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Introduction to Focus Issue: Dynamics of oscillator populations

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ABSTRACT

Even after about 50 years of intensive research, the dynamics of oscillator populations remain one of the most popular topics in nonlinear science. This Focus Issue brings together studies on such diverse aspects of the problem as low-dimensional description, effects of noise and disorder on synchronization transition, control of synchrony, the emergence of chimera states and chaotic regimes, stability of power grids, etc.

I. INTRODUCTION

The study of coupled autonomous, self-sustained oscillators is a traditional field in nonlinear sciences, pioneered by classical works of Appleton and van der Pol. Remarkably, these early studies essentially contributed to the development of chaos theory—after Cartwright and Littlewood, during WWII, got interested in the forced van der Pol equation, they could show in their 1945 paper that this dynamical system possesses an infinite number of unstable periodic orbits. Later, Smale revealed the essence of this finding by creating his horseshoe map. While electronic circuits were the primary objects for experimental studies of self-sustained oscillations in the first few decades, many more examples later appeared in physics (e.g., lasers, Josephson junctions, electrochemical oscillators) and life sciences (e.g., spiking neurons, pacemakers, circadian rhythms). This development boosted studies of coupled oscillators. The next step was done in the 1970s by Winfree and Kuramoto, who combined ideas of nonlinear science and statistical physics by considering the collective dynamics of large populations of oscillators and demonstrating the emergence of a collective mode with an increase in interaction strength, which can be interpreted as a nonequilibrium phase transition. Since 1980s, this field has been a flourishing realm of experimental, mathematical, and numerical studies at the edge of nonlinear dynamics and nonequilibrium statistical physics. We mention several articles and books reviewing main achievements in this field.[1–5]

In recent years, the scientific community witnessed essential advancements in studies of oscillator populations. First, there appeared novel applications, such as ensembles of cells with genetically implemented artificial clocks,[6] pedestrian dynamics,[7] nanomechanical and spin-torque oscillators,[8,9] power grids,[10] and other. Second, innovative theoretical approaches have been elaborated, providing a low-dimensional description of the collective dynamics.[11,12] The next important development was the inclusion of a disorder in the coupling, thus extending the analysis from the simplest case of all-to-all coupled ensembles to complex networks (see, e.g., a *Chaos* Focus Issue on Patterns of Network Synchronization[13]). Finally, several non-trivial unexpected phenomena, such as chimera patterns[14,15] and partial synchronization,[16,17] have been observed. All these achievements inspired us to organize the this Focus Issue. It is clear that delivering innovative research that meets a deadline for manuscript submissions is difficult; therefore, one should not consider this issue as a comprehensive review (moreover, some topics, such as the inverse problem of inferring interactions from observations of coupled oscillators, are missing; also, experiments definitely deserve more place than they have in this issue). On the other hand, the field is very vital, so even a "snapshot" of the contemporary research presented in this Focus Issue gives a good impression of current interests and trends.

II. THIS FOCUS ISSUE

Below, we tried to "classify" the contributions according to several topics. However, this classification is necessarily simplified, as many papers match several different keywords.

A. Globally coupled populations

Several contributions address different aspects of the classical problem of the dynamics of globally coupled oscillators. Four publications[18–21] have much in common: (i) the starting point is the model in the form of phase equations; (ii) the systems are treated in the thermodynamic limit of infinitely many oscillators, where a description via a probability distribution of the phases is adequate; and (iii) all four papers focus on a finite-dimensional reduction of the kinetic equation for the distribution density and the dynamical properties resulting from such a reduction.

Campa[18] considers the standard Kuramoto model, which, by virtue of the Ott–Antonsen ansatz, can be reduced for an arbitrary distribution of natural frequencies to a continuous set of differential equations for local order parameters, effectively coupled via the integral over the frequency distribution. While in the case of a Cauchy distribution of frequencies, the integral can be calculated via the residue method (and the whole set is then reduced to one Ott–Antonsen equation), for a Gaussian distribution, such a reduction is not possible. The paper suggests approximating a Gaussian function with a constant divided by a polynomial. As a result, one has a distribution with a finite number of poles allowing for the application of a residue-based integration and the representation of the dynamics by a finite set of modes; practically, the author takes 12 modes. Numerical simulations show that this approximation works rather well, even for initial phase distributions with singularities. We mention for an interested reader that yet another way to approximate the Kuramoto model for a Gaussian distribution of natural frequencies has been recently suggested by Pyragas and Pyragas.[22]

The model treated by León and Pazó[19] is practically identical to that of Campa:[18] they analyze a population of globally coupled phase oscillators with a Gaussian distribution of natural frequencies. León and Pazó apply a moment-based scheme that reduces the continuous set of equations to the doubly infinite set of Fourier–Hermite modes. This set is then truncated, which results in finite-dimensional dynamics. The paper explores the accuracy of different truncation schemes; furthermore, these truncations are applied to situations where exact solutions are not possible, like the case of the Kuramoto model with a bimodal frequency distribution, and the enlarged Kuramoto model endowed with non-pairwise interactions.

