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SYNCHRONIZATION REGIMES IN AN ENSEMBLE OF PHASE OSCILLATORS COUPLED THROUGH A DIFFUSION FIELD

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We consider an ensemble of identical phase oscillators coupled through a common diffusion field. Using the Ott–Antonsen reduction, we develop dynamical equations for the complex local order parameter and the mean field. The regions of the existence and stability are determined for the totally synchronous, partially synchronous, and asynchronous spatially homogeneous states. A procedure of searching for inhomogeneous states as periodic trajectories of an auxiliary system of the ordinary differential equations is demonstrated. A scenario of emergence of chimera structures from homogeneous synchronous solutions is described.

1. INTRODUCTION

Ensembles of coupled oscillators represent one of the key models of nonlinear dynamics. They are used to describe multiparticle self-oscillating systems in various fields of physics, chemistry, biology, and other areas of modern natural science [1–3]. In the case of a weak coupling, the oscillators are completely characterized by their phases, which allows one to describe their dynamics using the phase equations. When considering collective synchronous effects in the populations and media of the phase oscillators, the Kuramoto model of oscillators that are directly coupled via the phase difference plays the fundamental role [4]. This model and its various modifications were considered for both the systems of globally coupled elements [5, 6] and the spatially distributed chains and arrays with local and nonlocal coupling [7, 8].

In many actual multiparticle systems, the interaction between elements occurs indirectly via common medium or the so-called common field, rather than directly [9, 10]. The examples include the collective behavior of cold atoms interacting with an electromagnetic field [11] and synchronization of metronomes located on a common support [12]. In particular, works [13–16] present analytical results of the study of synchronous regimes in the systems with interaction of phase oscillators through a common inertial field. The emergence of chimeric modes, i.e., the spatial structures in which fully synchronous regions coexist with the regions whose oscillators exhibit asynchronous dynamics, is of particular interest in the field of nonlinear dynamics [7, 17]. The appearance of this behavior is directly related to such a fundamental phenomenon as the loss of symmetry [18]. At the same time, one can also observe a completely symmetric spatially homogeneous synchronous state in the system.

In this work, a detailed study of the spatially homogeneous states of different synchronization degrees is performed and their influence on the implementation of spatially inhomogeneous regimes, in particular chimeras, in a system of identical phase oscillators is analyzed. The coupling is ensured via a common field,

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which obeys the diffusion equation. The case of an infinitely fast diffusion has been considered in detail in a series of our previous works [19–24].

The article is organized as follows. Section 2 describes the model of nonlocally coupled identical phase oscillators interacting via the diffusion field. Using the Ott–Antonsen (OA) reduction, the model is reformulated in terms of the mesoscopic characteristics of the system, i.e., the complex local order parameter and the common field. The regions of existence and stability of synchronous regimes are determined in Sec. 3 on the basis of analysis of the stationary, spatially homogeneous solutions of the OA equations. Section 4 presents an original method for seeking inhomogeneous states, including the chimera ones, as periodic trajectories of an auxiliary system of the ordinary differential equations of the third order. A scenario of the chimera appearance from the spatially homogeneous solutions is described. The main results of the studies are formulated in Sec. 5.

2. MODEL OF AN ENSEMBLE OF PHASE OSCILLATORS

2.1. Description of a microscopic model

A wide range of fundamental phenomena inherent in systems consisting of a large number of self-oscillating elements can be understood and studied within the framework of the phase approximation [1]. This approximation is based on the parametrization of the state elements at the limit cycle and in its neighborhood by an angular variable, which is called the oscillation phase. The phase approximation is valid for a weak interaction between oscillators. In this work, we study synchronization regimes in a one-dimensional oscillating medium with periodic boundary conditions, i.e., the medium is closed in a ring of length L. The oscillators are assumed to be identical and their states are described by the phases $\varphi(x,t)$, where x is the coordinate and t is the time. Interaction is performed via the field $H(x,t)$ [17, 19, 25, 26]. The oscillator dynamics satisfies the equation

$$\frac{\partial\varphi(x,t)}{\partial t} = \omega + \mathrm{Im}\left(H(x,t)\exp\{-i[\varphi(x,t)+\alpha]\}\right), \tag{1}$$

where ω is the frequency of free oscillations, the parameter α determines the phase shift in the coupling, and the complex field $H(x,t)$ is supposed to be an external force which acts on the oscillation phase. In this case, an ensemble of oscillating particles itself is the source of the field $H(x,t)$. In their absence, any inhomogeneous profile $H(x,t)$ would gradually be blurred and its amplitude would be decreased to disappear because of the diffusion effects and the presence of dissipation in the medium. Therefore, from the mathematical viewpoint, evolution of the field $H(x,t)$ in the studied model obeys the following diffusion equation [19, 27, 28]:

$$\tau\frac{\partial H(x,t)}{\partial t} = \frac{\partial^2 H(x,t)}{\partial x^2} - H(x,t) + \exp[i\varphi(x,t)]. \tag{2}$$

The two coefficients in this equation are chosen equal to unity, which can always be reached by renormalization of x, t, and H (such that the remaining parameters ω and τ are dimensionless). The model of Eqs. (1) and (2) can describe the dynamics of a biological system containing a great number of identical oscillating elements (e.g., cells), which are distributed in a certain substance as in the experiment [29]. In this case, the states of the elements can be specified by multidimensional vectors whose coordinates can represent such physical quantities as the number densities of various chemical components, temperature, etc. The phases parametrize the states of the considered elements in the self-oscillating regime when the system moves along the limit cycle in the space of states of each oscillator. In this case, the interaction between the oscillators is organized using chemical substances, which diffuse in the surrounding medium, rather than by direct coupling between oscillators. Although such an interaction is similar to the chemotaxis, in our formulation it leads to modification of the phases of the internal cycles of microorganisms rather than to their motion. For example, for some possible modifications and generalizations, the system of Eqs. (1) and (2) can reproduce the dynamics of such populations of oscillating elements as yeast-cell suspensions during glycolysis [30] and amoebas in a certain phase of their life cycle [31]. We can also mention a certain similarity of the proposed model to the variant of the Belousov–Zhabotinsky reaction using the “water in oil” microemulsion [32]. It should be noted that work [16] deals with a system of two ensembles of phase oscillators, which are globally coupled via a common field with inertia. In the limit $\tau \to 0$ of the infinitely fast dynamics of the field $H(x,t)$, the system of Eqs. (1) and (2) is the classical Kuramoto–Battogtokh model with a nonlocal coupling specified by a kernel with exponentially decaying tails [19]. In this work, we consider the dynamics features, which are due to the finite time scale τ of the mean-field diffusion.

2.2. Ott–Antonsen equations

In the continuum limit, the system of Eqs. (1) and (2) can be characterized by the distribution density $\rho(\varphi,x,t)$ of the phases φ for fixed x and t. This function satisfies the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\varphi}\Big\{\omega\rho+\mathrm{Im}\big[H\exp(-i\varphi)\big]\rho\Big\}=0. \tag{3}$$

Based on the Ott–Antonsen works [33, 34] and the subsequent publications [17, 35], it can be shown that there exists a manifold (the OA manifold), which is invariant with respect to the studied-system dynamics. The idea of the OA method is as follows. By virtue of periodicity of the density $\rho(\varphi,x,t)$ with respect to the phase φ, the solution of Eq. (3) can be sought in the form of the Fourier series

$$\rho(\varphi,x,t)=\frac{1}{2\pi}\Bigg\{1+\sum_{m=1}^{+\infty}\Big[a_m(x,t)\exp(-im\varphi)+\text{c.c.}\Big]\Bigg\}. \tag{4}$$

In [33, 34], it is shown that the substitution $a_m(x,t)=Z^m(x,t)$ yields the exact solution of the continuity equation (3) if the field $Z(x,t)$ satisfies the following Ott–Antonsen equation:

$$\frac{\partial Z}{\partial t}=i\omega Z+\frac{1}{2}\left[\exp(-i\alpha)H-\exp(i\alpha)H^*Z^2\right],$$

where the asterisk * denotes complex conjugation. The physical meaning of the field $Z(x,t)$ is the following. It is the local order parameter which is defined as the mean field

$$Z(x,t)=\int_0^{2\pi}\mathrm{d}\varphi\;\rho(\varphi,x,t)\exp(i\varphi).$$

The order parameter characterizes the correlation degree of the oscillator phases $\varphi(x,t)$ at each point of the considered medium. In the case $|Z(x,t)|=1$, all oscillators in the neighborhood of the point x are synchronized with respect to the phase. Under the conditions $0<|Z(x,t)|<1$ or $|Z(x,t)|=0$, the system elements are said to be partially synchronized or totally asynchronous, respectively.

