with E < 0). If B = 0, the original Hamiltonian (1) is diagonal and features no special behavior; henceforth, we assume $B \neq 0$. The equations we need to solve are similar to Eqs. (10) and (11), with k^2 replaced by $(-\beta^2) \equiv 2mE/\hbar^2 < 0$. The solutions are (see Appendix)

$$u(z) = Ne^{-\beta|z|}, \quad v(z) = N'e^{-\beta'|z|},$$
(33)

and the constraint is now $\beta'^2 = \beta^2 + K_B^2$, where these parameters are as defined in Eq. (12). Inserting this *Ansatz* into the Schrödinger equation, the following two relations for the constants N, N' arise:

$$\frac{N'}{N} = \frac{w}{\beta' - \alpha} = \frac{\beta - \alpha}{w}.$$
(34)

From the second equality in this equation, we obtain the energy quantization condition,

$$(\beta' - \alpha)(\beta - \alpha) = w^2, \tag{35}$$

which can be cast in the form

$$\beta = \alpha_{\rm eff}(\beta'). \tag{36}$$

We see that the quantization condition is the same as for an attractive delta potential (see Appendix) with the effective coupling α_{eff} found in Eq. (19).

Figure 3 shows a plot of the solutions as a function of β' . Both the number of bound states and their energies depend on *B*: for $K_B \leq \beta'_c$ there are two bound states whereas for $K_B > \beta'_c$ there is just one.

In what follows, we consider a magnetic field just underneath the critical value B_c , which means a value of K_B just below β'_c , and we focus on the least bound of the two bound states that exist in this case. We call it the *halo* state for reasons that will become apparent immediately.



Fig. 3. Graphical solution [see Eq. (36)] of $\alpha_{\rm eff}(\beta') = \sqrt{\beta'^2 - K_B^2}$. The thin solid black lines represent $\alpha_{\rm eff}(\beta')$ and the thick grey lines represent $\sqrt{\beta'^2 - K_B^2}$ for two different values of K_B . The solutions are the β' values where the curves intersect (denoted by black dots). Depending on the value of K_B (the onset of the hyperbola on the β' axis), there are either two solutions or one; two solutions if $K_B \leq \beta'_c$ and one solution otherwise. In the figure, we have plotted the solutions for two such values of K_B .

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For simplicity we first rewrite Eq. (36) in a more convenient way for the analysis in terms of β as

$$\beta = \alpha + \frac{w^2}{\sqrt{\beta^2 + K_B^2} - \alpha} \,. \tag{37}$$

At $K_B = \beta'_c$, this equation admits a zero energy bound state with $\beta = 0$. In order to find the solution for slightly smaller values of K_B , we insert $\beta = 0$ in the right-hand-side of Eq. (37) and find

$$\beta = \frac{1}{a(B)} + O(B - B_c)^2.$$
 (38)

The leading contribution is given by the inverse scattering length a(B)—the same length parameter we found in Eq. (29)—plus very small corrections that are quadratic in the departure of the magnetic field from its critical value.

In the range of K_B just below β'_c , the value of a(B) is dominated by the pole term in Eq. (31) and is large and positive as corresponds to values of *B* just underneath B_c . Therefore, the *u*-channel bound-state wavefunction extends over a large distance equal to a(B), hence the name of halo state. Accordingly, it follows from Eq. (38) that its energy scales as the inverse square of the scattering length; the larger the a(B), the smaller the energy (in magnitude),

$$E_{\text{halo}}(B \le B_c) = -\frac{\hbar^2}{2m} \frac{1}{a(B)^2},$$
 (39)

and thus corresponds to a state that is weakly bound. This is the energy of a large molecule that has a tiny binding energy.

From Eq. (33), the halo state wave function becomes $u(z) = Ne^{-|z|/a(B)}$ and extends out to a (very large) distance a(B). However, the extent of v(z) in Eq. (33) is $1/\beta'$ and remains bounded. As *B* gets sufficiently close to B_c (from the left), a(B) becomes unboundedly large, whereas $1/\beta'$ approaches $1/\beta'_c$.