Medvedev *et al.*[20] study a population of active rotators with the Kuramoto-type coupling. Assuming the Cauchy distribution of the natural frequencies, they apply the Ott–Antonsen ansatz and reduce the problem to a second-order ordinary differential equation. Different regimes of the collective dynamics and the corresponding bifurcations are identified.

Skardal and Xu[21] consider a situation more complex than the standard Kuramoto model: first, they include time delays in the coupling, and second, they include coupling via the nonlinear functions of the order parameters. (In terms of the phase dynamics, this corresponds to so-called high-order interactions that are not pairwise but include triplets and quadruplets of the phases, cf. Ref. 23). The reduction of the original system relies on the Ott–Antonsen equations. The main result of this contribution is the bistability between weakly and strongly synchronized states that appears due to a combination of effects of delayed and higher-order interactions. The authors demonstrate their findings by analyzing bifurcations in the low-dimensional dynamics and comparing them to the simulations of large populations.

Two contributions of the Focus Issue explore the effects of external control action. Sarkar and Gupta[24] consider repeated random resettings of the Kuramoto ensemble. At the resetting, the old state of the system becomes forgotten, and a new state with prescribed properties (e.g., with a particular value of the Kuramoto order parameter) is created. Such resettings suppress synchrony if the new order parameter is small and facilitate if it is large. For the Lorentzian distribution of the natural frequencies, the authors ensure that the state after reset is on the Ott–Antonsen manifold; this allows for an exact solution of the dynamics between resettings. For a Gaussian distribution of the natural frequencies, the evolution of the population is followed numerically.

Toth and Wilson[25] address a problem of desynchronizing control in a large population of interacting neuronal oscillators. Their study is motivated by the problem of deep brain stimulation (DBS)—a medical procedure commonly used to treat Parkinson's disease and other neurological disorders. Following a popular line of model studies (see, e.g., Refs. 26 and 27), the authors investigate a feedback setup providing a phase-specific stimulation. The designed control action is nearly periodic; i.e., the stimuli have a slowly varying amplitude and a phase offset. If the underlying model equations are unavailable (and in a DBS application, they are undoubtedly unknown), the authors suggest a strategy to identify the reduced model of oscillating neurons from data.

B. Networks, disorder, and noise

Effects of disorder (e.g., in the form of random connectivity networks) and noise are essential in most applications of the oscillations dynamics. In this Focus Issue, oscillator models of different complexity are presented: phase oscillators,[28–31] one-dimensional models of neural oscillations,[32] as well as higher-dimensional ordinary differential equations with limit cycles.[33,34]

Hong and Martens[28] consider a random network of phase oscillators, where couplings between units are symmetric but can take one of two prescribed values—a positive or a negative one. In this way, one models a system with attractive and repulsive interactions. In the thermodynamic limit, they analyze stability of the decoherent state, both for deterministic oscillators and for noise-driven ones. The transition to synchrony is demonstrated to be discontinuous in the noise-free case and continuous in the presence of noise.

Gkogkas *et al.*[29] contribute a general mean-field theory for a noise-driven Kuramoto-type model on a random network. The approach uses a continuous thermodynamic limit of random connection graphs, formulated in terms of so-called graphons and graphops. Such an approach is expected to work well for dense networks, where mean-field calculations appear to be justified. Formulation of the generalized Fokker–Planck equation allows for a derivation of a stability criterion for the disordered state, where the network's properties enter via the spectrum of the graphon operator. Calculations of the incoherence–coherence transitions of large finite networks support these exact results.

Kassabov *et al.*[30] study a standard Kuramoto model with identical oscillators and random coupling (Erdős–Rényi random graphs with probability p that two oscillators interact). In such a system, for large p, one observes synchronization for almost all initial conditions, while other stable configurations can also appear for small values of p. The authors quantify this observation by proving that a network of n units is globally synchronizing (i.e., converges to an all-in-phase state for almost all initial conditions) with the probability larger than $1 - 4/n$ provided that $p \gg n^{-1}(\log n)^2$.

Lacerda *et al.*[31] analyze the effect of the complex network structure on the synchronization of sine-coupled phase oscillators. They generate networks of different topologies with high, intermediate, and low values of assortativity and clustering coefficient. Next, using an optimization algorithm, they generate different frequency patterns that are correlated with connectivity and quantify them by the total dissonance metric for neighborhood similarity; they obtain what they denote as similar (natural frequencies of adjacent nodes are close), dissimilar (frequencies of adjacent nodes are different), and neutral natural frequency patterns. The study's main finding is that low values of assortativity and clustering coefficient are generally favorable for phase locking of the network elements.

Di Volo *et al.*[32] analyze the dynamical regimes observed in a balanced network of identical quadratic integrate-and-fire neurons with sparse connectivity for homogeneous and heterogeneous in-degree distributions. This model is rather similar to the theta-model formulated in terms of the phase dynamics but is more suitable for spike-coupled neurons. The theoretical description relies on a mean-field model based on a self-consistent Fokker–Planck equation. While analytical predictions are possible in some cases, an approximate description based on the truncation of a hierarchy of equations for circular cumulants is developed to treat a broader class of situations. The authors discuss in detail the roles of connectivity and structural heterogeneity of the network on the appearance of coherent oscillations.

