Chapter 6
Non-pairwise Interaction in Oscillatory Ensembles: from Theory to Data Analysis

Arkady Pikovsky and Michael Rosenblum

Abstract In this chapter, we briefly review several cases when non-pairwise interaction naturally appears in oscillatory networks. First, we analyze globally coupled ensembles of phase oscillators. We demonstrate that nonlinear high-order mean-field coupling is equivalent to a hypernetwork with non-pairwise interactions of the population elements. Next, we consider small networks of limit-cycle oscillators. We show that pairwise interaction in the state variables description results in non-pairwise interaction on the level of phase dynamics, if one goes beyond the first order in the weak-coupling phase reduction. Finally, we discuss the implications for recovery of the network connectivity in terms of the phase dynamics from observations.

6.1 Introduction

Phase reduction is widely and efficiently exploited to investigate dynamics of interacting self-sustained oscillators [1, 2]. The main results of this approach can be summarized as follows. Consider a unit with frequency ω driven by a force with close frequency $\nu \approx \omega$. If the forcing is weak enough so that deviations of the state space trajectory from the limit cycle are small, then in the first approximation in the forcing amplitude, the dynamics of the phase is decoupled of that of the amplitudes, and obeys the equation

$$\dot{\phi} = \omega + Q(\phi, \psi), \tag{6.1}$$

where ϕ, ψ are the phases of the oscillator and the force, respectively, and Q is the coupling function. If the norm of Q is much smaller than ω, another approximation—averaging over the oscillation period—provides a description in terms of phase differences:

A. Pikovsky · M. Rosenblum (✉)
University of Potsdam, Potsdam, Germany
e-mail: mros@uni-potsdam.de

A. Pikovsky
e-mail: pikovsky@uni-potsdam.de

F. Battiston and G. Petri (eds.), *Higher-Order Systems*, Understanding Complex Systems,
https://doi.org/10.1007/978-3-030-91374-8_6

$$\dot{\phi} = \omega + \bar{Q}(\psi - \phi). \tag{6.2}$$

In a particular case when the forcing $F(t)$ of the oscillator is scalar and independent of the state of the oscillator, one can write the phase equation (6.1) in the Winfree form [3]:

$$\dot{\phi} = \omega + S(\phi)F(t), \tag{6.3}$$

where $S(\phi)$ is the phase sensitivity function, also known as the phase response curve.

A generalization to large oscillatory networks typically implies that interaction is pairwise and additive. Another standard assumption is that the form of the coupling function is the same for all pairs. The simplest case of equally strong sine-like coupling represents the famous Kuramoto-Sakaguchi model [4, 5]:

$$\dot{\phi}_k = \omega_k + \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_k + \beta), \tag{6.4}$$

where ϵ is the interaction strength, N is the population size, k is the oscillator index, and β is a phase shift. In terms of the complex order parameter $Z = N^{-1}\sum_{k=1}^{N} e^{i\phi_k}$, often called the mean field, the model reads

$$\dot{\phi}_k = \omega_k + \epsilon \mathrm{Im}\left(Ze^{i(-\phi_k+\beta)}\right). \tag{6.5}$$

Numerous generalizations of the Kuramoto model [6–8] also rely on the assumption of pairwise interaction. In this Chapter, we go beyond this assumption and consider general mean-field coupled systems. We demonstrate that higher-order mean-field coupling naturally results in the interaction of oscillatory triplets, quadruplets, etc. In this sense the network of the interactions is hypernetwork. Furthermore, we show that such high-order interaction naturally appears in the phase description of pairwise- but non-weakly coupled oscillators. Finally, we discuss the implication for data analysis, namely for the reconstruction of the network connectivity from observations.

6.2 Theory of Higher-Order Mean-Field Phase Coupling

6.2.1 General Nonlinear Mean-Field Coupling

Here we follow [9, 10] and outline a general approach to nonlinear mean-field coupling in the context of phase dynamics. To simplify the presentation, we assume that there are M populations of oscillators, each population labeled by index $n = 1, \ldots, M$. All elements of a population have the same natural frequencies ω_n; all other properties of the oscillators are assumed to be identical across populations. The dynamics of the phase in the first-order approximation in coupling is, according to (6.3)

$$\dot{\phi}_n[a] = \omega_n + \epsilon F_n S(\phi_n[a]), \tag{6.6}$$

where $\phi_n[a]$ denotes the phase of the ath oscillator within the nth population, $S(\phi) = \sum_p s_p \exp[ip\phi]$ is the phase sensitivity function, and F_n is the force from all other oscillators acting on elements $\phi_n[a]$ [11]. The equations are valid if parameter ϵ is small. We assume a mean-field coupling, what means that the force F is a function of complex mean fields of the populations. These complex mean fields (a.k.a. Kuramoto-Daido order parameters) are defined as

$$Z_k^{(n)} = \langle e^{ik\phi_n[a]} \rangle\big|_a = N_n^{-1} \sum_{a=1}^{N_n} e^{ik\phi_n[a]} \tag{6.7}$$

where averaging is over all N_n units in population n. We assume that the force F_n can be represented as power series in $Z_k^{(n)}$:

$$F_n = \sum_{k,m} h_{k,m}^{(n)} Z_k^{(m)} + \sum_{k,m,l,s} g_{k,m;l,s}^{(n)} Z_k^{(m)} Z_l^{(s)} + \sum_{k,m;l,s;j,r} d_{k,m;l,s;j,r}^{(n)} Z_k^{(m)} Z_l^{(s)} Z_j^{(r)} + \dots \tag{6.8}$$

where we explicitly write linear, quadratic, and cubic terms. Substitution of (6.8) and (6.7) in (6.6) already yields equations with coupling terms containing complex combinations of phase variables. However, not all of these terms are really relevant and lead to essential effects; to reveal important terms one has to perform an averaging over the basic oscillation period.