The local order parameter can be considered as a result of the averaging of the profile $\varphi(x,t)$ (generally speaking, nonsmooth) over the immediate neighborhood δ (whose size should be smaller than all other characteristic spatial scales) [33, 34, 36]:

$$Z(x,t)=\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\exp[i\varphi(\widetilde{x},t)]\,\mathrm{d}\widetilde{x}. \tag{5}$$

This representation means that the same local averaging of the field $H(x,t)$ corresponds to the replacement $\exp[i\varphi(x,t)]\to Z(x,t)$ in Eq. (2) for the field $H(x,t)$. As a result, we obtain the system of partial differential equations

$$\frac{\partial Z}{\partial t}=i\omega Z+\frac{1}{2}\left[\exp(-i\alpha)H-\exp(i\alpha)H^*Z^2\right],\qquad \tau\frac{\partial H}{\partial t}=\frac{\partial^2 H}{\partial x^2}-H+Z \tag{6}$$

with the periodic boundary conditions

$$H(0,t) = H(L,t), \qquad \frac{\partial H}{\partial x}(0,t) = \frac{\partial H}{\partial x}(L,t). \tag{7}$$

Compared with the initial model of Eqs. (1) and (2), the advantage of Eq. (6) is that the fields Z and H are continuous (as distinct from the field φ), which allows us to apply the standard methods for analyzing the partial differential equations to Eq. (6). The OA approach has been used in many situations including those encountered when studying the chimera states [17, 19–21, 23]. However, its justification still remains a partially unsolved problem since it is closely related to the properties of stability of the OA manifold, which are not totally established. Obviously, this manifold is not attracting for the ensembles of identical oscillators with global coupling [37], i.e., for the elements exposed to the common-field action. In our case, we consider the medium of identical oscillators in the nonuniform field $H(x,t)$. This allows one to assume an at least weak stability of the OA manifold, which is confirmed by some works [17, 37, 38].

Therefore, to analyze the spatiotemporal dynamics in the system of phase oscillators, described by Eqs. (1) and (2), one can consider such a mesoscopic characteristic of the collective behavior of the system elements as the complex order parameter $Z(x,t)$ whose dynamics is described by the system of partial differential equations (6) with boundary conditions (7). These equations have four significant parameters (after the introduction of the dimensionless time and coordinate), namely, the phase shift α, the diffusion time scale τ, the eigenfrequency ω of the oscillators, and the length L of the medium. Note that the parameter ω becomes insignificant for the infinitely high diffusion rate ($\tau \to 0$), i.e., it can be omitted by moving to a rotating reference frame.

3. SPATIALLY HOMOGENEOUS REGIMES

3.1. Totally synchronous, asynchronous, and partially synchronous states

Let us first consider the regimes characterized by the same phase-correlation degree of the elements at each point of the studied medium. We search for such states as solutions of system (6) with the boundary conditions as follows (7):

$$Z(x,t) = z_0 \exp[i\,(\omega+\Omega)\,t], \qquad H(x,t) = h_0 \exp[i\,(\omega+\Omega)\,t], \tag{8}$$

where z_0 and h_0 denote complex amplitudes of the fields, and Ω is an as yet unknown parameter, which determines a correction to the rotation frequency (the sum $\omega+\Omega$ yields the total oscillation frequency). Let us substitute Eq. (8) into system (6) and obtain the following expressions relating z_0, h_0, and Ω:

$$2i\Omega z_0\left[1+\tau^2(\omega+\Omega)^2\right] = \exp(-i\alpha) z_0\left[1-i\tau(\omega+\Omega)\right] - \exp(i\alpha) z_0 |z_0|^2 \left[1+i\tau(\omega+\Omega)\right], \tag{9}$$

$$z_0 = h_0\left[1+i\tau(\omega+\Omega)\right]. \tag{10}$$

Equation (9) has three solutions with respect to the complex quantity z_0.

The first solution corresponds to the totally asynchronous regime (TAR) when

$$z_0 = z_{\mathrm{as}} = 0, \qquad h_0 = h_{\mathrm{as}} = 0. \tag{11}$$

In this case, the parameter Ω is not specified and can take any value. This solution exists for all values of the control parameters τ, α, ω, and L. In this case, the phases $\varphi(x,t)$ of the elements are uniformly distributed at each point of the medium (see Figs. 1e and 1h).

The second solution corresponds to the totally synchronous regime (TSR) for which

$$z_0 = z_{\mathrm{s}} = 1, \quad h_0 = h_{\mathrm{s}} = \frac{1 - i(\omega+\Omega_{\mathrm{s}})\,\tau}{1+(\omega+\Omega_{\mathrm{s}})^2\tau^2}, \tag{12}$$

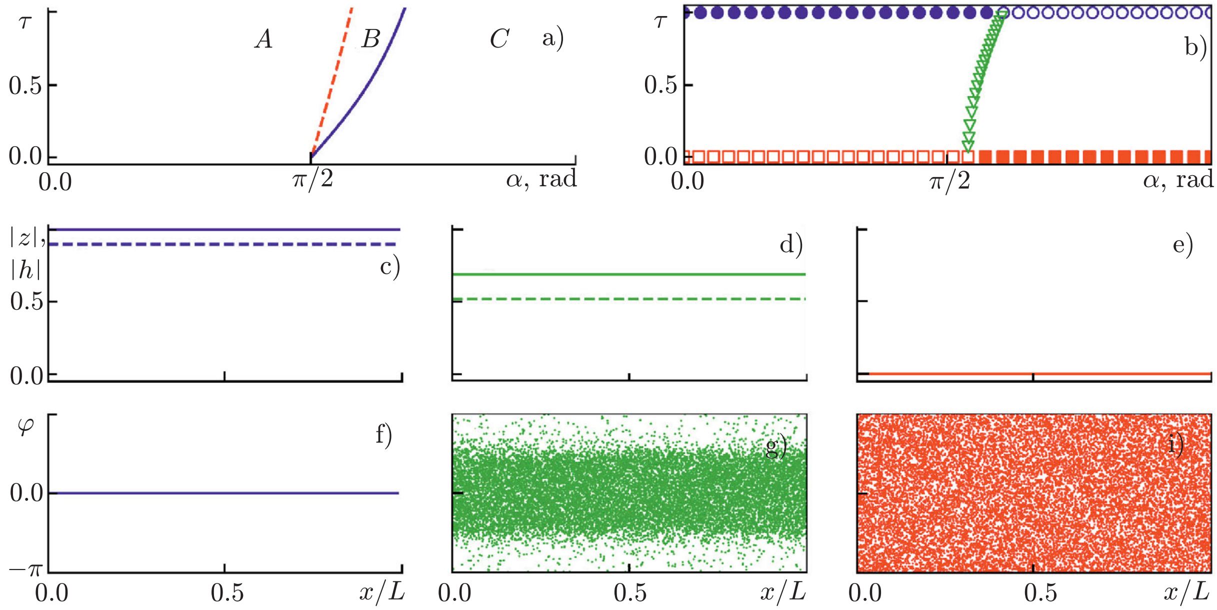

Fig. 1. Spatially homogeneous states of the system of Eqs. (1) and (2) for $\omega = 0.25$. (*a*) The parameter plane $(\alpha,\ \tau)$: the TSR is stable in the region $A \cup B$, the TAR is stable in the region $B \cup C$, and the PSR exists only in the region B and is unstable. (*b*) The bifurcation diagram of the spatially homogeneous regimes for $\omega = 0.25$ and $\tau = 0.5$. The blue circular markers denote the TSR, the green triangular markers denote the PSR, and the red quadratic markers denote the TAR. The solid and hollow markers denote the stable and unstable regimes, respectively. Panels *c*–*e* show the spatial profiles $|z(x)|$ (solid lines) and $|h(x)|$ (dashed lines) for spatially homogeneous solutions of system (6), while the panels *f*–*h* demonstrate the corresponding phase distributions $\varphi(x)$ of the system of Eqs. (1) and (2). The panels *c* and *f* show the TSR, the panels *d* and *g* show the PSR, and the panels *e* and *h* represent the TAR.

whereas the parameter $\Omega = \Omega_{\rm s}$ is determined as a real root of the cubic equation

$$\tau^2\Omega_{\rm s}^3 + 2\omega\tau^2\Omega_{\rm s}^2 + \left(1 + \tau\cos\alpha + \omega^2\tau^2\right)\Omega_{\rm s} + \omega\tau\cos\alpha + \sin\alpha = 0. \qquad (13)$$

Solution (12) also exists for any values of τ, α, ω, and L. Depending on the number of the real solutions of Eq. (13), there may exist either one or three TSRs with various values of the parameter $\Omega_{\rm s}$. In this case, the system-element phases coincide, such that $\varphi(x,t) = \varphi(t)$, and perform uniform rotation with the frequency $\omega + \Omega_{\rm s}$ (see Fig. 1*c* and 1*f*). The spatially homogeneous totally synchronous and asynchronous states, which correspond to solutions (12) and (11), exist for the system of Eqs. (1) and (2), and they were studied in [21, 24] for $\tau = 0$.

