

Exact finite-dimensional reduction for a population of noisy oscillators and its link to Ott–Antonsen and Watanabe–Strogatz theories

Cite as: Chaos 32, 113126 (2022); doi: 10.1063/5.0106171

Submitted: 27 June 2022 · Accepted: 10 October 2022 ·

Published Online: 8 November 2022



View Online



Export Citation



CrossMark

Rok Cestnik^{a)}  and Arkady Pikovsky 

AFFILIATIONS

Department of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Strasse 24/25, 14476 Potsdam-Golm, Germany

^{a)} Author to whom correspondence should be addressed: rokcestn@uni-potsdam.de

ABSTRACT

Populations of globally coupled phase oscillators are described in the thermodynamic limit by kinetic equations for the distribution densities or, equivalently, by infinite hierarchies of equations for the order parameters. Ott and Antonsen [Chaos 18, 037113 (2008)] have found an invariant finite-dimensional subspace on which the dynamics is described by one complex variable per population. For oscillators with Cauchy distributed frequencies or for those driven by Cauchy white noise, this subspace is weakly stable and, thus, describes the asymptotic dynamics. Here, we report on an exact finite-dimensional reduction of the dynamics outside of the Ott–Antonsen subspace. We show that the evolution from generic initial states can be reduced to that of three complex variables, plus a constant function. For identical noise-free oscillators, this reduction corresponds to the Watanabe–Strogatz system of equations [Watanabe and Strogatz, Phys. Rev. Lett. 70, 2391 (1993)]. We discuss how the reduced system can be used to explore the transient dynamics of perturbed ensembles.

Published under an exclusive license by AIP Publishing. <https://doi.org/10.1063/5.0106171>

Large ensembles of globally coupled oscillators can be described by means of kinetic equations for the evolution of the distribution densities. These equations take the simplest form if the oscillators are described by their phases only. Alternatively, one can write an infinite set of ordinary differential equations for the set of order parameters (Fourier modes of the distribution). For oscillators driven by Cauchy white noise or for a Cauchy distribution of natural frequencies, the system of equations for the order parameters (for coupling in the first harmonics of the phase) has a remarkable property first discovered by Ott and Antonsen in 2008: it possesses an invariant manifold on which the dynamics reduces to just one complex equation. In this paper, we extend this result by showing that for arbitrary initial conditions the dynamics reduces to that of three complex variables. In the noise-free case of identical oscillators, our equations are equivalent to the system derived by Watanabe and Strogatz in 1993. The finite-dimensional reduction allows the exact calculation of transients to the attracting Ott–Antonsen states, by solving a simple six-dimensional system of equations.

I. INTRODUCTION

Ensembles of coupled oscillators is a popular object in studies of complex systems, with a wide range of applications; from physical systems (lasers,¹ Josephson junctions,^{2,3} chemical reactions⁴), to engineering (pedestrians on a bridge⁵) and life sciences (neurons,^{6,7} nephron cells,⁸ genetic circuits⁹). A common theoretical approach includes different levels of reductions and idealizations. If the units are self-sustained periodic oscillators, and the coupling is weak, one can perform a phase reduction, neglecting variations of the oscillators' amplitudes that appear in the higher orders in coupling strength.¹⁰ As a result, each oscillator is described by just one variable on a unit circle—the phase, which enormously simplifies the analysis. Another idealization, which is appropriate for large ensembles, is the thermodynamic limit of an infinite number of units. This allows for a formulation of the evolution in terms of kinetic equations for the distribution of the phases. An important class of models are those with global (or mean-field) coupling. Such models appear naturally, e.g., for Josephson junctions with a common load and for pedestrians on a bridge; in other cases (e.g., for neural ensembles),

they are justified by a huge number of interconnections between the units.

