Classical Yang-Mills theory

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We consider classical Yang-Mills theory with point sources and derive equations of motion for the coupled particle-field system. As an example, we discuss the special case of Yang-Mills theory coupled to point particles in $(1+1)$ dimensions. We formulate the initial value problem for the system and describe several example solutions.
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I. INTRODUCTION

Yang-Mills theory plays a central role in explaining fundamental interactions, because both the strong and weak interactions are described by Yang-Mills theories.[1,2] Students are usually introduced to Yang-Mills theory in the context of quantum field theory and never encounter Yang-Mills theory in its classical form. In contrast, students do not study quantum electrodynamics until they have thoroughly mastered classical electrodynamics. In this paper, we fill this gap by discussing classical Yang-Mills theory coupled to point sources. We derive the equations of motion for the coupled particle-field system and present several example solutions to these equations for Yang-Mills theory in $(1+1)$ dimensions.

In Sec. II, we briefly review ordinary classical electrodynamics. In Sec. III, we generalize ordinary electrodynamics to what we call color electrodynamics, which can be thought of as electrodynamics with three different types of charge. In Sec. IV, we generalize from color electrodynamics to Yang-Mills theory by means of symmetry principles. We then specialize to the case of $(1+1)$ dimensions. In Sec. V, we discuss ordinary electrodynamics in $(1+1)$ dimensions, and in Sec. VI, we discuss Yang-Mills theory in $(1+1)$ dimensions. In Sec. VII, we present several example solutions to the equations of motion for Yang-Mills theory in $(1+1)$ dimensions.

The following notation is used. The sign function $\epsilon(x)$ is defined such that $\epsilon(x)=1$ if $x>0$, $\epsilon(x)=0$ if $x=0$, and $\epsilon(x)=-1$ if $x<0$. The step function $\theta(x)$ is defined such that $\theta(x)=1$ if $x>0$, $\theta(x)=1/2$ if $x=0$, and $\theta(x)=0$ if $x<0$.

II. ORDINARY ELECTRODYNAMICS

We begin by briefly reviewing ordinary classical electrodynamics. For simplicity, we consider the case of a single point particle source. Let m and q denote the mass and electric charge of the particle, and let $z^\mu(\tau)$ and $w^\mu(\tau)=dz^\mu(\tau)/d\tau$ denote its position and velocity at proper time τ. The electromagnetic field is described by a vector potential A^μ from which we derive the field strength tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1}$$

The field strength tensor satisfies the field equation

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu, \tag{2}$$

where the current density J^μ is given by[3]

$$J^\mu(x) = q\int w^\mu(\tau)\,\delta^{(4)}(x-z(\tau))\,d\tau. \tag{3}$$

The particle equation of motion is

$$m\frac{dw^\mu}{d\tau} = qF^{\mu\nu}w_\nu. \tag{4}$$

Equations (1)–(4) give a complete description of electrodynamics.

An important property of electrodynamics is that it is invariant under the transformation $A^\mu \to A^\mu + \partial^\mu\lambda$, where λ is an arbitrary position-dependent parameter. Such a transformation is called a local abelian gauge transformation: local, because λ can be different at different points in spacetime, and abelian, because the net effect of two such transformations is independent of the order in which they are performed.

III. COLOR ELECTRODYNAMICS

We will now generalize electrodynamics so that instead of one type of charge, electric charge, we have three types of charge, which we call color charge.[4] We denote the three color charges of the particle by q_x, q_y, and q_z, and we define a vector $\vec{q} \equiv (q_x, q_y, q_z)$, which we call the charge vector. Note that $\vec{q}$ is a vector in color space, not physical space. We will refer to vectors in color space as color vectors and will denote them by using horizontal arrows.

In color electrodynamics there are three vector potentials, one for each type of color charge, which we collect to form a color vector $\vec{A}^\mu$. The corresponding field strength tensor $\vec{F}^{\mu\nu}$ is given by

$$\vec{F}^{\mu\nu} = \partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu. \tag{5}$$

The field strength tensor satisfies the field equation

$$\partial_\mu \vec{F}^{\mu\nu} = 4\pi \vec{J}^\nu, \tag{6}$$

where the color current density $\vec{J}^\mu$ is given by

$$\vec{J}^\mu(x) = \vec{q}\int w^\mu(\tau)\,\delta^{(4)}(x-z(\tau))\,d\tau. \tag{7}$$

The particle equation of motion is

$$m\frac{dw^\mu}{d\tau} = \vec{q}\cdot\vec{F}^{\mu\nu}w_\nu. \tag{8}$$

Equations (5)–(8) give a complete description of color electrodynamics.

The physical interpretation of color electrodynamics is simple: each type of color charge produces a corresponding color field that obeys the laws of ordinary electrodynamics, and charges and fields of different colors do not couple to each other. The force on a particle is the sum of the forces exerted by each of the three color fields.

Color electrodynamics is closely analogous to ordinary electrodynamics. For example, two particles with charge vectors $\vec{q}_1$ and $\vec{q}_2$ experience an attractive force if $\vec{q}_1 \cdot \vec{q}_2$ is negative and a repulsive force if $\vec{q}_1 \cdot \vec{q}_2$ is positive. Also, the field equation for the theory is linear and therefore obeys the superposition principle: the field generated by a collection of particles is the sum of the fields generated by the individual particles.