To perform averaging, it is convenient to introduce slow phases according to $\varphi_n = \phi_n - \omega_n t$. (For brevity, we omit the index for individual oscillators.) Complex order parameters $z_k^{(n)}$ for these variables are also slow. They are expressed in terms of original order parameters (6.7) as follows

$$z_k^{(n)} = \langle e^{ik\varphi_n} \rangle\big| = Z_k^{(n)} e^{-ik\omega_n t}. \tag{6.9}$$

The equations for the slow phases read

$$\begin{aligned}\dot{\varphi}_n =& \epsilon \sum_p s_p e^{ip\varphi_n} e^{ip\omega_n t} \\ & \Big[\sum_{k,m} h_{k,m}^{(n)} z_k^{(m)} e^{ik\omega_m t} + \sum_{k,m,l,s} g_{k,m;l,s}^{(n)} z_k^{(m)} z_l^{(s)} e^{i(k\omega_m + l\omega_s)t} + \\ & + \sum_{k,m;l,s;j,r} d_{k,m;l,s;j,r}^{(n)} z_k^{(m)} z_l^{(s)} z_j^{(r)} e^{i(k\omega_m + l\omega_s + j\omega_r)t} + \dots \Big].\end{aligned} \tag{6.10}$$

One can see that some terms on the r.h.s. contain explicit time dependence - these are fast terms. Other terms either do not depend explicitly on time, or contain a small frequency mismatch in the exponent—these are slow terms. It is convenient to work with exact resonances, therefore one shifts slightly natural frequencies (by terms of order ϵ), so that the combinations of these modified frequencies (like $k\omega_m +$

$p\omega_n$) vanish exactly. Due to these modifications, small terms $\sim \epsilon\delta\omega_n$ appear in the dynamics. Averaging means omitting fast terms, what leads to generic averaged phase coupled equations

$$\begin{aligned}\dot{\varphi}_n =& \epsilon\delta\omega_n + \epsilon \sum_{p;k,m} s_p h^{(n)}_{k,m} z^{(m)}_k \exp[ip\varphi_n]\Delta(k\omega_m + p\omega_n) + \\ & \epsilon \sum_{p;k,m,l,s} s_p g^{(n)}_{k,m;l,s} z^{(m)}_k z^{(s)}_l \exp[ip\varphi_n]\Delta(k\omega_m + l\omega_s + p\omega_n) + \\ & \epsilon \sum_{p;k,m;l,s;j,r} s_p d^{(n)}_{k,m;l,s;j,r} z^{(m)}_k z^{(s)}_l z^{(r)}_j \exp[ip\varphi_n]\Delta(k\omega_m + l\omega_s + j\omega_r + p\omega_n) + \dots\end{aligned} \tag{6.11}$$

where $\Delta(\omega) = 1$ if $\omega = 0$ and $\Delta(\omega) = 0$ otherwise.

Below we discuss different cases leading to pairwise and multiple couplings of the network elements in the phase approximation.

6.2.2 *One Population of Oscillators*

The simplest case is when there exist only one population of oscillators. In the context of Eq. (6.11) this means that all frequencies ω_n are equal (or nearly equal, as mentioned above, small frequency differences can be straightforwardly included in the consideration by adding deviations from the central frequency to the r.h.s.). The famous Kuramoto setup [12] belongs to this class. Below, to describe it, we omit index n.

6.2.2.1 Linear Coupling

In the case of linear coupling only terms $\sim hZ$ are present in (6.8). The function $\Delta(k\omega + p\omega)$ picks up from the sum in (6.11) only terms with $p = -k$. In this case the phase dynamics corresponds to a famous Kuramoto-Daido model [13]

$$\dot{\varphi} = \epsilon \sum_k s_{-k} h_k z_k \exp[-ik\varphi]. \tag{6.12}$$

Substitution here of the definition of the order parameters (6.9) reveals terms $\dot{\varphi}[a] \sim \exp[ik(\varphi[b] - \varphi[a])]$, where a and b are indexes of different oscillators in the population. This means that the couplings are pairwise.

6.2.2.2 First Harmonic Phase Sensitivity Function

In some cases, like e.g., the Stuart-Landau oscillators, the phase sensitivity function contains only the first harmonics of the phase, i.e., in (6.11) index p takes values ± 1 only (the term $p = 0$ would lead to a phase-independent frequency shift). Let us consider two cases.

Coupling via the main Kuramoto order parameter

In this case nonlinear terms in (6.11) contain only z_1. It is easy to see that the terms with quadratic nonlinearity vanish, and the only non-trivial, phase-dependent terms with qubic nonlinearity are $\sim z_1^2 z_1^* e^{-i\varphi}$ and its complex conjugate. Together with linear term in (6.11) this yields a Kuramoto model with nonlinearly corrected coupling: acting mean field is modified $z_1 \to z_1 + \alpha |z_1|^2 z_1$, with a complex parameter α. This model has been introduced in [14] and studied in more details in [15], for experimental realization see [16, 17]. The cubic nonlinearity in the order parameter leads to terms in the phase coupling containing four phases (quadruplets): $\dot{\varphi}[a] \sim \exp(i(\varphi[b] + \varphi[c] - \varphi[d] - \varphi[a]))$. If both linear and cubic terms are present, the coupling is a combination of pairs and quadruplets.

Coupling contains many order parameters

In this case already the quadratic terms in (6.11) contribute, provided the relations $k + l = \pm 1$ hold. The simplest case is where $k = 2$, $l = 1$. This corresponds to the coupling term $\sim z_2 z_1^* e^{-i\varphi}$. This coupling is organized in triplets $\dot{\varphi}[a] \sim \exp(i(2\varphi[b] - \varphi[c] - \varphi[a]))$.