However, for $\tau > 0$ there appears a possibility of occurrence of the third regime, i.e., the partially synchronous regime (PSR), for which we have

$$z_0 = z_{\rm ps} = \sqrt{\frac{2(\tau\omega\sin\alpha - \cos\alpha)}{\tau\sin^2\alpha} - 1}, \qquad h_0 = h_{\rm ps} = \frac{1 - i(\omega + \Omega_{\rm ps})\tau}{1 + (\omega + \Omega_{\rm ps})^2\tau^2} z_{\rm ps}, \qquad (14)$$

where $\Omega_{\rm ps} = \tau^{-1}\cot\alpha - \omega$. The existence region of this solution is specified by the inequality $0 < z_{\rm ps} < 1$ (see Figs. 1*a* and 1*b*). In this case, the phases $\varphi(x,t)$ of the elements of the system demonstrate identical values of the correlation degree at each point of the medium (see Figs. 1*d* and 1*g*).

Figures 1*a* and 1*b* show the results of the studies of stability of the spatially homogeneous regimes for $\omega = 0.25$ and $\tau = 0.5$ on the basis of direct numerical simulation of the system of Eqs. (1) and (2) of the phase-oscillators (a more rigorous analysis is given below). Obviously, in this case, the PSR originates from the TAR with increasing parameter α and then merges with the TSR. In this case, the PSR existence region coincides with the bistability region of the TAR and the TSR.

3.2. Stability of the spatially homogeneous regimes

In this section, we consider the issue of stability of the above-described regimes with the same degree of phase correlation at each point of the medium. To this end, we perform linear analysis of stability of the corresponding solutions of averaged OA equations (6). Let us write the complex fields $Z(x,t)$ and $H(x,t)$ in the following form:

$$Z(x,t) = [z_0 + \mathcal{Z}(x,t)]\exp[i(\omega+\Omega)t], \qquad H(x,t) = [h_0 + \mathcal{H}(x,t)]\exp[i(\omega+\Omega)t], \tag{15}$$

where the functions $\mathcal{Z}(x,t)$ and $\mathcal{H}(x,t)$ are the perturbations of the homogeneous state (8), which are small in amplitude and periodic with respect to the coordinate x.

Substituting Eq. (15) into Eq. (6) and linearizing the results near z_0 and h_0, we obtain the system of linear partial differential equations:

$$\frac{\partial \mathcal{Z}}{\partial t} = -[i\Omega + \exp(i\alpha) z_0 h_0^*]\,\mathcal{Z} + [\exp(-i\alpha)\mathcal{H} - \exp(i\alpha) z_0^2 \mathcal{H}^*]/2,$$
$$\tau\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial^2 \mathcal{H}}{\partial x^2} - [1 + i\tau(\omega+\Omega)]\mathcal{H} + \mathcal{Z}. \tag{16}$$

Then we use a version of the standard procedure for analyzing the stability of spatiotemporal structures. Namely, we seek $\mathcal{Z}(x,t)$ and $\mathcal{H}(x,t)$ as the superposition of two orthogonal components in the form of plane waves

$$\mathcal{Z}(x,t) = a\exp(iKx)\exp(\Lambda t) + b^*\exp(-iKx)\exp(\Lambda^* t),$$
$$\mathcal{H}(x,t) = c\exp(iKx)\exp(\Lambda t) + d^*\exp(-iKx)\exp(\Lambda^* t), \tag{17}$$

where Λ is the complex growth rate, while the wave numbers $K = 2\pi k/L$ ($k = 0, 1, 2, \dots$) specify the spatial period of the perturbation mode. After the substitution of Eq. (17) into Eq. (16), we obtain the problem of the eigenvectors $\boldsymbol{\xi} = (a, b, c, d)^{\mathrm{T}}$ and the eigenvalues Λ of the matrix $\boldsymbol{\mathcal{P}}$:

$$\Lambda\boldsymbol{\xi} = \boldsymbol{\mathcal{P}}\boldsymbol{\xi}, \qquad \boldsymbol{\mathcal{P}} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}, \tag{18}$$

where the subscript T denotes transposition,

$$p_{11} = -i\Omega - \frac{1 + i\,(\omega+\Omega)\,\tau}{1 + (\omega+\Omega)^2\tau^2}\exp(i\alpha)z^2, \quad p_{13} = \frac{1}{2}\exp(-i\alpha), \quad p_{14} = -\frac{1}{2}\exp(i\alpha)z^2,$$
$$p_{22} = i\Omega - \frac{1 - i\,(\omega+\Omega)\,\tau}{1 + (\omega+\Omega)^2\tau^2}\exp(-i\alpha)z^2, \quad p_{23} = -\frac{1}{2}\exp(-i\alpha)z^2, \quad p_{24} = \frac{1}{2}\exp(i\alpha),$$
$$p_{31} = \frac{1}{\tau}, \quad p_{33} = -\frac{k^2 + 1 + i\,(\omega+\Omega)\,\tau}{\tau}, \quad p_{42} = \frac{1}{\tau}, \quad p_{44} = -\frac{k^2 + 1 - i\,(\omega+\Omega)\,\tau}{\tau},$$
$$p_{12} = p_{21} = p_{32} = p_{34} = p_{41} = p_{43} = 0.$$

Therefore, to determine the stability of the spatially homogeneous regimes, one should determine a set of eigenvalues $\Lambda_1, \Lambda_2, \Lambda_3, \mathrm{and}\Lambda_4$. If the real part of at least one number Λ_i ($i = 1, \dots, 4$) turns out to be positive, the small perturbations $\mathcal{Z}(x,t)$ and $\mathcal{H}(x,t)$ exponentially increase with time and the corresponding spatially homogeneous regime (8) is unstable. The stability of each of the three spatially homogeneous states is considered below.

3.3. Stability of the asynchronous regime

For the totally asynchronous regime (11), the characteristic polynomial of the eigenvalue problem (18) has the form

$$\left[2\tau\Lambda^2 + 2\left(1 + K^2 - i\tau\omega\right)\Lambda - \exp(i\alpha)\right]^2 = 0. \tag{19}$$

From Eq. (19), one can obtain the expressions relating the parameters of the system of Eqs. (6) to (7) and the wave number K, for which $\operatorname{Re}\Lambda = 0$, as

$$1 + K^2 = \begin{cases} \dfrac{\tau\sin\alpha[\omega - \sqrt{\omega^2 - (2\cos\alpha)/\tau}]}{2\cos\alpha}, & \alpha_1^* \le \alpha \le \alpha_2^*; \\ \dfrac{\tau\sin\alpha[\omega + \sqrt{\omega^2 - (2\cos\alpha)/\tau}]}{2\cos\alpha}, & \alpha_3^* \le \alpha < \pi/2, \quad \alpha_4^* \le \alpha < -\pi/2, \end{cases} \tag{20}$$

where the quantity $\alpha_1^* = \arccos(\tau\omega^2/2)$ for $\tau\omega^2 \le 2$, while for $\tau\omega^2 \ge 2$ it is a root of the transcendental equation $\tau\sin[\omega - \sqrt{\omega^2 - (2\cos\alpha)/\tau}] - 2\cos\alpha = 0$; α_2^* is a root of the equation $\sin\alpha - \omega - \sqrt{\omega^2 - (2\cos\alpha)\tau} = 0$; and α_3^* and α_4^* are the solutions of the equation $\tau\sin\alpha[\omega + \sqrt{\omega^2 - (2\cos\alpha)\tau}] - 2\cos\alpha = 0$, such that $\{\alpha_1^*, \alpha_2^*, \alpha_3^*\} > 0$ and $\alpha_4^* < 0$. Note that the first branch of Eq. (20) does not exist for all values of the parameters α and τ at $\omega > 0.5$.