Like ordinary electrodynamics, color electrodynamics is invariant under local abelian gauge transformations, which in this case take the form

$$\vec{A}^\mu \rightarrow \vec{A}^\mu + \vec{\lambda}^\mu, \tag{9}$$

where the color vector $\vec{\lambda}$ is an arbitrary position-dependent parameter. But there is an additional symmetry of color electrodynamics that is not present in ordinary electrodynamics: we are free to globally rotate the color vectors. That is, the theory is invariant under the transformation

$$\vec{q} \rightarrow R\,\vec{q}, \qquad \vec{J}^\mu \rightarrow R\,\vec{J}^\mu, \tag{10}$$

$$\vec{A}^\mu \rightarrow R\vec{A}^\mu, \qquad \vec{F}^{\mu\nu} \rightarrow R\vec{F}^{\mu\nu}, \tag{11}$$

where R is an arbitrary rotation matrix.

This rotational symmetry implies that the theory depends only on the relative orientation of color vectors, not on their absolute orientation in color space. For example, we saw that the force between two particles depends only on the dot product $\vec{q}_1 \cdot \vec{q}_2$. Invariance under global color rotations is analogous to invariance under Lorentz transformations, because a Lorentz-invariant theory depends only on relative velocities, not on absolute velocities.

In what follows, it will be useful to restrict ourselves to infinitesimal transformations. This restriction does not entail a loss of generality, because finite transformations can always be composed out of sequences of infinitesimal transformations. Consider a rotation R of infinitesimal magnitude θ about an axis $\hat{n}$. For an arbitrary color vector $\vec{v}$, we have that $R\vec{v} = \vec{v} + \vec{\theta} \times \vec{v}$, where $\vec{\theta} \equiv \theta\hat{n}$. From Eqs. (9)–(11), it follows that color electrodynamics is invariant under the transformation

$$\vec{q} \rightarrow \vec{q} + \vec{\theta} \times \vec{q}, \tag{12}$$

$$\vec{J}^\mu \rightarrow \vec{J}^\mu + \vec{\theta} \times \vec{J}^\mu, \tag{13}$$

$$\vec{A}^\mu \rightarrow \vec{A}^\mu + \vec{\theta} \times \vec{A}^\mu + \partial^\mu\vec{\lambda}, \tag{14}$$

$$\vec{F}^{\mu\nu} \rightarrow \vec{F}^{\mu\nu} + \vec{\theta} \times \vec{F}^{\mu\nu}, \tag{15}$$

where $\vec{\lambda}$ and $\vec{\theta}$ are infinitesimal color vectors. The color vector $\vec{\theta}$ must be constant, but $\vec{\lambda}$ can be different at different points in spacetime. We will refer to this transformation, which combines a local abelian gauge transformation with a global color rotation, as a general transformation.

IV. YANG-MILLS THEORY

Color electrodynamics is invariant under local abelian gauge transformations and global color rotations. We will now generalize color electrodynamics so that it is invariant not just under global color rotations, but local color rotations as well; that is, we want our theory to be invariant under general transformations for which both $\vec{\lambda}$ and $\vec{\theta}$ depend on position.

We first introduce some terminology: a color vector $\vec{v}$ that transforms like $\vec{v} \rightarrow \vec{v} + \vec{\theta} \times \vec{v}$ under a general transformation is said to transform properly. So $\vec{q}$, $\vec{J}^\mu$, and $\vec{F}^{\mu\nu}$ should transform properly, but $\vec{A}^\mu$ should not. Our first task is to find an expression for the field strength tensor that transforms properly, given Eq. (14) for the transformation of the vector potential. We want to generalize color electrodynamics, so we start by considering the tensor

$$\vec{F}_0^{\mu\nu} \equiv \partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu. \tag{16}$$

From Eqs. (14) and (16), it follows that

$$\vec{F}_0^{\mu\nu} \rightarrow \vec{F}_0^{\mu\nu} + \vec{\theta} \times \vec{F}_0^{\mu\nu} + \partial^\mu\vec{\theta} \times \vec{A}^\nu - \partial^\nu\vec{\theta} \times \vec{A}^\mu, \tag{17}$$

so $\vec{F}_0^{\mu\nu}$ does not transform properly. Another natural tensor to consider is

$$\vec{F}_1^{\mu\nu} \equiv g\vec{A}^\mu \times \vec{A}^\nu, \tag{18}$$

where g is a constant with units of inverse charge. From Eqs. (14) and (18), it follows that

$$\vec{F}_1^{\mu\nu} \rightarrow \vec{F}_1^{\mu\nu} + \vec{\theta} \times \vec{F}_1^{\mu\nu} + g\partial^\mu\vec{\lambda} \times \vec{A}^\nu + g\vec{A}^\mu \times \partial^\nu\vec{\lambda}, \tag{19}$$

so $\vec{F}_1^{\mu\nu}$ does not transform properly either.[5] In fact it is not possible to define a viable field strength tensor if $\vec{\theta}$ and $\vec{\lambda}$ are taken to be independent parameters. But if we define

$$\vec{F}^{\mu\nu} \equiv \vec{F}_0^{\mu\nu} + \vec{F}_1^{\mu\nu} = \partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu + g\vec{A}^\mu \times \vec{A}^\nu, \tag{20}$$

and take $\vec{\lambda} = -(1/g)\vec{\theta}$, the unwanted terms in Eqs. (17) and (19) cancel, and $\vec{F}^{\mu\nu}$ transforms properly.[6]