Among the setups for ensembles of globally coupled phase oscillators, the paradigmatic Kuramoto model¹¹ and its generalizations^{12,13} are particularly popular. Here, one assumes a relatively simple coupling, where the dynamics of the oscillator's phase depends only on the first harmonics of the phase itself. To define the coupling, one introduces mean fields, which are the circular moments of the phase distribution. Different setups with identical deterministic units, as well as ones having different natural frequencies and/or being driven by noise, have been considered in the literature.

One of the striking properties of the Kuramoto-type models is the possibility to reduce the dynamics to a finite-dimensional one. Watanabe and Strogatz^{14,15} (WS) have demonstrated that ensembles of identical, noise-free units can be exactly reduced to three dynamical equations (plus constants of motion). Ott and Antonsen¹⁶ (OA) found a particular family of phase distributions (wrapped Cauchy distribution) that is invariant under the dynamical evolution. This holds not only for identical units, but also for ones with a Cauchy distribution of natural frequencies, and for ones driven by white Cauchy noise.^{17,18} In contradistinction to WS theory, the OA reduction is not valid for arbitrary initial states—they should belong to the OA invariant manifold. However, because there are arguments that the OA manifold is attracting (although not in a trivial sense, see discussion in^{19–21}), the OA equations correctly describe the asymptotic in time regimes.

The goal of this paper is to fill, at least partially, the gap between WS and OA theories. We will develop, in the thermodynamic limit, a low-dimensional description of the Kuramoto-type phase ensembles with Cauchy noise and/or Cauchy distribution of natural frequencies, valid for arbitrary initial conditions. Of course, this reduction contains WS and OA equations as particular cases.

The paper is organized as follows. In Sec. II, we formulate the problem. In Sec. III, we introduce our basic tools (generating functions) and define a family of finite-dimensional invariant manifolds (these results have been also presented in a short communication²²). Section IV contains the main result—we show how the evolution of generic states can be reduced to three complex variables plus a constant function. Here, we also discuss different possibilities of introducing these variables based on initial conditions. In Sec. V, we demonstrate the stability of the OA manifold in the presence of noise. In Sec. VI, we consider identical noise-free oscillators and demonstrate that the dynamics reduces to the WS equations. Section VII is devoted to the implications for the spectrum of the Lyapunov exponents. In Sec. VIII, we discuss how our approach allows for finding the evolution outside of the OA manifold. We conclude and discuss possible further developments in Sec. IX. Many technical details are shifted from the main text to appendixes.

II. PROBLEM FORMULATION

In this paper, we consider populations of phase oscillators, subject to global coupling or to a global common force, in the thermodynamic limit of an infinite number of units. Consequently, the proper description is in terms of the phase distribution functions. Our theory is valid for a restricted class of systems: (i) important

is that the coupling/forcing is proportional to the first harmonics of the phase only and (ii) the oscillators can differ from each other only in additive terms in their phase dynamics, which are either Cauchy-distributed white noise terms or Cauchy-distributed frequency constants, or a combination of both. In this section, we introduce these models.

A. Ensemble of phase oscillators with independent Cauchy noise forces

We consider an ensemble of noisy phase oscillators coupled in the first harmonic,

$$\dot{\varphi}_j = \omega(t) + \text{Im}[2h(t)e^{-i\varphi_j}] + \gamma\xi_j(t). \quad (1)$$

Here, ω is a combination of a natural frequency and a real-valued additive force, and $h(t)$ is a complex-valued force that includes the first harmonic of the phase. Both these quantities can potentially depend on the mean fields of the population, thus readily describing global coupling. There is no restriction on these forces, e.g., they can include noise which is then the common noise for all elements of the population (cf. Refs. 23 and 24). The term $\xi_j(t)$ represents independent, normalized Cauchy white noise forces, with γ being the real and positive noise strength.^{17,18,25,26}

We consider the thermodynamic limit of infinitely many oscillators. In this case, it is natural to describe the state with the phase density function $P(\varphi, t)$ and express the original dynamics in terms of the continuity equation, a partial differential equation (PDE) where the Cauchy noise begets a term with a fractional derivative on the right-hand side,

$$\frac{\partial}{\partial t}P + \frac{\partial}{\partial\varphi}(\dot{\varphi}P) = \gamma \left| \frac{\partial}{\partial\varphi} \right| P. \quad (2)$$