Let us analyze Eq. (20) for various ω, α, and τ, where $\alpha > 0$. If $\omega < 0.5$, both branches of Eq. (20) exist for all α and τ. Figures 2*a*–*c* show that for $\tau\omega^2 \le 2$ the asynchronous-regime instability is caused by the linear modes with $K = 0$. For $\tau\omega^2 > 2$, in the interval (α_4^*, α_3^*), the instability is also caused by the short-wave modes with large values of K. In this case, for fixed values of the parameters ω and τ, the asynchronous regime is unstable for $\alpha < \alpha_2^*$ and stable for $\alpha > \alpha_2^*$. For $0.5 < \omega < 1.0$, the instability of the asynchronous regime is caused by the short-wave modes (see Figs. 2*d*–2*g*). For $\alpha < \alpha_4^*$ and in the interval (α_3^*, α_2^*) (when the first branch of Eq. (20) exists), the linear mode with $K = 0$ is also unstable. For $\omega > 1$, the instability of the asynchronous regime is everywhere determined by the short-wave modes (see Figs. 3*a* and 3*b*). Therefore, for $\omega > 0.5$, the asynchronous regime is unstable for $\alpha < \pi/2$ and stable for $\alpha > \pi/2$, as in the case $\tau = 0$.

For the negative values of the phase-shift parameter α at all ω, the instability of the asynchronous regime is determined by the linear mode with $K = 0$. This regime is unstable for $\alpha > \alpha_4^*$ and stable for $\alpha < \alpha_4^*$ (see Fig. 3*c* and 3*d*).

3.4. Stability of the totally synchronous regime

For the totally synchronous regime (12), the characteristic polynomial of the eigenvalue problem (18) has the form

$$(\Lambda - \Lambda_1)(\Lambda^3 + S_2\Lambda^2 + S_1\Lambda + S_0) = 0, \tag{21}$$

where

$$\Lambda_1 = \frac{-\cos\alpha + (\omega + \Omega_\mathrm{s})\,\tau\sin\alpha}{1 + (\omega + \Omega_\mathrm{s})^2\,\tau^2},$$

and the coefficients S_0, S_1, and S_2 are the functions of the control parameters of the studied system and the wave number, $S_i = S_i(\omega, \alpha, \tau, K^2)$, such that $S_0(\omega, \alpha, \tau, 0) = 0$. Numerical analysis of the spectrum of eigenvalues Λ shows that the instability of the modes which are determined by the wave number K can appear in several ways. First, a linear mode with $K = 0$ can become unstable and then the synchronous regime becomes unstable for any length L of the medium. Second, the long-wave modes with $0 < K < K_\mathrm{s}^*$ can become unstable and then for the asynchronous regime there exists such a critical length $L_\mathrm{s}^* = 2\pi/K_\mathrm{s}^*$

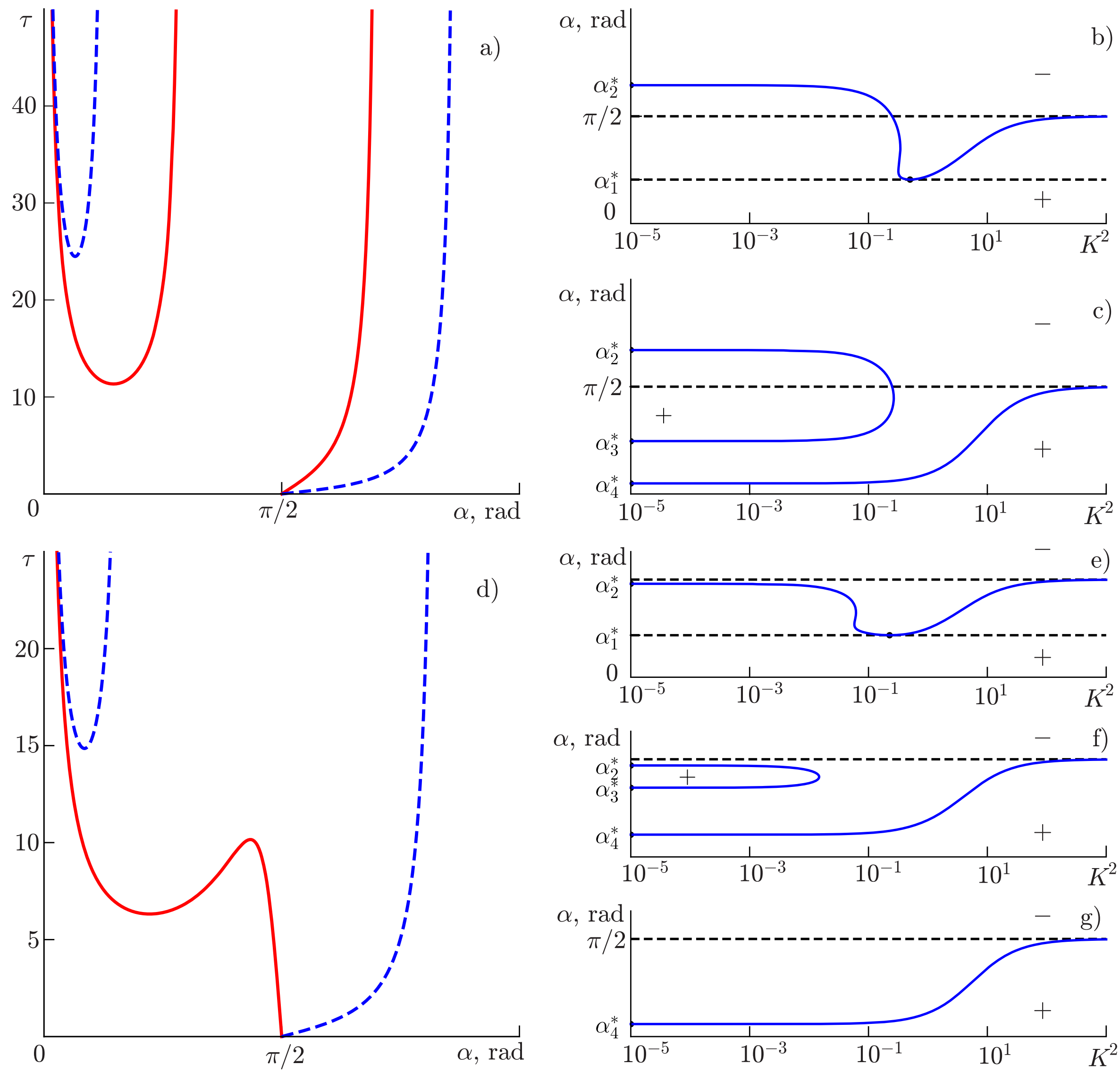

Fig. 2. The left-hand column: the TAR stability as a function of the parameters α and τ for $\omega = 0.4$ (*a*) and $\omega = 0.51$ (*d*) if $\alpha > 0$. The red solid lines denote the set $|z_{\rm ps}| = 0$, where the PSR originates from the TAR. The blue dashed lines show the set $|z_{\rm ps}| = 1$, where the PSR merges with the TSR. The right-hand column: the dependences (20) (blue curves) for $\omega = 0.4$ at $\tau = 10$ (*b*) and $\tau = 20$ (*c*), and for $\omega = 0.51$ at $\tau = 6.0$ (*e*), $\tau = 8.0$ (*f*), and $\tau = 12.0$ (*g*). The signs + and − denote the sign of the quantity $\max \operatorname{Re} \Lambda(K^2, \alpha)$, which characterizes the TAR stability in the corresponding subregion (K^2, α).

that the regime is stable for $L < L_{\rm s}^*$ and unstable for $L > L_{\rm s}^*$. Note that in the case of intersection of the imaginary axis by the eigenvalue Λ_1 (which corresponds to the excitation of modes for all K since Λ_1 is independent of K), the totally synchronous regime gives rise to the partially synchronous solution (14).

Figure 4*a* shows the stability region of the synchronous regime for $\omega = 0$. In this case, for small τ, there exists the only synchronous solution whose instability appears because of the linear-mode excitation once the partially synchronous regime appears (see Fig. 4*b*). As τ increases, there may additionally appear two synchronous regimes with different frequencies $\Omega_{\rm s}$, which are unstable in this case (see Fig. 4*c*).