Our theory is thus not invariant under independent local abelian gauge transformations and local color rotations, but is invariant under general transformations for which $\vec{\lambda} = -(1/g)\vec{\theta}$:

$$\vec{q} \rightarrow \vec{q} + \vec{\theta} \times \vec{q}, \tag{21}$$

$$\vec{J}^\mu \rightarrow \vec{J}^\mu + \vec{\theta} \times \vec{J}^\mu, \tag{22}$$

$$\vec{A}^\mu \rightarrow \vec{A}^\mu + \vec{\theta} \times \vec{A}^\mu - (1/g)\partial^\mu\vec{\theta}, \tag{23}$$

$$\vec{F}^{\mu\nu} \rightarrow \vec{F}^{\mu\nu} + \vec{\theta} \times \vec{F}^{\mu\nu}. \tag{24}$$

Such transformations are called local nonabelian gauge transformations: nonabelian, because the net effect of two such transformations depends on the order in which they are performed. In what follows, we will consider only nonabelian gauge transformations rather than arbitrary general transformations, and we will say that a vector $\vec{v}$ transforms properly if it transforms like $\vec{v} \rightarrow \vec{v} + \vec{\theta} \times \vec{v}$ under local nonabelian gauge transformations.

Now that we have an expression for the field strength tensor, we would like to generalize the field equation (6). We immediately encounter a problem due to the presence of the derivative ∂_μ, because if an arbitrary color vector $\vec{v}$ transforms properly then its derivative $\partial_\mu\vec{v}$ does not:

$$\partial_\mu \vec{v} \rightarrow \partial_\mu \vec{v} + \partial_\mu \vec{\theta} \times \vec{v} + \vec{\theta} \times \partial_\mu \vec{v}. \tag{25}$$

Let us define an operator D_μ such that

$$D_\mu \vec{v} = \partial_\mu \vec{v} + g\vec{A}_\mu \times \vec{v}. \tag{26}$$

From Eqs. (23), (25), and (26), it follows that if $\vec{v}$ transforms properly, then so does $D_\mu \vec{v}$:

$$D_\mu \vec{v} \rightarrow D_\mu \vec{v} + \vec{\theta} \times D_\mu \vec{v}. \tag{27}$$

Thus, we will take the field equation to be

$$D_\mu \vec{F}^{\mu\nu} = \vec{F}^{\mu\nu} + g\vec{A}_\mu \times \vec{F}^{\mu\nu} = 4\pi \vec{J}^\nu. \tag{28}$$

The particle equation of motion (8) for color electrodynamics is already invariant under local nonabelian gauge transformations and thus requires no modification:

$$m\frac{dw^\mu}{d\tau} = \vec{q} \cdot \vec{F}^{\mu\nu} w_\nu. \tag{29}$$

There is one more complication we must address to complete the theory: whereas in color electrodynamics the charge vector $\vec{q}$ is a fixed quantity, in our generalized theory it must be allowed to depend on the proper time τ. We can understand why from the following considerations. By using Eqs. (20) and (26), we can relate the commutator of D_μ and D_ν to the field strength tensor $\vec{F}_{\mu\nu}$:

$$[D_\mu, D_\nu]\vec{v} = D_\mu D_\nu \vec{v} - D_\nu D_\mu \vec{v} = g\vec{F}_{\mu\nu} \times \vec{v}. \tag{30}$$

If we apply D_ν to both sides of Eq. (28), we find that

$$4\pi D_\nu \vec{J}^\nu = D_\nu D_\mu \vec{F}^{\mu\nu} = (1/2)[D_\nu, D_\mu]\vec{F}^{\mu\nu} = 0, \tag{31}$$

where we have used the antisymmetry of $\vec{F}^{\mu\nu}$ and substituted for $[D_\mu, D_\nu]$ using Eq. (30). Thus, the source $\vec{J}^\mu$ must satisfy the current conservation relation

$$D_\mu \vec{J}^\mu = \partial_\mu \vec{J}^\mu + g\vec{A}_\mu \times \vec{J}^\mu = 0. \tag{32}$$

The current density is given by

$$\vec{J}^\mu(x) = \int \vec{q}(\tau) w^\mu(\tau)\, \delta^{(4)}(x - z(\tau))\, d\tau, \tag{33}$$

where we have allowed for the possibility that $\vec{q}$ depends on τ. If we apply ∂_μ to both sides of Eq. (33) and integrate by parts, we find that

$$\partial_\mu \vec{J}^\mu(x) = -\int \vec{q}(\tau) \frac{d}{d\tau} \delta^{(4)}(x - z(\tau))\, d\tau \tag{34}$$

$$= \int \frac{d\vec{q}(\tau)}{d\tau} \delta^{(4)}(x - z(\tau))\, d\tau\,. \tag{35}$$

Here we have used the fact that

$$\frac{d}{d\tau}\delta^{(4)}(x - z(\tau)) = -w^\mu(\tau)\partial_\mu \delta^{(4)}(x - z(\tau)). \tag{36}$$

If we substitute Eqs. (33) and (35) into the current conservation equation (32), we find that

$$\int \left(\frac{d\vec{q}}{d\tau} + g\vec{A}_\mu \times \vec{q} w^\mu \right) \delta^{(4)}(x - z(\tau)) d\tau = 0. \tag{37}$$

Thus, the equation of motion for the charge vector $\vec{q}$ is

$$\frac{d\vec{q}}{d\tau} = -g w^\mu \vec{A}_\mu \times \vec{q}. \tag{38}$$

Therefore to satisfy current conservation, as described by Eq. (32), we must allow the charge vector to vary in time as described by Eq. (38). From Eq. (38), it follows that $d|\vec{q}|^2/d\tau = 2\vec{q} \cdot d\vec{q}/d\tau = 0$, so the magnitude of the charge vector is constant in time.