6.2.2.3 Second-Harmonic Phase Sensitivity Function

This case is described by the phase sensitivity function possessing the terms $p = \pm 2$ only. The simplest resonance here is provided by the quadratic terms in (6.11) satisfying condition $k + l + p = 0$. One can see that the main complex order parameter ($k = l = 1$) will contribute; the coupling term is $\sim z_1^2 e^{-i\varphi}$. In terms of the phases, the coupling is arranged in triplets $\dot{\varphi}[a] \sim \exp(i(\varphi[b] + \varphi[c] - 2\varphi[a]))$. The dynamics of populations of oscillators with this triplet coupling has been studied in details in [18].

6.2.3 Several Populations of Oscillators

Here we consider a situation where several populations of oscillators with different frequencies ω_n interact. The novel aspect compared to the above-studied case is interaction *across* populations. Because the resonance conditions have to be fulfilled, essential are relations between the basic frequencies.

6.2.3.1 Incommensurate Basic Frequencies

Let us start with the simplest case of two populations with incommensurate frequencies ω_1 and ω_2. This means that nontrivial resonances (i.e., resonances with nonvanishing p, k, l, s) in Eq. (6.11) are impossible in linear and quadratic terms. The first possible nontrivial term appears in the third order; it corresponds to the condition $k\omega_{1,2} + l\omega_{2,1} - s\omega_{1,2} - p\omega_{2,1} = 0$. The simplest term of this form with $k = l = s = p = 1$ corresponds to the quadruplet coupling $\dot{\varphi}_{1,2}[a] \sim \exp(i(\varphi_{1,2}[b] + \varphi_{2,1}[c] - \varphi_{2,1}[d] - \varphi_{1,2}[a]))$. This is in fact a non-resonant coupling because it does not depend on the values of the frequencies. In the case of three incommensurate frequencies, the lowest-order term involving all three populations would be a six-plet $\dot{\varphi}_{1,2,3}[a] \sim \exp(i(\varphi_{1,2,3}[b] + \varphi_{2,3,1}[c] + \varphi_{3,1,2}[d] - \varphi_{3,1,2}[e] - \varphi_{2,3,1}[f] - \varphi_{1,2,3}[a]))$. Effects of such a nonresonant interaction in several globally coupled populations have been explored in [9].

6.2.3.2 Commensurate Basic Frequencies

The simplest case of resonance between two populations is $\omega_2 = 2\omega_1$. Inspecting Eq. (6.11) one can see that there is already a possibility for nontrivial interaction via linear in order parameters terms. This corresponds to the phase coupling terms $\dot{\varphi}_1[a] \sim \exp[i(\varphi_2[b] - 2\varphi_1[a])]$, $\dot{\varphi}_2[a] \sim \exp[i(2\varphi_1[b] - \varphi_2[a])]$. Such a coupling has been treated in [10, 19]. Additionally, there can exist quadratic in the order parameters resonant terms corresponding to triplet couplings $\dot{\varphi}_1[a] \sim \exp[i(\varphi_2[b] - \varphi_1[c] - \varphi_1[a])]$, $\dot{\varphi}_2[a] \sim \exp[i(\varphi_1[b] + \varphi_1[c] - \varphi_2[a])]$.

The next nontrivial case is of three populations with basic frequencies in resonance $\omega_3 = \omega_1 + \omega_2$. In this case there is no linear (pairwise) coupling between populations, and the first nontrivial terms are triplets $\dot{\varphi}_1[a] \sim \exp[i(\varphi_3[b] - \varphi_2[c] - \varphi_1[a])]$, $\dot{\varphi}_2[a] \sim \exp[i(\varphi_3[b] - \varphi_1[c] - \varphi_2[a])]$, $\dot{\varphi}_3[a] \sim \exp[i(\varphi_1[b] + \varphi_2[c] - \varphi_3[a])]$. The dynamics of three resonant populations has been studied in [20].

6.3 Multiple Effective Phase Coupling Appearing in Higher Orders of Phase Reduction

Section 6.2 discussed how couplings, nonlinear in the mean-field order parameters, result in hypernetworks with triplets, quadruplets, etc., of interacting phases. These nonlinear terms may be intrinsic for the problem, like in physical situations discussed in [14] and experimentally implemented in [16, 17]. Here we demonstrate that such terms also appear in high orders in the phase reduction from the original nonlinear equations having only pairwise linear interactions. Here we only sketch the derivation; for the complete analysis, we refer the reader to Ref. [21].

We consider interacting nonlinear oscillators with variables $\mathbf{y}_k$ possessing stable limit cycles $\mathbf{y}_k^0(t) = \mathbf{y}_k^0(t + T_k)$. For each of these cycles a phase φ_k can be introduced, satisfying $\dot{\varphi}_k = \omega_k = 2\pi/T_k$. The phases are functions of the variables $\varphi_k = \Phi_k(\mathbf{y}_k)$, but only on the limit cycle the variables $\mathbf{y}$ are unique functions of the phases. In the basin of attraction of the limit cycles one has to account for amplitude deviations $\delta\mathbf{y}$. We write the system of coupled oscillators as

$$\dot{\mathbf{y}}_k = \mathbf{F}_k(\mathbf{y}_k) + \epsilon \sum_{m \neq k} \mathbf{G}_{mk}(\mathbf{y}_m, \mathbf{y}_k), \tag{6.13}$$

so that only pairwise couplings are present. For the phases, the equations read

$$\dot{\varphi}_k = \frac{d}{dt}\Phi_k(\mathbf{y}_k) = \omega_k + \epsilon \frac{\partial \Phi_k}{\partial \mathbf{y}_k} \sum_{m \neq k} \mathbf{G}_{mk}(\mathbf{y}_m, \mathbf{y}_k). \tag{6.14}$$

This equation is, of course, not a closed equation for the phases, and one solves it iteratively in powers of ϵ. In parallel, one also represents the deviations from the limit cycle $\delta \mathbf{y}_k$ in powers of ϵ.