As ω increases, a region in which one or several synchronous regimes are stable only in some intervals of the lengths L appears on the parameter plane. In this case, the existence region of three synchronous regimes is realized for $\tau \to +\infty$ (see Figs. 4*d* and 5*a*). Should the eigenfrequency ω increase further, this region becomes limited (Fig. 5*e*) and then disappears (Fig. 6). It should be noted that the synchronous

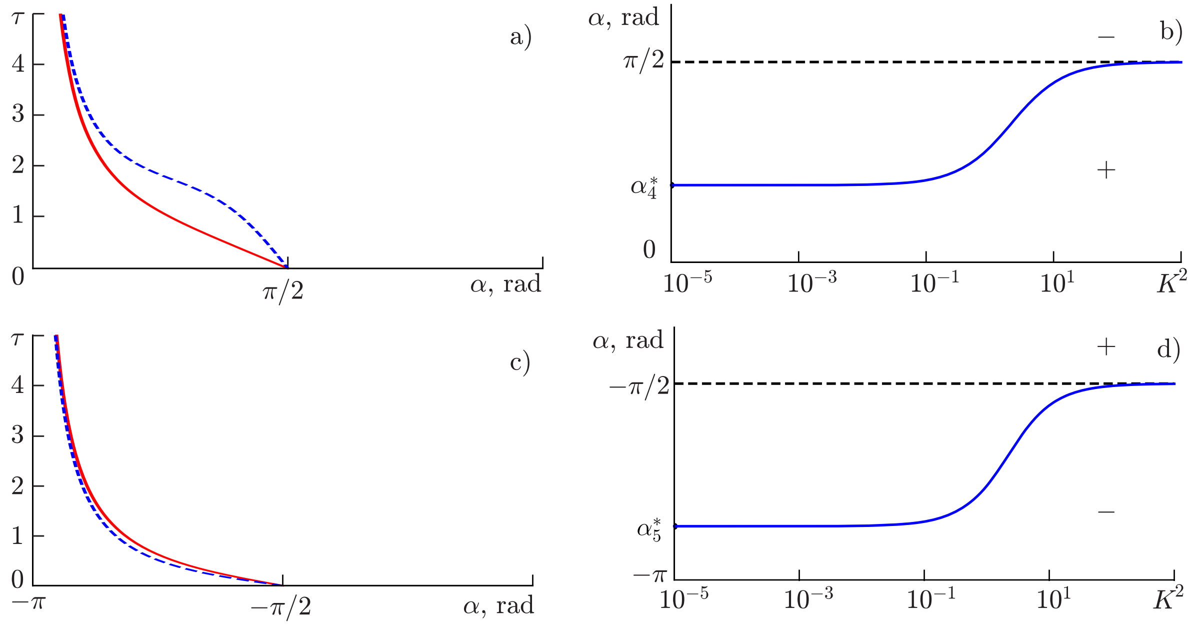

Fig. 3. Same as in Fig. 2 but for $\omega = 1.25$ at $\alpha > 0$ (*a* and *b*) and $\alpha < 0$ (*c* and *d*). On panels *b* and *d*, $\tau = 1.5$.

regime is stable in the case of repelling coupling for the negative phase shift $-\pi < \alpha \leq -\pi/2$ in a wide range of control parameters. For $\omega > 1$ and the large values of the diffusion time scale τ at $\alpha > 0$, the synchronous regime is almost always unstable (see Figs. 5*e* and 6).

3.5. Stability of the partially synchronous regime

For the partially synchronous regime (14), the characteristic polynomial of the eigenvalue problem has the form

$$K^4 + P_3K^3 + P_2K^2 + P_1K + P_0 = 0, \tag{22}$$

where the coefficients P_0, P_1, P_2, and P_3 are the functions of the control parameters of the studied system and the eigenvalue Λ, i.e., $P_i = P_i(\omega, \alpha, \tau, \Lambda)$. The numerical analysis of the equation $P_0(\omega, \alpha, \tau, \Lambda) = 0$ in the region of determining the partially synchronous regime has shown the existence of the root Λ_{ps} with the real part $\operatorname{Re}\Lambda_{\text{ps}} > 0$. Therefore, the linear mode with the wave number $K = 0$ is always unstable. Thus, the partially synchronous solution is always unstable (see Fig. 7).

4. SPATIALLY INHOMOGENEOUS STATES

In this section, we consider the issue of the existence and stability of spatially inhomogeneous states, which correspond to the stationary solutions of the OA equations (6) and Eq. (7). Such regimes are characterized by different degrees of the element correlation with respect to the phases at various points of the studied medium of the phase oscillators described by Eqs. (1) and (2). One of the most interesting and nontrivial inhomogeneous states is the chimera state, having the form of clusters of totally synchronous oscillators ($|Z(x,t)| = 1$), which coexist with the regions of asynchronous dynamics ($|Z(x,t)| < 1$) [7, 17, 19]. Within the framework of the studied model, the character of stability of the totally synchronous and asynchronous states and the region of existence of the partially synchronous regime, which have been described in the previous section, considerably influence the possibility of realizing chimera regimes and their properties.

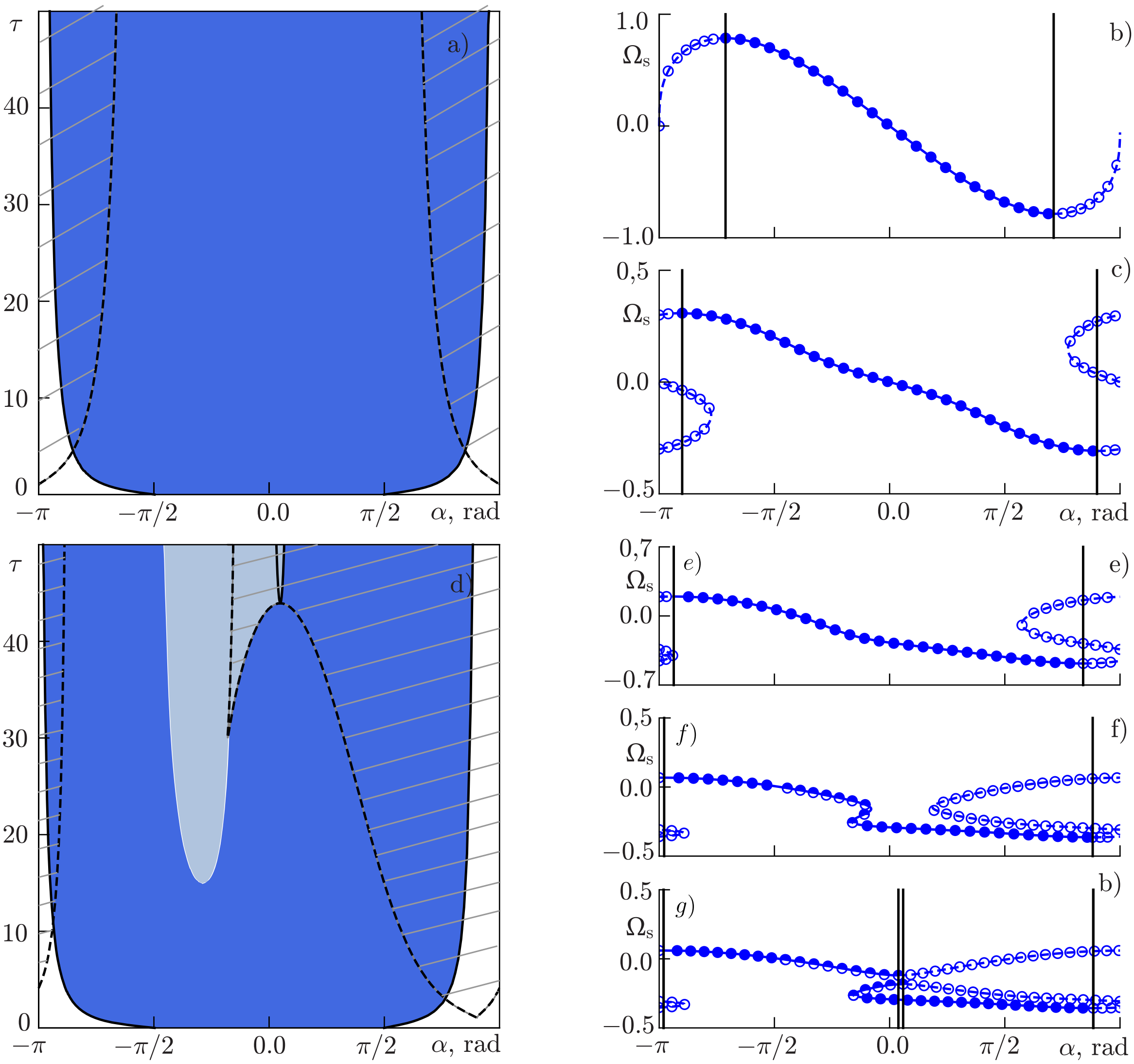

Fig. 4. The left-hand column: the TSR stability as a function of the parameters α and τ for $\omega = 0.0$ (*a*) and $\omega = 0.3$ (*b*). The black solid lines denote the set $|z_{\text{ps}}| = 1$, where the PSR originates from the TSR. The black dashed lines show the boundary of the region (the gray shading) in which three TSRs coexist. In the dark-blue region, the TSR is stable for any length L of the medium. In the bright blue region, the TSR is stable in the interval $(0, L_{\text{s}}^{*})$ of this length . In the white region, the TSR is unstable. The right-hand column: the dependences $\Omega_{\text{s}}(\alpha)$ for $\omega = 0.0$ at $\tau = 1.0$ (*b*) and $\tau = 10.0$ (*c*), and for $\omega = 0.3$ at $\tau = 10.0$ (*e*), and $\tau = 40.0$ (*f*), $\tau = 46.0$ (*g*). The totally solid markers denote regimes, which are stable for any L, the hollow markers denote unstable regimes and the half-painted markers show that the TSR is stable in the interval $(0, L_{\text{s}}^{*})$. The black vertical lines are used to show the values of α for which $|z_{\text{ps}}| = 1$.