We have now completed our generalization of color electrodynamics. The field strength tensor and current density are given by Eqs. (20) and (33), the field equation is given by Eq. (28), and the evolution of the particle is described by Eqs. (29) and (38). The resulting theory is known as Yang-Mills theory.[7] It is invariant under local nonabelian gauge transformations, as described by Eqs. (21)–(24).

Yang-Mills theory differs from color electrodynamics because there are nonlinear terms in the field strength tensor (20) and the field equation (28), and because the charge vector is rotated by the vector potential as described by Eq. (38). The presence of these new features is dictated by the requirement that Yang-Mills theory be invariant under local nonabelian gauge transformations. The nonlinear terms and the rotation rate of the charge vector are all proportional to g, so for $g = 0$, Yang-Mills theory reduces to color electrodynamics. The greater the value of g, the more important these new features become, and the more Yang-Mills theory deviates from color electrodynamics. Because of the nonlinear terms, Yang-Mills theory does not obey the superposition principle.

V. ELECTRODYNAMICS IN (1 + 1) DIMENSIONS

For simplicity, we will consider the special case of Yang-Mills theory in $(1+1)$ dimensions. For comparison, we first consider ordinary electrodynamics in $(1+1)$ dimensions.[8] We define

$$A^\mu = (V, A), \qquad J^\mu = (\rho, J), \tag{39}$$

$$z^\mu = (t, z), \qquad w^\mu = (\gamma, \gamma v), \tag{40}$$

where $v = dz/dt$ and $\gamma = (1 - v^2)^{-1/2}$. In $(1+1)$ dimensions, the electromagnetic field is described by the single electric field $E \equiv F^{10} = -F^{01}$. From the definition of the field strength tensor given in Eq. (1), we find that

$$E = -\partial_x V - \partial_t A. \tag{41}$$

In $(1+1)$ dimensions, the field equation (2) can be expressed as the pair of equations[9]

$$\partial_t E = -2J, \tag{42}$$

$$\partial_x E = 2\rho, \tag{43}$$

and the particle equation of motion (4) can be expressed as

$$du/dt = (q/m)E, \tag{44}$$

where $u \equiv \gamma v = (1 - v^2)^{-1/2} v$. From Eq. (3), we find that ρ and J are given by

$$\rho(t,x) = q\delta(x - z(t)), \tag{45}$$

$$J(t,x) = qv(t)\delta(x - z(t)). \tag{46}$$

Equations (41)–(46) give a complete description of electrodynamics in $(1+1)$ dimensions. If we compare this theory to electrodynamics in $(3+1)$ dimensions, we see that Eqs. (42) and (43) are analogous to Maxwell's equations, and Eq. (44) is analogous to the Lorentz force law.

VI. YANG-MILLS THEORY IN (1 + 1) DIMENSIONS

Now let us consider Yang-Mills theory in $(1+1)$ dimensions. We define

$$\vec{A}^\mu = (\vec{V}, \vec{A}), \qquad \vec{J}^\mu = (\vec{\rho}, \vec{J}), \tag{47}$$

$$z^\mu = (t, z), \qquad w^\mu = (\gamma, \gamma v). \tag{48}$$

From the definition of the field strength tensor $\vec{F}^{\mu\nu}$ given in Eq. (20), we find that $\vec{F}^{10} = -\vec{F}^{01} = \vec{E}$, where

$$\vec{E} = -\partial_x\vec{V} - \partial_t\vec{A} + g\vec{A} \times \vec{V}. \tag{49}$$

In $(1+1)$ dimensions, Eq. (28) can be expressed as the pair of equations

$$\partial_t\vec{E} = -2\vec{J} - g\vec{V} \times \vec{E}, \tag{50}$$

$$\partial_x\vec{E} = 2\vec{\rho} + g\vec{A} \times \vec{E}, \tag{51}$$

and Eqs. (29) and (38) can be expressed as

$$du/dt = \vec{q} \cdot \vec{E}/m, \tag{52}$$

$$d\vec{q}/dt = -g(\vec{V} - v\vec{A}) \times \vec{q}. \tag{53}$$

From Eq. (33) for the charge density, we find that $\vec{\rho}$ and $\vec{J}$ are given by

$$\vec{\rho}(t,x) = \vec{q}(t)\delta(x - z(t)), \tag{54}$$

$$\vec{J}(t,x) = v(t)\vec{q}(t)\delta(x - z(t)). \tag{55}$$

Equations (49)–(55) give a complete description of Yang-Mills theory in $(1+1)$ dimensions.

Let us now consider the initial value problem for the system. We will take the dynamical variables for the field to be $\vec{E}$, $\vec{V}$, and $\vec{A}$, and the dynamical variables for the particle to be z^μ, w^μ, and $\vec{q}$. The first field equation (50) gives us an equation of motion for $\vec{E}$. We can obtain an equation of motion for $\vec{A}$ by using the definition of $\vec{E}$ given in Eq. (49):

$$\partial_t\vec{A} = -\partial_x\vec{V} - \vec{E} + g\vec{A} \times \vec{V}. \tag{56}$$

To obtain an equation of motion for $\vec{V}$, we impose the Lorentz gauge condition $D_\mu A^\mu = \partial_\mu\vec{A}^\mu = 0$:

$$\partial_t\vec{V} = -\partial_x\vec{A}. \tag{57}$$

The equations of motion (50), (52), (53), (56), and (57) completely determine the evolution of the system: given initial conditions for the dynamical variables, we can integrate these equations to evolve the system in time.