In the first order in the small parameter ϵ, one can neglect the deviations $\delta \mathbf{y}_k$, then the coupling terms

$$\left.\frac{\partial \Phi_k}{\partial \mathbf{y}_k}\right|_{\mathbf{y}_k^0} \mathbf{G}_{mk}(\mathbf{y}_m^0, \mathbf{y}_k^0)$$

depend on two phases φ_k, φ_m and one obtains pairwise interactions in the phase dynamics in form of Eqs. (6.1). In this order $\sim \epsilon$ also $\delta \mathbf{y}_k$ can be represented as a sum of terms depending on two phases only.

In the second order in the small parameter ϵ, when one substitutes the expressions of the first approximation $\delta \mathbf{y}_k = \epsilon \sum Q_{km}(\varphi_k, \varphi_m)$ in (6.14), one obtains terms containing three phases $\varphi_k, \varphi_m, \varphi_l$, i.e., an effective triplet interaction. In higher-order approximations in ε also the quadruplet, etc., interactions appear in the phase dynamics equations. One can complete this analysis in an exceptional case of the Stuart-Landau oscillators, where the phases and their derivatives are known explicitly. Reference [21] derives phase equations for three Stuart-Landau units organized in a chain, $1 \leftrightarrow 2 \leftrightarrow 3$. As expected, already the second-order phase approximation provides the terms depending on the phases of all three oscillators. Thus, on the level of the phase dynamics, unit 1 interacts with unit 3, though there is no direct link between them, and the simple motif $1 \leftrightarrow 2 \leftrightarrow 3$ becomes a hypernetwork.

For general oscillators, the high-order phase reduction can be performed only numerically. The interested reader can find the corresponding techniques for computation of phases and instantaneous frequencies in Ref. [21]. For three van der Pol oscillators, also coupled in a $1 \leftrightarrow 2 \leftrightarrow 3$ motif, the analysis yields phase reduction equations in different orders, similar to the theoretical findings. Like in the Stuart-Landau systems, the second-order reduction already represents a hypernetwork, with the coupling terms depending on three phases. Contrary to the Stuart-Landau case, the phase dynamic equations for the van der Pol model also contain the terms with the phases' sums.

6.4 Non-pairwise Interactions in the Network Reconstruction Problem

Reconstruction of the coupled oscillatory models from data is an efficient tool for experimental studies of interacting rhythmical objects. A particular example is the recovery of brain connectivity from multichannel measurements of brain activity [22–24]. Another example is the analysis of mutual influences of the cardiac, respiratory, and brain rhythms [25–27]. In this approach, one assumes that the registered time series represent outputs of interacting self-sustained oscillating units. These series allow for estimating of phases and instantaneous frequencies of all oscillators. Typically, one computes these quantities exploiting the Hilbert transform. Finally, one uses these estimates to construct the observed network's phase dynamics model and exploits this model to quantify the strength and directionality of all connections. For technical details of phase estimation and equation reconstruction, we refer to Refs. [27–29]. An essential issue is that the described approach yields the effective phase connectivity that generally differs from the structural connectivity. The latter is determined by physical connections between the oscillators, while the former represents the approximately equivalent phase model's connections. Below we illustrate that the difference is precisely due to the appearance of the non-pairwise interaction on the level of phase reduction.

This approach's main idea is that the dynamics of N interacting oscillators are represented by a torus in the N-dimensional space if the coupling is not too strong. Since the coupling function Q (cf. Eq. (6.1)) is 2π-periodic with respect to its arguments, it can be written as an N-dimensional Fourier series, and the coefficients of this series can be determined by fit.

Consider a simple motif of three pairwise coupled oscillators, described by Eqs. (6.13). Our goal is to determine the network structure, i.e., to quantify all connecting links' strength. While doing this, we shall distinguish between direct or indirect links. It is convenient to quantify first all incoming connections to one of the units and then repeat it for other network elements. Without loss of generality, we can consider the first oscillator. Its phase equation reads:

$$\dot{\phi}_1 = \omega_1 + Q(\phi_1, \phi_2, \phi_3), \tag{6.15}$$

where Q does not contain the constant term. The simplest and straightforward approach, used in many studies, is to perform a pairwise analysis of the network. It means that to quantify the link $1 \leftarrow 2$, we neglect the third unit entirely and reconstruct the equation in the form $\dot{\phi}_1 = \omega_1 + Q(\phi_1, \phi_2)$ with a two-dimensional coupling function $Q(\phi_1, \phi_2)$. Then we compute the norm of the coupling function $\|Q\|$ and use it as a measure of the action exerted by the second oscillator on the first one. To emphasize, that this quantity comes from a pairwise analysis, we denote it as $\mathcal{P}_{1\leftarrow 2}$.

However, this estimation may yield spurious effective phase connections. Indeed, suppose three oscillators are organized in a chain, $1 \leftrightarrow 2 \rightarrow 3$. Because φ_1 is correlated with φ_2 and φ_2 acts on φ_3, pairwise analysis for φ_1, φ_3 will yield spurious non-zero coupling for the $1 \to 3$ link. Only the full phase dynamics given by Eq. (6.15) would reveal the absence of a direct connection between the nodes 1 and 3 and the presence of indirect coupling $1 \to 2 \to 3$. Indeed, as argued in the previous section, the second-order phase approximation yields the terms depending on all three phases. It means that the reconstructed from data coupling function Q in Eq. (6.15) generally contains Fourier components depending on all three phases, and these components describe the indirect connection $1 \to 2 \to 3$. The direct (pairwise) interaction can be quantified by its total strength

$$\mathcal{T}_{1\leftarrow 2} = \left[\sum_{k,l \neq 0} \left| F_{k,l,0} \right|^2 \right]^{1/2} , \tag{6.16}$$

where $F_{k,l,m}$ are Fourier coefficients of $Q(\phi_1, \phi_2, \phi_3)$ and the summation is performed over the terms which do not depend on the third phase φ_3. Correspondingly, the joint action of the second and third oscillators on the first one, i.e., the triplet interaction, that appears in the higher-order approximation can be quantified by the triplet norm

$$\mathcal{T}_{j\leftarrow 2,3} = \left[\sum_{k,l,m \neq 0} \left| F_{k,l,m} \right|^2 \right]^{1/2} , \tag{6.17}$$

where summation is performed over terms depending on three phases. Numerical experiments in Refs. [30] demonstrate that coefficient $\mathcal{P}_{1\leftarrow 2}$ that describes direct, structurally existing, connections scales linearly with coupling strength ϵ, see Eq. (6.15). On the contrary, the scaling of $\mathcal{T}_{j\leftarrow 2,3}$ reveals high-order dependence on ϵ, in full agreement with the theory outlined in the previous section.