The constants N, N' are fixed by the normalization condition

$$\int_{-\infty}^{\infty} dz \left(|u(z)|^2 + |v(z)|^2 \right) = N^2 a(B) + \frac{N^2}{\beta'} = 1, \quad (40)$$

in addition to the relation $N'/N = w/(\beta' - \alpha)$ in Eq. (34). These conditions lead to

$$N^{2}\left[a(B) + \left(\frac{w}{\beta' - \alpha}\right)^{2} \frac{1}{\beta'}\right] = 1 \quad \Rightarrow \quad N \simeq a(B)^{-1/2}.$$
(41)

We conclude that near the Feshbach resonant value B_c , the properties of the halo state are determined by a single large parameter, the correlation length a(B); its energy in Eq. (39), as well as the *u*-channel wave-function $u(z) = a(B)^{-1/2} e^{-|z|/a(B)}$, adopt universal forms in terms of a(B), which hold independently of the details of the coupling with the *v*-channel.

Due to the large spatial extent of u(z), there is only a tiny probability that the atoms will be present in the *v*-channel, a probability that reduces to

$$\frac{N^{\prime 2}}{\beta^{\prime}} \simeq \left(\frac{w}{\beta^{\prime} - \alpha}\right)^2 \frac{1}{\beta^{\prime} a(B)} \ll 1.$$
(42)

To summarize, we see that the scattering length a(B) not only determines the behavior of the low-energy scattering amplitude in Eq. (29) but also the halo-state energy as well as its *u*-channel wavefunction. This is because the energy spectrum of the halo-state energy, although negative, lies very close to the positive low energies in the continuum of the unbounded scattering states.

V. CONCLUSIONS

In recent years, a lot of activity has been devoted to the field of dilute, ultracold alkali atoms. Part of this activity has relied on quantum-mechanical effects that appear in the presence of an applied magnetic field. One such effect is the Feshbach resonance that we have exemplified with an exactly solvable, one-dimensional toy model. We have shown how the coupling to a closed channel produces an effective interaction the intensity of which, as well as its attractive or repulsive character, can be changed by tuning the magnetic field. For values of the magnetic field close to the Feshbach resonant value B_c , the model also has a halo bound state with the characteristic universal dependence on the (diverging) scattering length.

In an experiment at extremely low energies, such as in a dilute ultracold gas, let the magnetic field vary in the neighborhood of the Feshbach resonant critical value B_c . Assume we start with a magnetic field above B_c and decrease it very slowly, in an adiabatic manner. If the change in *B* is slow enough when crossing the critical field B_c , the atoms will remain in the lowest energy state (which is the halo state for $B < B_c$). In this way, by starting from dissociate atoms and by means of an adiabatic decrease of the magnetic field across the critical value B_c weakly bound molecules are produced.

Notice that this transition is accompanied by an abrupt change in the reflection amplitude behavior, which switches from totally reflective, $r(k \rightarrow 0, B \ge B_c) = -1 + O(k)$, to reflectionless at the exact critical value,

$$r(k \to 0, B_c) = \frac{i}{2} \frac{\alpha/w^2}{\left[1 - (w/\alpha)^2\right]} k + O(k^2),$$
(43)

prior to the formation of the halo molecules.

As for the scattering length, its behaviour is dominated by the pole contribution in Eq. (31). It flips its sign from negative $[a(B > B_c) < 0]$ to positive $[a(B < B_c) > 0]$ and diverges at the critical value B_c , which is the distinctive sign of a Feshbach resonance in this context. For this critical value of the field, a zero-energy bound state $E_{halo}(B = B_c) = 0$ appears in the spectrum [see Eq. (39)].

In recent years, the Feshbach resonance has become a fundamental tool used to tune the strength of interactions between ultracold atoms over several orders of magnitude with unprecedented control, simply by tuning a magnetic field.^{1,2}

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APPENDIX: MATHEMATICAL DETAILS

All the solutions we have presented can be easily checked. We have used the following two facts:

$$\left(\frac{d^2}{dz^2} + k^2\right)\frac{e^{ik|z|}}{2ik} = \delta(z), \quad \left(\frac{d^2}{dz^2} - \beta^2\right)\frac{e^{-\beta|z|}}{2\beta} = -\delta(z),$$
(A1)

the first one being suitable for outgoing waves.