4.1. Stationary solutions of the Ott–Antonsen equations

The stationary spatially inhomogeneous solutions of the OA equations (6) and Eq. (7), which are uniformly rotating with the frequency $\omega + \Omega$, can be written as follows:

$$Z(x,t) = z(x)\exp[i\,(\omega + \Omega)\,t], \quad H(x,t) = h(x)\exp[i\,(\omega + \Omega)\,t], \tag{23}$$

where Ω is the parameter determining the rotation frequency, and $z(x)$ and $h(x)$ are the complex functions of the coordinate x, which characterize the spatial profile of the inhomogeneous regime. The solutions given

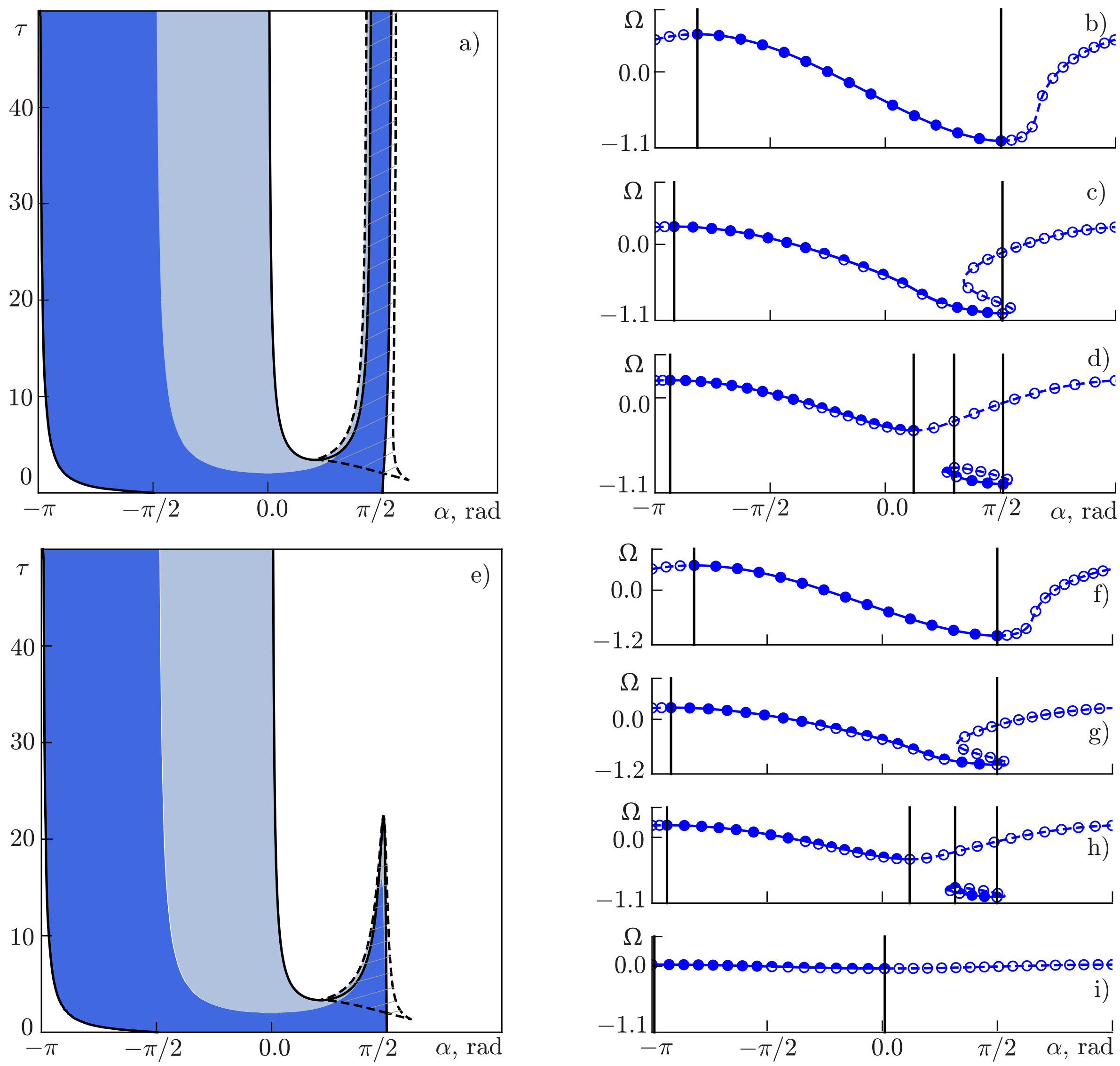

Fig. 5. Same as in Fig. 4 but for $\omega = 0.99$ (*a–d*) and $\omega = 1.001$ (*e–i*): $\tau = 1.0$ (*b*), $\tau = 3.0$ (*c*), $\tau = 4.0$ (*d*), $\tau = 1.0$ (*e*), $\tau = 3.0$ (*g*), $\tau = 4.0$ (*h*), and $\tau = 40.0$ (*i*).

by Eq. (23) are called stationary in what follows assuming that the spatial profiles $z(x)$ and $h(x)$ are time-constant. Substituting Eq. (23) into the Eqs. (6) and (7), we arrive at the following system, which consists of an algebraic equality and an ordinary differential equation with respect to $z(x)$ and $h(x)$:

$$\exp(i\alpha)h^* z^2 + 2i\Omega z - \exp(-i\alpha)h = 0, \tag{24a}$$

$$h'' - [1 + i\,(\omega + \Omega)\,\tau]\,h = -z. \tag{24b}$$

Hereinafter, the prime denotes the derivative with respect to the coordinate x.

First, let us express $z(x)$ from the quadratic equation (24*a*). This equation determines the order parameter $z(x)$ of a set of oscillators, which move under the action of the field $h(x) = r(x)\exp[i\theta(x)]$. For each point x, the solution is determined by the coupling between r and Ω (it is assumed that the parameter $\Omega < 0$). Then, on the basis of the physical meaning of the order parameter ($|z| \leq 1$), we choose only one solution of Eq. (24*a*):

$$z = \frac{-(i\Omega - \sqrt{r^2 - \Omega^2})}{r}\exp[i\,(\theta - \alpha)], \quad |r| \geq |\Omega|, \tag{25a}$$

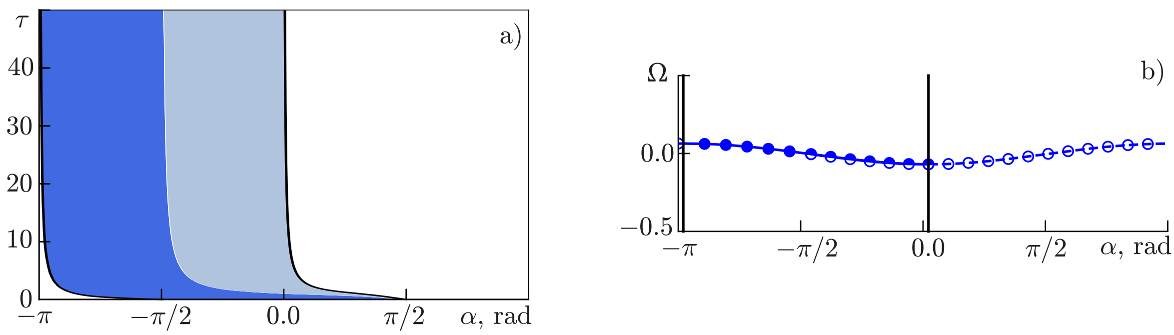

Fig. 6. Same as in Fig. 4 but for $\omega = 1.5$. On panel *b*, $\tau = 10.0$.