Extension of the connectivity analysis through partial norms to the case of $N > 3$ oscillators seems to be straightforward; in Sect. 6.5 we provide such an example. However, reconstruction of the coupling function for more than three variables requires very long data sets. As shown in [29], the triplet analysis performed for moderate lengths of time series, can eliminate this difficulty. Suppose the goal is to quantify the link $j \leftarrow k$. The solution is to consider all possible $N-2$ triplets of oscillators j, k, m, where $m = 1, 2, \ldots, N$, $m \neq j, k$. For each triplet one reconstructs the coupling function $Q_j(\varphi_j, \varphi_k, \varphi_m)$, ignoring all other phases, and computes the partial norm $\mathcal{T}^{(m)}_{j\leftarrow k}$ like in Eq. (6.17). The minimal value of these estimates yields the final triplet-based measure of the binary (pairwise) effective phase connectivity $\mathcal{T}_{j\leftarrow k} = \min_m \mathcal{T}^{(m)}_{j\leftarrow k}$. We illustrate this approach to reconstruction of phase dynamics hypernetworks by an example of $N = 5$ and $N = 9$ randomly coupled van der Pol oscillators [29]:

$$\ddot{x}_k - \mu(1 - x_k^2)\dot{x}_k + \omega_k^2 x_k = \varepsilon \sum_l \sigma_{kl}(x_l \cos\Theta_{kl} + \dot{x}_l \sin\Theta_{kl}). \tag{6.18}$$

The data was generated in many runs, and then the coupling structure was reconstructed by estimating the strength of all links as already discussed. For each run, the random frequencies ω_k were taken from the uniform distribution $0.5 < \omega < 1.5$. The asymmetric connection matrix σ_{kl} composed of zeros and ones was also randomly generated, with four incoming connections. Another coupling parameter Θ was taken from a uniform distribution $0 \le \Theta < 2\pi$. The results, presented in Figs. 4–7 of Ref. [29] demonstrate that the phase dynamics reconstruction using hypernetworks provides enhanced separation between truly existing and absent structural connections. For application of this approach to a hypernetwork with triplet interactions of 12 phase oscillators see Ref. [31].

6.5 Example of Phase Dynamics Reconstruction in a Network with Triplet Couplings

Here we consider a simple example of a hypernetwork of oscillators with triplet coupling. It consists of four FitzHugh-Nagumo units [32], and the force acting on a unit is a product of observables of two other units:

$$
\begin{aligned}
\dot{u}_1 &= u_1 - \frac{u_1^3}{3} - v_1 + 0.9 + 0.4\epsilon u_2 u_3, \\
\dot{v}_1 &= 0.35(u_1 - 0.8v_1 + 0.7), \\
\dot{u}_2 &= u_2 - \frac{u_2^3}{3} - v_2 + 0.8 - 0.6\epsilon u_1 u_3 + 0.7\epsilon u_3 u_4, \\
\dot{v}_2 &= 0.5(u_1 - 0.8v_1 + 0.7), \\
\dot{u}_3 &= u_3 - \frac{u_3^3}{3} - v_3 + 1.1 + 0.8\epsilon u_1 u_2 - 0.3\epsilon u_2 u_4, \\
\dot{v}_3 &= 0.42(u_1 - 0.8v_1 + 0.7), \\
\dot{u}_4 &= u_1 - \frac{u_1^3}{3} - v_1 + 1 + 0.5\epsilon u_2 u_3, \\
\dot{v}_4 &= 0.45(u_1 - 0.8v_1 + 0.7).
\end{aligned}
\tag{6.19}
$$

This configuration is schematically presented in Fig. 6.1. It should be noted, that although the coupling looks like a pure triplet one, because the interaction terms are products of the corresponding variables (cf. [33]), in fact the pairwise coupling is also present, because average values of observables u_i are generally nonzero. Thus, if one separates these average values by writing $u_i = \overline{u}_i + \tilde{u}_i$, then, e.g., the forcing for the first unit will be written as $\overline{u}_2\overline{u}_3 + \overline{u}_2\tilde{u}_3 + \overline{u}_3\tilde{u}_2 + \tilde{u}_2\tilde{u}_3$, i.e. with terms which can be effectively considered as a pairwise coupling.

For the model (6.19) we performed numerically the phase reduction analysis, as described above in Sect. 6.3 and in Ref. [21], for a range of values of the coupling parameter $0.001 \le \epsilon \le 0.05$. We have looked for the phase dynamics equations in the form

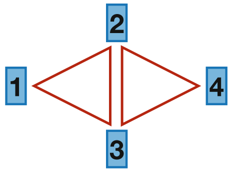

Fig. 6.1 A schematic representation of network (6.19), with two triplet couplings

$$\dot{\varphi}_s = \omega_s + \sum_{jklm} \left[A^{(s)}_{jklm} \cos(j\varphi_1 + k\varphi_2 + l\varphi_3 + m\varphi_4) + B^{(s)}_{jklm} \sin(j\varphi_1 + k\varphi_2 + l\varphi_3 + m\varphi_4) \right] \tag{6.20}$$

In the reconstruction we took into account all terms with $|j, k, l, m| \leq 2$. To represent the strength of each coupling mode, we calculated $C^{(s)}_{jklm} = \left[(A^{(s)}_{jklm})^2 + (B^{(s)}_{jklm})^2 \right]^{1/2}$. Altogether, for each oscillator this gives 310 coupling terms.