Khramenkov *et al.*[33] address the problem of power grid stability; in particular, they investigate whether an addition of a transmission line enhances the network stability or reduces it (the reduction of the entire grid stability because of adding a connection is known as the Braess paradox). The primary considered example treats a power grid as a network of second-order active rotators, with a tree-like three-element motif at the periphery. The authors consider two scenarios for the stability loss. In the first, previously known scenario, the synchronous mode disappears, while in the novel scenario, the stability reduction is due to the emergence of an asynchronous mode. The authors derive the necessary conditions for the stable operation of the grid.

Emelianova *et al.*[34] investigate the phenomenon of disordered quenching in an array of bistable oscillators. For the model, they take the Bautin oscillator, i.e., a version of a Stuart–Landau system with more complex amplitude dynamics. These oscillators have a stable fixed point and a stable limit cycle for the chosen parameters, separated by an unstable limit cycle. With increasing of the coupling strength, the system becomes quenched. The authors show analytically the existence of stable regimes with amplitude disorder for identical oscillators. Furthermore, they demonstrate numerically that the disordered oscillation quenching holds for rings and chains of systems with nonidentical natural frequencies.

C. Chimera states, chaos, and complex coupling

Many recent studies of coupled oscillators treat the effects of symmetry breaking in the form of chimera states and of chaotic dynamics. In this Focus Issue, several contributions address these issues dealing either with large populations[35,36] or with small sets.[37–39]

Bi and Fukai [35] investigated the role of amplitude dynamics in the emergence of chimera states in nonlocally coupled Stuart–Landau oscillators. The oscillators are arranged in a one-dimensional array (with periodic boundary conditions) with symmetric nonlocal coupling. In the weak coupling regime, the dynamics reduce to the previously studied case of chimeras for phase oscillators. For strong coupling, the amplitude effects become essential; here, the authors report on novel amplitude-mediated multicluster chimera states.

Clusella *et al.*[36] start with quadratic integrate-and-fire neurons (the same objects as in Ref. 32) but re-formulate the model as phase oscillators. The resulting Kuramoto-type model combines electrical and chemical couplings of neurons in one term possessing a phase shift, which depends on the relation between different coupling channels. Two coupled populations of neurons can be described within the Ott–Antonsen ansatz as a three-dimensional system possessing stable asymmetric solutions, where one population is synchronous while another one is partially synchronous. Such a chimera regime is demonstrated to be less probable for large couplings.

Chimera states are not possible in globally coupled identical phase oscillators but can appear due to a breakup of the permutation symmetry, like in Ref. 36. Burylko *et al.*[37] analyze the smallest possible network, where permutation symmetry can be broken, composed of four phase oscillators. They demonstrate how the breakup of full symmetry leads to weak chimera states, where different oscillators have different frequencies. Moreover, these states exhibit a period-doubling cascade resulting in the chaotic dynamics.

Grines *et al.*[38] explore chaotic dynamics in a population of five identical phase oscillators coupled via a biharmonic function. The system, described by four ordinary differential equations, exhibits Shilnikov spiral attractors, with two Lyapunov exponents indistinguishable from zero in numerical studies. The authors explain the observed phenomenon by analyzing a three-dimensional Poincaré map. They show that chaos develops near a codimension-three bifurcation, where the fixed point of the map has the triplet of unity multipliers. The authors relate the observed dynamics to those of the normal form of this bifurcation, known as the Arneodó–Coullet–Spiegel–Tresser system.

Coupling between oscillators is often modeled via additional terms in the equations for the oscillating systems. It can, however, happen that coupling requires separate additional equations. Goldsztein *et al.*[39] revisit such a situation in the context of the classical Huygens problem of coupled clocks. They analyze behavior of two metronomes on a common support, where the motion of the support is governed by dynamical equations, incorporating dry friction terms (this type of friction is not covered by previous studies). The analysis of this new model reveals different synchronous and asynchronous states, including in-phase and anti-phase locking, as well as suppression of oscillation amplitude so that one or both metronomes no longer engage their escapement mechanisms. The authors also present experiments supporting theoretical findings.

III. CONCLUSION

Early studies on coupled oscillators focused on synchronization due to increasing interaction strength, typically in a setup of few units. The seminal works by Winfree and Kuramoto extended the consideration to the realm of (infinitely) large oscillator ensembles. Subsequent research over several decades accounted for more complicated setups, such as complex networks, delayed or nonlinear interaction, etc., and revealed different non-trivial collective dynamical states. To a large extent, the actual progress is due to approaches providing a low-dimensional description of large systems. However, there are still many unexplored theoretical and applied problems, and we foresee that this branch of nonlinear science will remain an active field in the coming years. The 18 papers of the Focus Issue certainly do not cover all the current developments but reflect the main trends, and we hope that this issue will be of interest both for experts in the field, as well as for those outside of it, who want to have a look at a concise set of current hot topics.

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