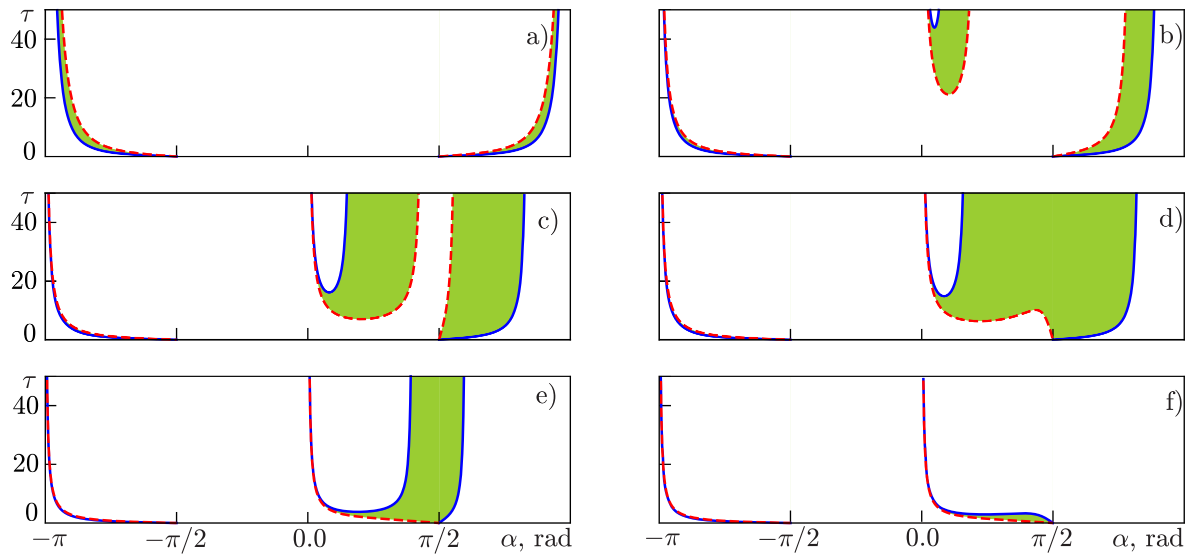

Fig. 7. The PSR stability. The solid blue line denotes the set $|z_{\rm ps}| = 1$ where the PSR merges with the TSR. The red dashed line shows the set $|z_{\rm ps}| = 0$ where the PSR merges with the TAR. The green color is used to denote the PSR existence region in which this regime is everywhere unstable. The parameter $\omega = 0.0$ (*a*), $\omega = 0.3$ (*b*), $\omega = 0.49$ (*c*), $\omega = 0.51$ (*d*), $\omega = 0.95$ (*e*), and $\omega = 1.05$ (*f*).

$$z = \frac{-i(\Omega + \sqrt{\Omega^2 - r^2})}{r} \exp[i\,(\theta - \alpha)], \quad |r| < |\Omega|. \tag{25b}$$

If $|r| \geq |\Omega|$, then the phases of the elements are synchronized and $|z| = 1$. In the opposite case, the oscillators are partially synchronized such that $0 < |z| < 1$. Let us substitute Eq. (25) into Eq. (24*b*). Although $h(x)$ is a complex function, by virtue of the invariance of the relative phase shift $\theta(x) \to \theta(x) + \theta_0$, the obtained equation can be written in the form of an ordinary differential equation of the third order with respect to the real functions $r(x)$ and $q(x) = r^2(x)\theta'(x)$ as

$$r'' = r + \frac{q^2}{r^3} + \frac{\Omega}{r}\sin\alpha - \frac{\sqrt{r^2 - \Omega^2}}{r}\cos\alpha, \tag{26}$$

$$q' = \tau(\omega + \Omega)r^2 + \Omega\cos\alpha + \sqrt{r^2 - \Omega^2}\sin\alpha \tag{27}$$

in the region where $|r| \geq |\Omega|$, and as

$$r'' = r + \frac{q^2}{r^3} + \frac{\Omega + \sqrt{\Omega^2 - r^2}}{r}\sin\alpha, \tag{28}$$

$$q'\tau(\omega + \Omega)r^2 + (\Omega + \sqrt{\Omega^2 - r^2})\cos\alpha \tag{29}$$

at the points with $|r| < |\Omega|$.

Therefore, the problem of seeking the stationary solutions of the OA equations, which satisfy periodic boundary conditions, is reduced to the problem of seeking the periodic orbits of the system of Eqs. (26)–(29) under the conditions $r(x+L) = r(x)$ and $q(x+L) = q(x)$. In this case, it is convenient to use the frequency Ω as a parameter of the system of Eqs. (26)–(29), rather than fix the period L. In what follows, we seek the periodic trajectories of the system of Eqs. (26)–(29) with the length L depending on Ω. After the dependence inversion, one can obtain the function $\Omega = \Omega(L)$, which relates the parameter Ω of the solution of the type of Eq. (23), to the length parameter L of the studied system. The system of Eqs. (26)–(29) is also invariant with respect to the involution transformation $(x \to -x, r \to r, q \to -q)$, which allows us to confine ourselves to searching for only periodic solutions that are symmetric with respect to the center of the interval $[0, L)$ and satisfy the conditions $r(0) = r(L)$, $r'(0) = r'(L) = 0$, and $q(0) = q(L) = 0$. These solutions should be corresponded to the spatial profiles $z(x)$ and $h(x)$, which, along with the appropriate values of the parameter Ω, determine the form of the stationary structures (23). Note that the periodic solutions of the system of Eqs. (26)–(29) determine only the existence of the stationary spatiotemporal regimes of the type of Eq. (23). To perform a linear analysis of stability of these solutions, one should consider the system of partial differential equations (6) with boundary conditions (7). Linearizing equations (6) in the neighborhood of the solutions (23), where the solution can be represented in the form

$$Z(x,t) = \Big[z(x) + \mathcal{Z}(x,t)\Big] \exp[i\,(\omega + \Omega)\,t], \qquad H(x,t) = \Big[h(x) + \mathcal{H}(x,t)\Big] \exp[i\,(\omega + \Omega)\,t], \tag{30}$$

we obtain the system of partial differential equations with respect to the perturbations $\mathcal{Z}(x,t)$ and $\mathcal{H}(x,t)$, which are relatively small in amplitude and periodic with respect to the coordinate x:

$$\frac{\partial}{\partial t}\boldsymbol{\zeta}(x,t) = \hat{\mathbf{L}}\boldsymbol{\zeta}(x,t), \tag{31}$$

where $\boldsymbol{\zeta}(x,t) = (\operatorname{Re}\mathcal{Z}, \operatorname{Im}\mathcal{Z}, \operatorname{Re}\mathcal{H}, \operatorname{Im}\mathcal{H})$ and $\hat{\mathbf{L}}$ is the linear operator [17, 19]. In this case, the stability of solution (23) is determined by the spectrum of eigenvalues of the linear operator $\hat{\mathbf{L}}$. However, this problem is characterized by high calculational complexity because it requires high degree of the spatial discretization ($N > 10^4$) to reveal the spectrum components which are located near the imaginary axis and responsible for the possible development of instability of the studied structures. Therefore, in what follows, the stability of the considered inhomogeneous solutions is determined on the basis of the long-term numerical simulation within the framework of the phase-oscillator system of Eqs. (1) and (2) for the times of the order of 10^4.

4.2. Chimera states

The chimera regimes are spatial structures with coexisting clusters of the totally synchronous oscillators and the regions of the partially synchronous elements. The study of such regimes using the OA reduction has allowed us to determine the key bifurcation mechanisms of the chimera appearance in both the systems of the globally coupled oscillators and the spatially distributed media [7, 17, 33, 34]. A scenario of the chimera-state appearance is considered below as an example.