After finding the coupling modes, we sorted them according to dependence on ϵ. We performed a fit $C \sim \epsilon^p$, and interpreted terms with $|p - 1| < 0.1$ as linear in ϵ, and those with $|p - 2| < 0.1$ as quadratic in ϵ. We present the dependencies on ϵ of all these terms for oscillators 1 and 2 in Fig. 6.2.

It is instructive to see, which effective phase coupling terms appear in the first and the second orders in the coupling strength ϵ. Here are all 34 first-order terms for the oscillator 1, sorted in descending order in their strength:

$$\begin{gathered}
(1,1,1,0),\ (1,-1,1,0),\ (1,1,-1,0),\ (1,-1,-1,0),\ (1,1,0,0),\ (1,-1,0,0),\\
(1,-1,-2,0),\ (1,1,2,0),\ (1,1,-2,0),\ (1,-1,2,0),\ (1,0,-1,0),\ (1,0,1,0),\\
(1,-2,1,0),\ (1,2,-1,0),\ (1,2,1,0),\ (1,-2,-1,0),\ (2,1,1,0),\ (2,-1,-1,0),\\
(2,1,-1,0),\ (0,1,1,0),\ (0,1,-1,0),\ (2,-1,1,0),\ (1,0,0,0),\ (1,2,0,0),\\
(1,0,-2,0),\ (1,-2,0,0),\ (1,0,2,0),\ (2,1,0,0),\ (1,2,-2,0),\ (1,-2,-2,0),\\
(1,2,2,0),\ (2,1,-2,0),\ (0,1,0,0),\ (2,-1,-2,0).
\end{gathered} \tag{6.21}$$

One can see that the largest terms describe triplet coupling $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1$ and the pairwise couplings. There is no term that includes the phase of oscillator 4. Such terms appear in the second order in ϵ. Altogether, there are 63 terms:

$$\begin{gathered}
(1,2,0,2),\ (1,2,-2,2),\ (1,-2,0,-2),\ (1,-2,2,-2),\ (1,0,-2,-2),\ (1,2,-2,-2),\\
(1,0,2,2),\ (1,-2,2,2),\ (1,2,-1,2),\ (1,-2,1,-2),\ (1,2,1,2),\ (1,-2,-1,-2),\\
(1,-2,1,-1),\ (1,2,-1,1),\ (1,-2,-1,-1),\ (2,2,2,0),\ (1,2,1,1),\ (1,2,2,1),\\
(1,-2,-2,-1),\ (1,0,0,1),\ (1,0,0,-1),\ (1,0,1,1),\ (1,-2,-2,-2),\ (1,2,1,-1),\\
(1,2,2,2),\ (1,0,-1,-1),\ (1,-2,-1,1),\ (1,-1,2,2),\ (1,0,-1,1),\ (1,2,0,1),\\
(1,-2,0,-1),\ (1,2,-2,-1),\ (1,0,1,-1),\ (1,-2,2,1),\ (1,2,2,-1),\ (1,0,-2,1),\\
(1,-1,-2,-2),\ (1,0,2,-1),\ (1,0,-2,-1),\ (1,1,2,2),\ (1,0,2,1),\ (2,-2,0,-2),\\
(1,-2,-2,1),\ (1,0,0,-2),\ (1,0,0,2),\ (2,2,-2,-2),\ (0,2,0,2),\ (1,-2,0,1),\\
(0,0,2,2),\ (0,2,-2,-2),\ (1,-2,-2,2),\ (1,-1,-2,1),\ (2,2,1,1),\ (1,0,-2,2),\\
(1,1,-2,1),\ (1,2,-1,-1),\ (2,-2,1,-2),\ (1,-2,0,2),\ (1,0,1,2),\ (1,0,1,-2),\\
(1,0,-1,-2),\ (1,-2,-1,2),\ (0,0,1,-1).
\end{gathered} \tag{6.22}$$

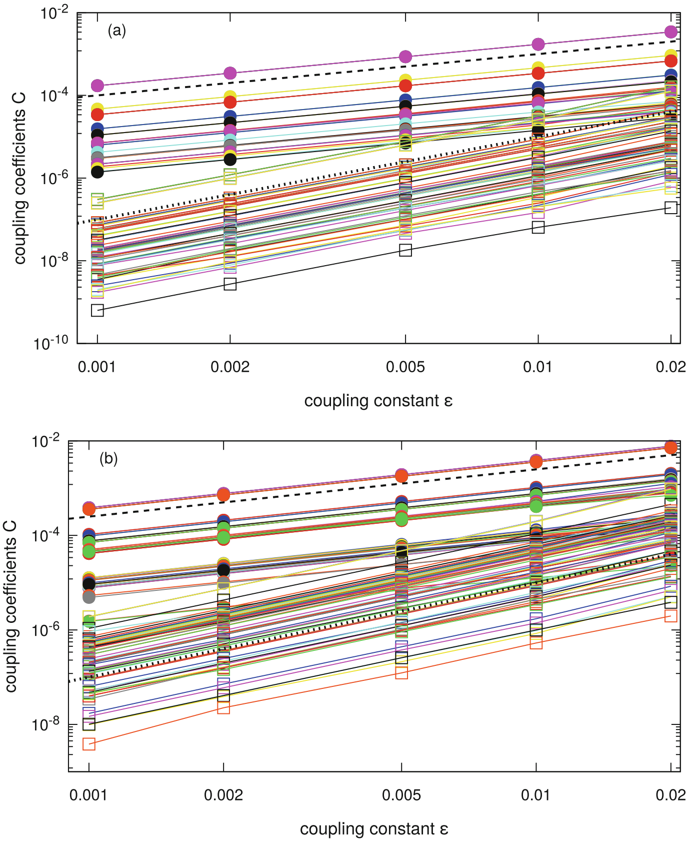

Fig. 6.2 Scaling of coupling terms with ϵ for oscillators 1 (panel (a)) and 2 (panel (b)). The dashed and the dotted lines show scalings $\sim \epsilon$ and $\sim \epsilon^2$, respectively. The corresponding coupling coefficients are depicted with filled circles and open squares

Thirty-three of them include all four phases and therefore describe effective quadruplet coupling.