Note that we have not yet used the second field equation (51). This equation does not serve as an equation of motion; rather it acts as a constraint on the allowed initial conditions. We define a color vector $\vec{G}$ to measure the violation of this constraint:

$$\vec{G} \equiv \partial_x\vec{E} - 2\vec{\rho} - g\vec{A} \times \vec{E}. \tag{58}$$

By using the equations of motion (50) and (56) for $\vec{E}$ and $\vec{A}$, together with the current conservation equation (32), one can show that

$$\partial_t\vec{G} = -g\vec{V} \times \vec{G}. \tag{59}$$

Thus, if the constraint is satisfied by the initial conditions ($\vec{G}(0,x) = 0$), it will remain satisfied as the system evolves in time ($\vec{G}(t,x) = 0$).

So far we have considered only the case of a single particle, but it is straightforward to generalize to the case of N particles. Let $z_n^\mu = (t, z_n)$, $w_n^\mu = (\gamma_n, \gamma_n v_n)$, and $\vec{q}_n$ denote the position, velocity, and charge vector of particle n. The equations of motion for particle n are

$$du_n/dt = \vec{q}_n \cdot \vec{E}/m, \tag{60}$$

$$d\vec{q}_n/dt = -g(\vec{V} - v_n\vec{A}) \times \vec{q}_n, \tag{61}$$

where $u_n \equiv \gamma_n v_n$. The equations of motion for the field variables $\vec{E}$, $\vec{V}$, and $\vec{A}$ are still given by Eqs. (50), (56), and (57), but the charge density $\vec{\rho}$ and current density $\vec{J}$ are now given by

$$\vec{\rho}(t,x) = \sum_n \vec{q}_n(t)\delta(x - z_n(t)), \tag{62}$$

$$\vec{J}(t,x) = \sum_n v_n(t)\vec{q}_n(t)\delta(x - z_n(t)). \tag{63}$$

VII. EXAMPLE SOLUTIONS

Let us now consider some example solutions to Yang-Mills theory in $(1+1)$ dimensions. We will first describe a two-particle solution. Let us assume that at time $t=0$ the particles have equal and opposite charge vectors:

$$\vec{q}_1(0) = -\vec{q}_2(0) = \vec{Q}_0. \tag{64}$$

It is straightforward to show that the following expressions satisfy the initial conditions (64), the equations of motion (50), (56), (57), and (61), and the constraint equation $\vec{G}(0,x) = 0$:

$$\vec{q}_1(t) = -\vec{q}_2(t) = \vec{Q}_0, \tag{65}$$

$$\vec{E}(t,x) = \vec{Q}_0(\epsilon(x - z_1(t)) - \epsilon(x - z_2(t))), \tag{66}$$

$$\begin{aligned}\vec{A}^\mu(t,x) = \vec{Q}_0 \int &(w_1^\mu(\tau)D_r(x^\nu - z_1^\nu(\tau)) \\ &- w_2^\mu(\tau)D_r(x^\nu - z_2^\nu(\tau)))\, d\tau.\end{aligned} \tag{67}$$

Here $x^\mu \equiv (t, x)$, and $D_r(x^\nu) = \theta(t - |x|)$ is the retarded Green function for the inhomogeneous wave equation in $(1+1)$ dimensions. The remaining equation of motion (60) becomes

$$du_1/dt = -du_2/dt = -a\epsilon(z_1 - z_2), \tag{68}$$

where $a \equiv |\vec{Q}_0|^2/m$. From Eq. (68), we see that the particles feel an attractive force whose strength is independent of the particle separation.

Let us assume that the particles start out at rest at the origin with equal and opposite velocities, so that

$$z_1(0) = z_2(0) = 0, \qquad u_1(0) = -u_2(0) = U_0. \tag{69}$$

We can integrate Eq. (68) subject to these initial conditions to obtain

$$z_1(t) = -z_2(t) = a^{-1}((1+U_0^2)^{1/2} - (1+U_0^2 f^2(2t/T)))\epsilon(f(t/T)), \tag{70}$$

where $f(x) \equiv 1 - 2(x - [x])$ and $T \equiv 4U_0/a$. Here $[x]$ denotes the integer part of x; that is, $[x]$ is the largest integer less than or equal to x. Thus, the particles undergo periodic oscillations with period T and amplitude $a^{-1}((1+U_0^2)^{1/2} - 1)$. In Fig. 1, we plot the particle trajectories for $U_0 = 0.5$ and $a = 1$.

Note that this solution is independent of the coupling constant g. This independence follows from the fact that g enters into the equations of motion only as a prefactor to cross products of color vectors such as $\vec{E}$, $\vec{V}$, $\vec{A}$, and $\vec{q}_n$. Because these vectors are all oriented along $\vec{Q}_0$, the cross products all vanish, and the solution is therefore independent of g.

In particular, the solution satisfies the equations of motion for $g = 0$ Yang-Mills theory, which we have seen is equivalent to color electrodynamics. By a suitable choice of gauge, we can always take $\vec{Q}_0$ to lie entirely along the $\hat{x}$-axis, in which case only one of the three types of color charge is actually present. Thus, the particle trajectories are a solution to the equations of motion for ordinary electrodynamics, where the electric charges of the particles are taken to be $\pm|\vec{Q}_0|$. As we would expect, the two particles undergo periodic oscillations due to their mutual Coulomb attraction.