Let us briefly review the spectrum of a one-dimensional Hamiltonian with an attractive delta potential given by

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(z)}{dz^2} - g\delta(z)\chi(z) = E\chi(z).$$
 (A2)

For E > 0, this equation becomes

$$\left(\frac{d^2}{dz^2} + k^2\right)\chi(z) = -2\alpha\chi(0)\delta(z),\tag{A3}$$

where $\alpha = mg/\hbar^2$ and $k = \sqrt{2mE/\hbar^2}$. Considering an entering plane wave plus an outgoing scattered wave, the solution, according to Eq. (A1), is of the form

$$\chi(z) = e^{ikz} - 2\alpha\chi(0) \frac{e^{ik|z|}}{2ik},$$
(A4)

where, using self-consistency, one finds $\chi(0) = -(1 + \alpha/ik)^{-1}$, so that

$$\chi(z) = e^{ikz} - \frac{\alpha}{ik + \alpha} e^{ik|z|}.$$
(A5)

The reflection r(k) and transmission t(k) amplitudes defined as

$$\chi(z \to -\infty) \sim e^{ikz} + r(k)e^{-ikz},$$

$$\chi(z \to +\infty) \sim t(k)e^{ikz},$$
(A6)

can be read off immediately, and using the notation $a_0 = 1/\alpha$ as in Eq. (32), they read

$$r(k) = -\frac{1}{1+ika_0}, \quad t(k) = \frac{ika_0}{1+ika_0}.$$
 (A7)

If E < 0, the Hamiltonian (A2) always has a single bound state. Using $\beta = \sqrt{2m|E|/\hbar^2}$, Eq. (A2) becomes

$$\left(\frac{d^2}{dz^2} - \beta^2\right)\chi(z) = -2\alpha\chi(0)\delta(z),\tag{A8}$$

and according to Eq. (A1), we have

$$\chi(z) = \frac{\alpha}{\beta} \chi(0) e^{-\beta|z|}.$$
 (A9)

Setting z = 0, one is led to conclude that

$$\beta = \alpha, \tag{A10}$$

which is the quantization condition for the bound-state energy $E = -\hbar^2 \alpha^2 / 2m$, whereas $\chi(0)$ remains free for normalization of the wavefunction. The end result is

$$\chi(z) = \sqrt{\alpha} \, e^{-\alpha |z|},\tag{A11}$$

showing that the wavefunction of the bound state extends over a distance $a_0 = 1/\alpha$ around z = 0.

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⁸The Zeeman states are *B* independent in our toy-model, but in general this is not the case. This can be seen by repeating the calculations after adding, say, a splitting term to Eqs. (2) and (3) of the form $H_{\text{split}} = \sigma_z \Delta/2$. It is not difficult to find that now the Zeeman states depend on *B*. (The potentials depend on *B* as well.)

⁹The dissociation threshold of a given potential is the energy above which the two atoms are not bounded by it.

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- ¹²This picture is valid when the coupling among the channels is weak and may be viewed as a perturbation of the scenario with the channels uncoupled. However, multichannel physics is in general richer than what perturbation theory provides and the picture given above is not necessarily valid in situations when the coupling is strong. The exact results we presented in Secs. III and IV do not rely on perturbation theory.
- ¹³A general procedure to reduce a multichannel problem to a single-channel-equivalent problem with an effective potential (the so called Optical Potential) follows from Feshbach's analysis (see Ref. 10). This procedure also covers the case with the two channels open; in our example this boils down to the substitution $\beta' \rightarrow -ik'$ in Eq. (18). The effective potential in Eq. (20) becomes complex with an imaginary part that reflects the inelastic transition to the *v*-channel that is now allowed, with the absorption of an energy $2\mu B$. Indeed, we find that $\text{Im}(V_{\text{eff}}) = -\delta(z) (\hbar^2/m)[w^2k'/(k'^2 + \alpha^2)] \leq 0$. Being non-Hermitian, this potential leads by itself to probability non-conservation. The physical interpretation is clear and reflects the fact that part of the probability at the entrance channel leaks through the other channel that is also open. It should be mentioned that the correspondence of a two-channel problem and a single-channel-equivalent one is not always free from ambiguities (see Refs. 14 and 15). Several examples on how some of such difficulties have been dealt with can be found (see Refs. 6, 16, and 17).
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- ¹⁸The correction can only contain even powers of w. There is one power of w for each channel swap and the number of swaps cannot be odd because the process has to end up in the original channel.
- ¹⁹At $\beta'_0 = \alpha$ we find $v(z) = e^{-\alpha |z|} ik/w$, which coincides with the bare *v*-channel bound-state wavefunction, except for normalization. In general one expects a further contribution from the bare *v*-channel continuum states too, which turns out to vanish in our example.
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Toepler-Holtz Machine

I photographed this large, multiple-disk Toepler-Holtz electrostatic machine at Fort Hayes State University in Hayes, Kansas in the spring of 1986. It dates from the early years of the 20th century, and was used to generate the high voltage necessary to produce diagnostic x rays. Sonia Greenslade serves as a scale to show the size of the machine. The picture of a much smaller version is in *Am. J. Phys.*, **76**, 186 (2008). (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)