The second oscillator participates in two triplet couplings, and therefore the number of phase interaction terms in the first and in the second order is larger, 65 and 69, respectively. The first-order terms are

$$
\begin{gathered}
(0,1,1,1),\ (0,1,-1,1),\ (0,1,-1,-1),\ (0,1,1,-1),\ (1,-1,-1,0),\ (1,1,-1,0),\\
(1,-1,1,0),\ (1,1,1,0),\ (0,1,0,-1),\ (0,1,0,1),\ (1,-1,0,0),\ (1,1,0,0),\\
(0,1,2,1),\ (0,1,-2,-1),\ (0,1,2,-1),\ (0,1,-2,1),\ (1,-1,-2,0),\ (1,1,2,0),\\
(1,-1,2,0),\ (1,1,-2,0),\ (0,2,-1,1),\ (0,0,1,-1),\ (0,1,1,0),\ (1,0,-1,0),\\
(0,2,1,-1),\ (0,0,1,1),\ (1,0,1,0),\ (0,1,-1,0),\ (0,2,-1,-1),\ (1,-2,-1,0),\\
(1,-2,1,0),\ (0,1,1,2),\ (1,2,-1,0),\ (1,2,1,0),\ (0,1,-1,2),\ (0,1,1,-2),\\
(0,1,-1,-2),\ (0,2,1,1),\ (1,0,0,0),\ (0,2,0,-1),\ (0,1,0,2),\ (0,1,0,0),\\
(1,2,0,0),\ (0,1,2,2),\ (0,1,-2,2),\ (1,0,2,0),\ (1,-2,0,0),\ (0,1,2,-2),\\
(1,-2,2,0),\ (2,-1,-1,0),\ (0,2,2,-1),\ (2,1,-1,0),\ (0,1,0,-2),\ (1,0,-2,0),\\
(1,2,2,0),\ (2,-1,1,0),\ (0,0,2,-1),\ (2,1,1,0),\ (1,2,-2,0),\ (0,1,2,0),\\
(0,1,-2,0),\ (0,2,-1,0),\ (0,2,-1,-2),\ (2,-1,-2,0),\ (2,1,-2,0)
\end{gathered}
\tag{6.23}
$$

and the second-order terms are

$$
\begin{gathered}
(1,-2,2,-2),\ (1,2,0,2),\ (1,2,-2,2),\ (1,-2,0,-2),\ (1,-2,-1,1),\ (1,-2,-1,-1),\\
(1,-2,1,-2),\ (1,0,2,2),\ (1,2,-2,-2),\ (1,-2,2,2),\ (1,2,-1,2),\ (0,2,-2,0),\\
(1,0,-2,-2),\ (1,-2,0,1),\ (1,-2,-1,-2),\ (1,-2,2,1),\ (1,2,1,1),\ (1,-2,-2,-1),\\
(1,2,0,1),\ (1,-2,-2,1),\ (1,0,0,1),\ (1,-2,2,-1),\ (1,0,2,1),\ (1,-2,0,-1),\\
(1,2,1,2),\ (1,2,0,-1),\ (1,2,-2,1),\ (1,0,-2,1),\ (1,2,-2,-1),\ (1,0,1,1),\\
(1,2,2,1),\ (2,2,0,0),\ (2,-2,-2,0),\ (1,0,0,-1),\ (1,-2,1,-1),\ (1,0,2,-1),\\
(1,2,2,2),\ (1,0,-2,-1),\ (1,0,1,-1),\ (1,0,-1,1),\ (1,2,2,-1),\ (1,1,0,1),\\
(1,2,1,-1),\ (1,1,2,1),\ (1,2,-1,-1),\ (2,2,-2,0),\ (1,-1,2,1),\ (1,-2,1,1),\\
(1,0,-1,-1),\ (1,1,-2,-1),\ (1,-1,2,-1),\ (1,-2,0,2),\ (1,-2,-1,2),\ (1,1,2,-1),\\
(1,-1,1,1),\ (1,2,1,-2),\ (1,0,0,2),\ (1,0,1,2),\ (1,-1,-2,1),\ (2,2,-2,2),\\
(1,1,0,-1),\ (2,-2,2,-2),\ (1,1,1,-1),\ (2,2,1,-1),\ (2,0,1,-1),\ (2,0,-2,-2),\\
(2,2,1,2),\ (2,2,-2,-2),\ (2,2,2,2)
\end{gathered}
\tag{6.24}
$$

6.6 Conclusions

This mini-review demonstrates that hypernetworks naturally appear in the phase dynamics description of ensembles of coupled oscillators. There are two main scenarios. First, the hypernetworks arise due to nonlinear mean-field coupling. Second, simple pairwise connections on the level of state variables result in hypernetworks of phase oscillators in the process of high-order phase reduction. This fact is significant for a practical problem, namely, to determine the network connectivity from measurements. Fitting a hypernetwork of phase oscillators to experimental data essentially improves the recovery of the structural connectivity.

Acknowledgements The work was supported by the Russian Science Foundation (grant Nr. 17-12-01534).