Let us now consider a four-particle solution. We will take the initial conditions for the particles to be

$$z_1 = z_2 = -z_3 = -z_4 = Z_0, \tag{71}$$

$$u_1 = -u_2 = u_3 = -u_4 = U_0, \tag{72}$$

$$\vec{q}_1 = -\vec{q}_2 = -Q_0\hat{x}, \tag{73}$$

$$\vec{q}_3 = -\vec{q}_4 = -Q_0\hat{y}. \tag{74}$$

First, let us take $g = 0$, for which Yang-Mills theory reduces to color electrodynamics. Because the charge vectors for particles 1 and 2 lie along the $\hat{x}$-axis, and the charge vectors for particles 3 and 4 lie along $\hat{y}$-axis, the particle pairs (1, 2) and (3, 4) carry different types of color charge and thus do not interact with each other. The two pairs therefore evolve independently: the two particles in each pair undergo periodic oscillations due to their mutual Coulomb attraction, as described by the two-particle solution. The resulting four-particle solution is plotted in Fig. 2(a) for $Z_0 = 0.1$, $U_0 = 0.5$, and $a \equiv Q_0^2/m = 1$.

Next, we consider $g \neq 0$. We can evolve the system in time by numerically integrating the equations of motion subject to the initial conditions for the particle variables given in

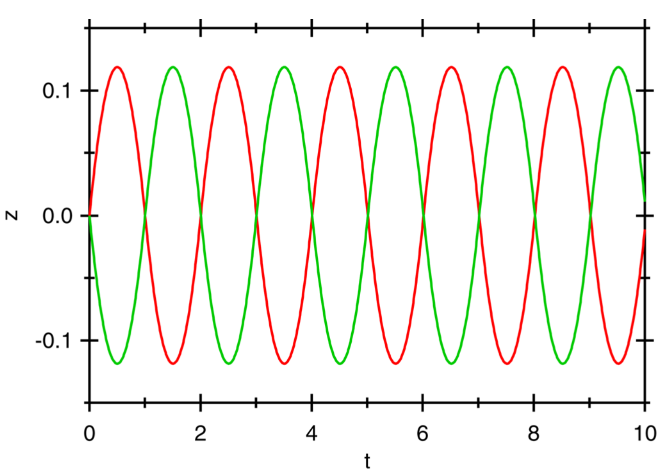

Fig. 1 Particle positions z_1 and z_2 versus time t.

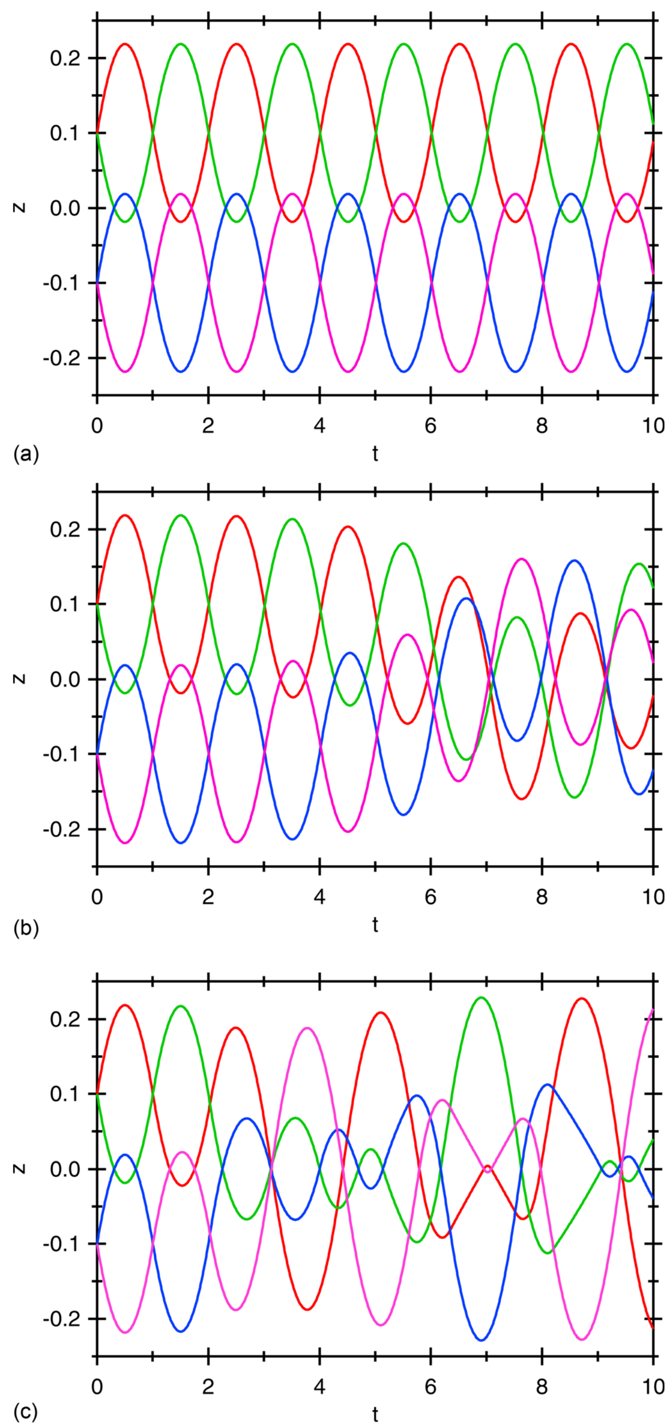

Fig. 2 Particle positions z_1, z_2, z_3, and z_4 versus time t. (a) $g = 0$. (b) $g = 1$. (c) $g = 5$.