Let us consider the parameter region (ω, τ, α), where the stable TSR coexists with the PSR (always unstable). For example, such a situation takes place for $\omega = 0.25$, $\tau = 0.5$, and $\alpha = 1.74$. Figure 8 shows the bifurcation diagram of the spatially homogeneous and inhomogeneous stationary regimes on the plane (L, Ω) for the above-given parameter values. Here, there exist the only TSR with $\Omega = \Omega_\mathrm{s} \approx -0.934$ and the TAR, which are stable for any length L of the medium (see Fig. 8*b*). On the plane (L, Ω), the curve CH of the chimera states asymptotically tends to the branch S of the totally synchronous regime for $\Omega \to \Omega_\mathrm{s} + 0$. In this case, the medium length $L \to \infty$ and the length of the chimera subregion, in which the partially synchronous dynamics occurs, tends to zero. For $\Omega_\mathrm{s} < \Omega < \Omega_1$, where $\Omega_1 \approx -0.825$, the chimera is unstable and evolves to either totally synchronous regime or the stable chimeras with $\Omega > \Omega_1$ (see Fig. 8*c*). Then, in

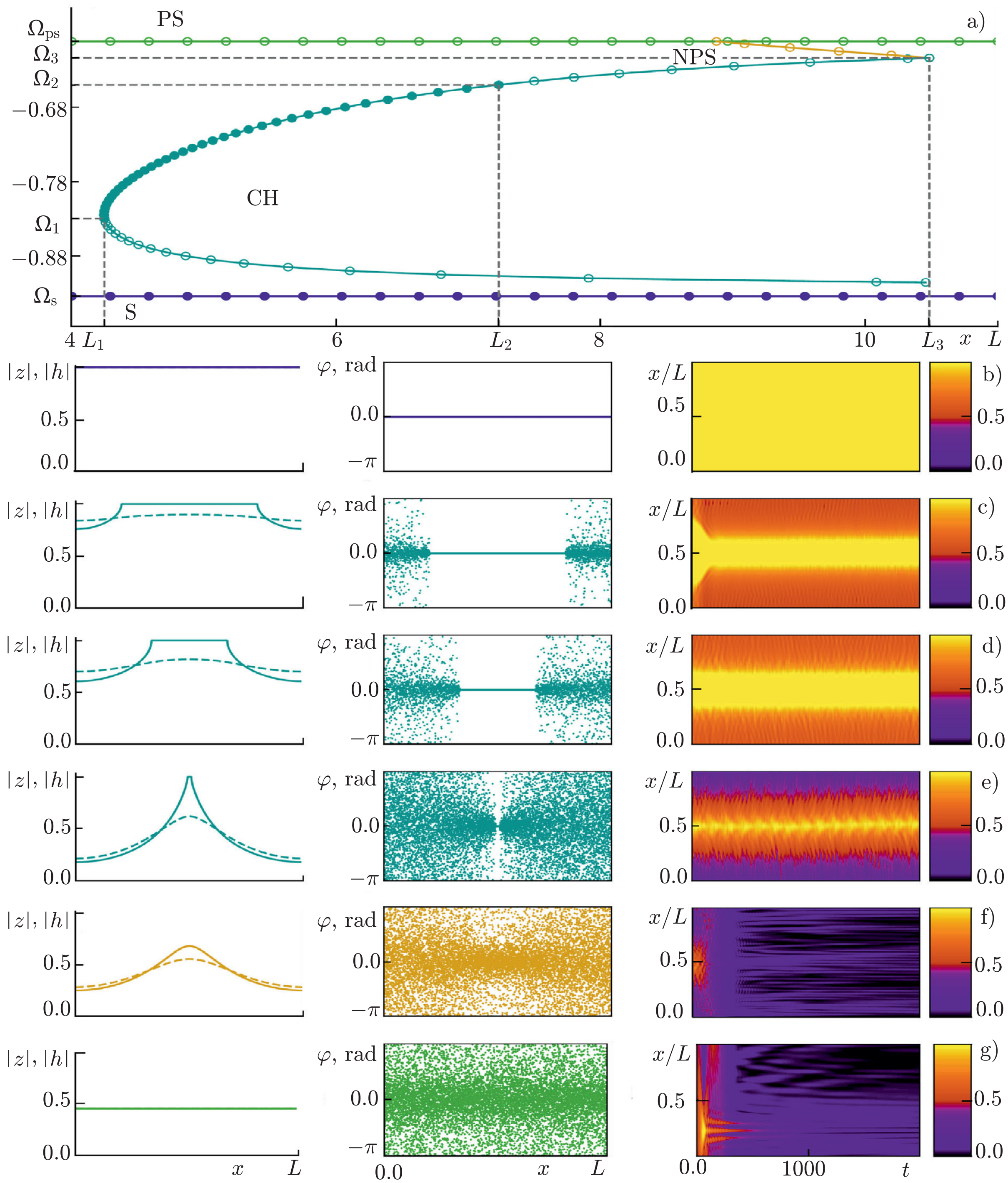

Fig. 8. (*a*) Bifurcation diagram of the spatially homogeneous and inhomogeneous stationary regimes for $\omega = 0.25$, $\tau = 0.5$, and $\alpha = 1.74$. Branch S corresponds to the TSR, PS to the PSR, CH to the chimera regime, and NPS to the inhomogeneous PSR. The solid (hollow) markers denote stable (unstable) regimes. (*b*–*g*) The left-hand column shows the spatial profiles $|z(x)|$ (solid lines) and $|h(x)|$ (dash lines) for the spatially inhomogeneous solutions of the system of Eqs. (6) and the central column shows the corresponding phase distributions $\varphi(x)$, which are used as the initial conditions for numerical simulation. The right-hand column shows the spatiotemporal diagram $|Z(x,t)|$, which was obtained as a result of direct numerical simulation of the system (1) and (2). (*b*) The stable TSR. (*c*) Unstable chimera $\to$ stable chimera. (*d*) Stable chimera. (*e*) Unstable chimera $\to$ breather chimera. (*f*) The inhomogeneous PSR $\to$ the TAR. (*g*) The PSR $\to$ the TAR.

the frequency interval (Ω_1, Ω_2), where $\Omega_2 \approx -0.635$, the stable chimera regimes are observed (see Fig. 8d). For $\Omega = \Omega_2$, the stationary chimera loses its stability and, for $\Omega_2 < \Omega < \Omega_3 \approx -0.617$, one can observe a transition to the unstationary breather chimera regime when the absolute values of the averaged fields $Z(x,t)$ and $H(x,t)$ demonstrate periodic dynamics at each point of space (see Fig. 8e). Note that the region of the existence of breather chimera states in the studied system of Eqs. (1) and (2) with allowance for the finite time of the diffusion process ($\tau > 0$) is considerably wider compared with the Kuramoto–Battogtokh canonical model ($\tau = 0$) [19]. A similar effect has been observed in [20, 21] in the case of the phase shift α depending on the correlation degree of the oscillator phases (i. e., on $|H|^2$). Then, for $\Omega = \Omega_3$, the chimera structure disappears since the size of the synchronous cluster turns to zero. In the interval $\Omega_3 < \Omega < \Omega_{\rm ps}$, there exists an unstable inhomogeneous partially synchronous solution (see Fig. 8f), which, for $\Omega = \Omega_{\rm ps}$, "sticks" into the branch of partially synchronous homogeneous states, which are also always unstable and go over to the TAR in the resolution process (see Fig. 8g).

Therefore, for the considered values of the control parameters, the phase-oscillator system of Eqs. (1) and (2) has the TSR, which is stable for any length of the medium. A stable stationary chimera with the rotation frequency $\omega + \Omega_{\rm ch}$ of the coherent cluster, where $\Omega_s < \Omega_{\rm ch} < \Omega_{\rm ps}$, can be realized in the length interval $L_1 < L < L_2$ whereas the breather chimera regime occurs for $L_2 < L < L_3$. In this case, the coexistence of the totally synchronous and partially synchronous spatially homogeneous regimes is considered to be the property which determines the possibility of existence of the chimera regime.

5. CONCLUSIONS

Let us formulate the main results which have been presented in this work. We have considered an ensemble of identical phase oscillators. The coupling in the system is ensured using an "intermediary," namely, a diffusion field, which is created by the oscillators and influences them. The basic attention has been paid to studying the properties of the spatially homogeneous (from the viewpoint of the local order parameter) regimes and their influence on the formation of the chimera inhomogeneous states. Using the OA reduction procedure, we have obtained dynamical equations with respect to the mesoscopic fields of the system. On the basis of the analysis of these equations, it has been established that the totally synchronous, asynchronous, and partially synchronous regimes can occur in the system. The parameter regions, in which three totally synchronous regimes with different rotation frequencies can coexist, have been determined. The synchronous regime can be stable, stable in a certain length interval in the case of a long-wave instability, and unstable because of the short-wave or linear-mode instability. By analogy, the asynchronous regime loses its stability simultaneously for all possible lengths of the medium because of the short-wave or zero-harmonic instability. The region of the existence of the partially synchronous regime, which is, however, always unstable, has been established. As a rule, the region of existence of the partially synchronous regime coincides with the region of bistability of the totally synchronous and asynchronous regimes.

In this work, we have also presented an original method for seeking the spatially inhomogeneous regimes, which, in particular, comprise chimeras. It has been shown that these states correspond to the periodic trajectories of an auxiliary system of the differential equations of the third order. The scenario of appearance of the stable chimera regimes from the spatially homogeneous states has been described. The proposed method allows one to study chimeras in an infinite medium. The detailed analysis of the chimera solitons will be given elsewhere.

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