References

1. H. Nakao, Phase reduction approach to synchronisation of nonlinear oscillators. Contemp. Phys. **57**, 188–214 (2016)
2. B. Pietras, A. Daffertshofer, Network dynamics of coupled oscillators and phase reduction techniques. Phys. Rep. **819**, 1–105 (2019)
3. A.T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators. J. Theor. Biol. **16**, 15 (1967)
4. Y. Kuramoto, *Chemical Oscillations*. Waves and Turbulence (Springer, Berlin, 1984)
5. H. Sakaguchi, Y. Kuramoto, A soluble active rotator model showing phase transition via mutual entrainment. Prog. Theor. Phys. **76**(3), 576–581 (1986)
6. J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, R. Spigler, The Kuramoto model: a simple paradigm for synchronization phenomena. Rev. Mod. Phys. **77**(1), 137–175 (2005)
7. A. Pikovsky, M. Rosenblum, Dynamics of globally coupled oscillators: Progress and perspectives. Chaos **25**, 097616 (2015)
8. S. Chandra, M. Girvan, E. Ott, Continuous versus discontinuous transitions in the d-dimensional generalized kuramoto model: odd d is different. Phys. Rev. X **9**, 011002 (2019)
9. M. Komarov, A. Pikovsky, Effects of nonresonant interaction in ensembles of phase oscillators. Phys. Rev. E **84**(1), 016210 (2011)
10. M. Komarov, A. Pikovsky, Intercommunity resonances in multifrequency ensembles of coupled oscillators. Phys. Rev. E **92**, 012906 (2015)
11. A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001)
12. Y. Kuramoto, Self-entrainment of a population of coupled nonlinear oscillators, in *International Symposium on Mathematical Problems in Theoretical Physics*, ed. by H. Araki, p. 420, New York, 1975. Springer Lecture Notes Phys, v. 39
13. H. Daido, Onset of cooperative entrainment in limit-cycle oscillators with uniform all-to-all interactions: bifurcation of the order function. Physica D **91**, 24–66 (1996)
14. M. Rosenblum, A. Pikovsky, Self-organized quasiperiodicity in oscillator ensembles with global nonlinear coupling. Phys. Rev. Lett. **98**, 064101 (2007)
15. A. Pikovsky, M. Rosenblum, Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators. Physica D **238**(1), 27–37 (2009)
16. A.A. Temirbayev, ZZh. Zhanabaev, S.B. Tarasov, V.I. Ponomarenko, M. Rosenblum, Experiments on oscillator ensembles with global nonlinear coupling. Phys. Rev. E **85**, 015204 (2012)
17. A.A. Temirbayev, Y.D. Nalibayev, ZZh. Zhanabaev, V.I. Ponomarenko, M. Rosenblum, Autonomous and forced dynamics of oscillator ensembles with global nonlinear coupling: an experimental study. Phys. Rev. E **87**, 062917 (2013)
18. M. Komarov, A. Pikovsky, Finite-size-induced transitions to synchrony in oscillator ensembles with nonlinear global coupling. Phys. Rev. E **92**, 020901 (2015)
19. S. Lück, A. Pikovsky, Dynamics of multi-frequency oscillator ensembles with resonant coupling. Phys. Lett. A **375**(28–29), 2714–2719 (2011)
20. M. Komarov, A. Pikovsky, Dynamics of multifrequency oscillator communities. Phys. Rev. Lett. **110**, 134101 (2013)
21. E. Gengel, E. Teichmann, M. Rosenblum, A. Pikovsky, High-order phase reduction for coupled oscillators. J. Phys.: Complexity **2**, 015005 (2021)
22. K. Lehnertz, Assessing directed interactions from neurophysiological signals—an overview. Physiol. Measurement **32**, 1715–1724 (2011)

23. T.B. Leergaard, C.C. Hilgetag, O. Sporns, Mapping the connectome: multi-level analysis of brain connectivity. Front. Neuroinformatics, **6**(14) (2012)
24. O. Sporns, Making sense of brain network data. Nature Methods **10**(6), 491–493 (2013)
25. R. Mrowka, L. Cimponeriu, A. Patzak, M.G. Rosenblum, Directionality of coupling of physiological subsystems—age related changes of cardiorespiratory interaction during different sleep stages in babies. Ame. J. Physiol. Regul. Comp. Integr. Physiol. **145**, R1395–R1401 (2003)
26. B. Musizza, A. Stefanovska, P.V.E. McClintock, M. Paluš, J. Petrovčič, S. Ribarič, F.F. Bajrović, Interactions between cardiac, respiratory, and EEG-δ oscillations in rats during anæsthesia. J. Physiol. (Londn.) **580**(1), 315–326 (2007)
27. B. Kralemann, M. Frühwirth, A. Pikovsky, M. Rosenblum, T. Kenner, J. Schaefer, M. Moser, In vivo cardiac phase response curve elucidates human respiratory heart rate variability. Nature Commun. **4**, 2418 (2013)
28. B. Kralemann, L. Cimponeriu, M. Rosenblum, A. Pikovsky, R. Mrowka, Phase dynamics of coupled oscillators reconstructed from data. Phys. Rev. E **77**, 066205 (2008)
29. B. Kralemann, A. Pikovsky, M. Rosenblum, Reconstructing effective phase connectivity of oscillator networks from observations. New J. Phys.**16**, 085013 (2014)
30. B. Kralemann, A. Pikovsky, M. Rosenblum, Reconstructing phase dynamics of oscillator networks. Chaos **21**, 025104 (2011)
31. A. Pikovsky, Reconstruction of a random phase dynamics network from observations. Phys. Lett. A **382**(4), 147–152 (2018)
32. E.M. Izhikevich, R. FitzHugh, FitzHugh-Nagumo model. Scholarpedia, **1**(9), 1349 (2006). revision #123664
33. J. Petereit, A. Pikovsky, Chaos synchronization by nonlinear coupling. Comm. Nonl. Sci. Num. Simul. **44**, 344–351 (2017)