Eqs. (71)–(74).[10,11] We also need initial conditions for the field variables, which we take to be

$$\vec{E}(0,x) = \vec{V}(0,x) = \vec{A}(0,x) = 0. \tag{75}$$

Note that the initial conditions satisfy the constraint equation $\vec{G}(0,x) = 0$. In Figs. 2(b) and 2(c), we plot the resulting particle trajectories for $g=1$ and $g=5$, where Z_0, U_0, and a have the same values as in Fig. 2(a). At early times, the two pairs of particles oscillate independently, and the solution is well approximated by the solution for $g=0$. At late times, the nonlinear terms in the equations of motion become important and the solutions deviate from the $g=0$ solution. The time at which the deviation begins is $t \simeq 4$ for the $g=1$ solution and $t \simeq 2$ for the $g=5$ solution. As we would expect, the deviation is more pronounced for larger values of g. In Figs. 3(a) and 3(b), we plot the components of the charge vectors $\vec{q}_1$ and $\vec{q}_2$ for the $g=5$ solution. As expected, the charge vectors are rotated by the vector potential. Figures 2 and 3 show that the nonlinearity of $g \neq 0$ Yang-Mills theory can produce very complicated dynamics.

As a consistency check, we repeat the calculation of the $g=5$ solution using a different choice of gauge. We use the same initial conditions for the particle variables as before, but we take the initial conditions for the field variables to be

$$\vec{E}(0,x) = \vec{A}(0,x) = 0, \qquad \vec{V}(0,x) = V_0\hat{x}, \tag{76}$$

where $V_0 = 1$. We perform the numerical integration and find that the particle trajectories are still given by Fig. 2(c). This result is expected, because the particle trajectories are independent of the choice of gauge. The charge vectors, however, do depend on the choice of gauge, and for the new initial conditions they evolve as shown in Fig. 4. Note that although the directions of the charge vectors depend on the choice of gauge, the magnitudes do not: a gauge transformation rotates the charge vectors, but does not scale them. The magnitudes of the charge vectors are thus independent of time and independent of the choice of gauge.

These solutions help illustrate how Yang-Mills theory can be viewed as a two-step generalization of ordinary electrodynamics. The solution shown in Fig. 1 can be understood entirely in terms of ordinary electrodynamics: the particles undergo periodic oscillations due to their mutual Coulomb attraction. In Fig. 2(a), we move from ordinary electrodynamics to color electrodynamics: the two particles in each pair undergo periodic oscillations due to their mutual Coulomb attraction, but the pairs do not interact with each other because they carry different types of color charge. The solutions shown in Figs. 2(b) and 2(c) show how the nonlinear terms in the Yang-Mills field equations give rise to complicated particle dynamics when $g \neq 0$.

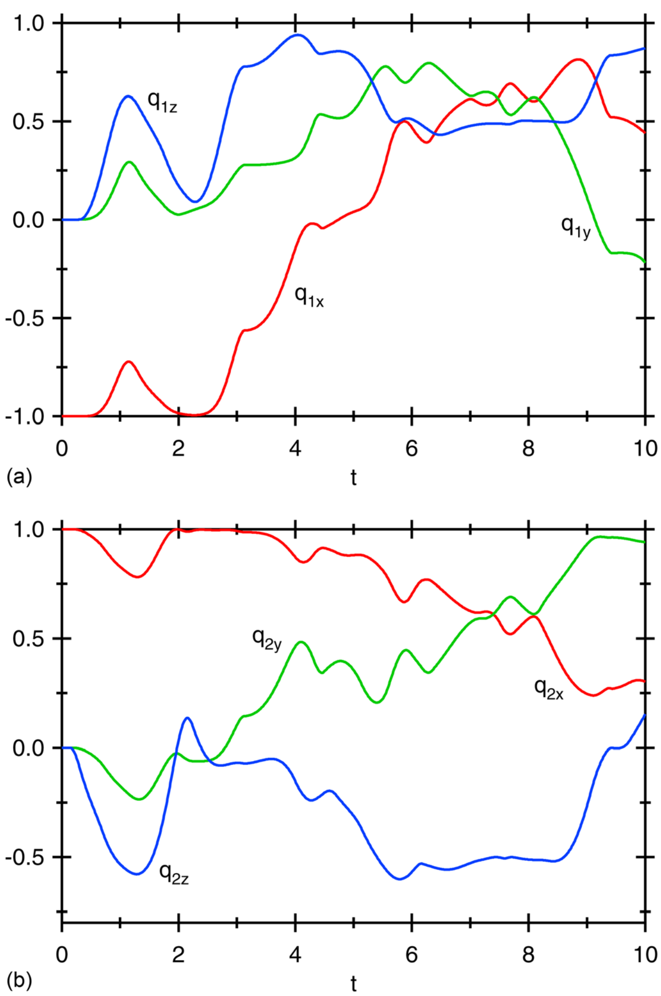

Fig. 3 Charge vector components versus time t. (a) Charge vector $\vec{q}_1$. (b) Charge vector $\vec{q}_2$.

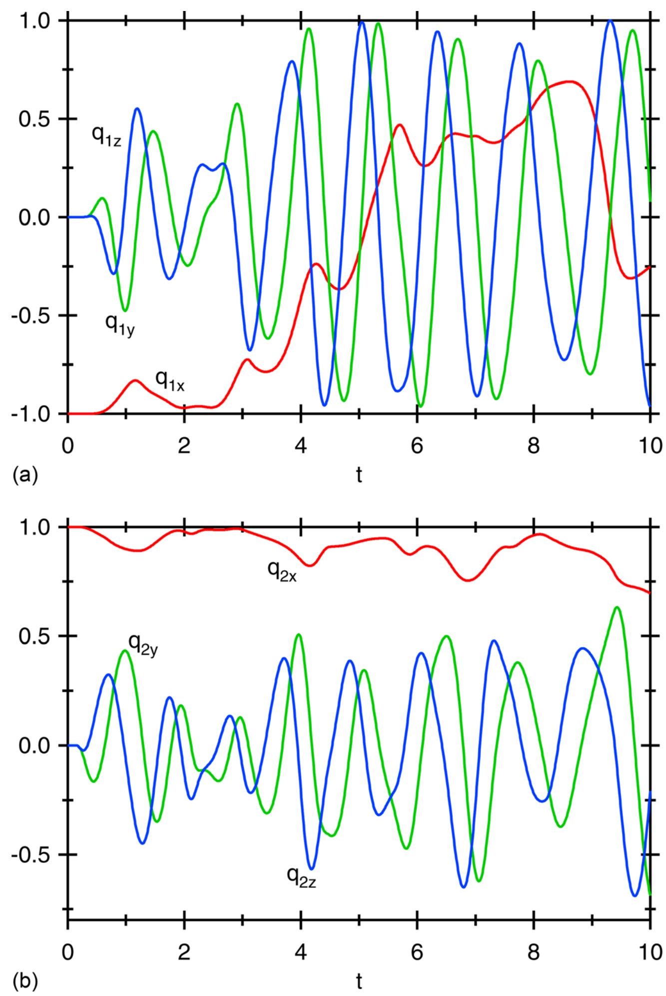

Fig. 4 Charge vector components versus time t. (a) Charge vector $\vec{q}_1$. (b) Charge vector $\vec{q}_2$.

VIII. CONCLUSIONS

We have described classical Yang-Mills theory with particle sources and derived the equations of motion for the coupled particle-field system. As an example, we described in detail the special case of Yang-Mills theory coupled to particles in $(1+1)$ dimensions. We presented several example solutions and showed that the nonlinearity of the equations of motion leads to complicated particle dynamics. The investigation of the particle dynamics in this $(1+1)$-dimensional system could form the basis of some interesting research projects for students.

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[1]An overview of the literature on gauge theory is given in T. P. Cheng and Ling-Fong Li, "Resource Letter: GI-1 gauge invariance," Am. J. Phys. **56**(7), 586–600 (1988).

[2]The role of gauge theory in describing fundamental interactions is described in C. Quigg, *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions* (Addison-Wesley, New York, 1993).

[3]The current density given in Eq. (3) is discussed in J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley & Sons, New York, 1975), pp. 611–612.

[4]We generalize to three types of charge, rather than some other number, so as to obtain the simplest possible version of Yang-Mills theory, which has gauge group SU(2). More complicated versions of Yang-Mills theory with larger gauge groups can be obtained by increasing the number of types of charge. Our definition of "color" is somewhat different from the definition of "color" in QCD, which has gauge group SU(3).

[5]Because $\vec{\theta}$ and $\vec{\lambda}$ are infinitesimal, we have neglected terms of second order in these parameters.

[6]An alternative derivation of the Yang-Mills field strength tensor is given in Palash B. Pal and K. S. Sateesh, "The field strength and the Lagrangian of a gauge theory," Am. J. Phys. **58**(8), 789–790 (1990).

[7]Yang-Mills theory was first proposed in C. N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," Phys. Rev. **96**, 191–195 (1954).The coupling of point particles to classical Yang-Mills fields was first discussed in S. K. Wong, "Field and particle equations for the classical Yang-Mills field and particles with isotopic spin," Nuovo Cimento A **65**, 689–694 (1970).

[8]Electrodynamics in $(1+1)$ dimensions is discussed in many places. See, for example, I. Bialynicki-Birula, "Classical electrodynamics in two dimensions: Exact solution," Phys. Rev. D **3**, 864–866 (1971) and the appendix of Ref. 10.

[9]We have replaced 4π, the appropriate solid angle factor for $(3+1)$ dimensions, by 2, the appropriate solid angle factor for $(1+1)$ dimensions. The appropriate solid angle factor for $(d+1)$ dimensions is given by the area of a d-dimensional sphere, which is $2\pi^{d/2}/\Gamma(d/2)$.

[10]The methods used to perform the numerical integration are described in A. D. Boozer, "Simulating a toy model of electrodynamics in $(1+1)$ dimensions," Am. J. Phys. **77**(3), 262–269 (2009).

[11]The program used to perform the numerical integration is available upon request.

Crystal Set. This crystal set was assembled ca. 1940 from some commercial parts and with a hand-wound coil. The parallel-resonant circuit is formed by the capacitor (made of aluminum foil interleaved with sheets of mica and located between the terminals in front) and the coil. Tuning is accomplished by sliding the contact along the coil, thus adjusting the number of turns in the circuit. The galena crystal is behind the capacitor, and the metallic cats-whisker in contact with the surface of the crystal forms the required metal-semiconductor rectifying junction. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)