With $\left| \frac{\partial}{\partial\varphi} \right|^\alpha$ one denotes an operator, which in the Fourier representation reduces to a multiplication with $|n|^\alpha$: $\left| \frac{\partial}{\partial\varphi} \right|^\alpha e^{in\varphi} = -|n|^\alpha e^{in\varphi}$, cf. Ref. 27. In this representation, the usual Gaussian noise corresponds to $\alpha = 2$, while the Cauchy noise corresponds to $\alpha = 1$ (other non-integer values of α describe different α -stable distributions).

The phase density is commonly expressed as a Fourier series,

$$P(\varphi, t) = \frac{1}{2\pi} \left(1 + \sum_{n=1}^{\infty} Z_n(t) e^{-in\varphi} + \text{c.c.} \right), \quad (3)$$

$$Z_n(t) = \langle e^{in\varphi} \rangle = \int_0^{2\pi} d\varphi e^{in\varphi} P(\varphi, t).$$

Quantities Z_n represent complex order parameters, also known as Kuramoto–Daido order parameters.^{11,28} These circular moments of the phase distribution are in fact the “mean fields,” which may govern the ensemble. In terms of these order parameters, the dynamics is represented as an infinite set of ordinary differential equations (ODE),

$$\frac{1}{n} \dot{Z}_n = (i\omega - \gamma)Z_n + hZ_{n-1} - h^*Z_{n+1}, \quad n \geq 1, \quad (4)$$

(here it suffices to consider positive n only, so we replace $|n|$ in the noisy term by n). These equations have been discussed in Ref. 18 and

represent the exact dynamics of system (1) in the thermodynamic limit, without any approximation or assumption. Normalization of the phase density implies $Z_0 \equiv 1$.

B. Ensemble with a Cauchy distribution of natural frequencies

Equivalent equations can also be derived to represent the case of Cauchy distributed natural frequencies, the situation widely considered starting from the initial formulation by Kuramoto.^{10,29} In this case, we consider the terms ξ_j in Eq. (1) as constants with a normalized Cauchy distribution $g(\xi) = \pi^{-1}(1 + \xi^2)^{-1}$. The total additive force $\omega(t) + \gamma\xi_j$ can be interpreted as an instantaneous frequency of oscillator j . If $\omega = \omega_0$ is a constant, then $\omega_0 + \gamma\xi_j$ is the natural frequency of oscillator j . Parameter γ describes the width of the distribution of natural frequencies (as we will see below, this parameter plays the same role as the strength of the Cauchy noise above, thus we use the same letter to describe it). In our further derivation, we follow the way presented recently in Ref. 21. One introduces the parameter ξ into the distribution of phases $P(t, \varphi; \xi)$, and the equation for this distribution (2) then reads

$$\frac{\partial}{\partial t} P|_{\xi} + \frac{\partial}{\partial \varphi} \left([\omega + \gamma\xi - ihe^{-i\varphi} + ih^*e^{i\varphi}] P|_{\xi} \right) = 0, \quad (5)$$

where we used compact notation $P|_{\xi} \equiv P(\varphi, t; \xi)$. Of interest are the order parameters (circular moments), averaged over the additions to the frequency ξ ,

$$Z_n(t) = \int_{-\infty}^{\infty} d\xi \int_0^{2\pi} d\varphi e^{in\varphi} P|_{\xi} g(\xi). \quad (6)$$

The main assumption allowing for explicit equations for these order parameters is analyticity of the distribution $P(\varphi, t; \xi)$ in the upper halfplane of complex ξ . This assumption has been first introduced by Ott and Antonsen in their seminal paper.¹⁶ The main reason behind it is the possibility to calculate the integrals via residue integration. Indeed, employing the residue theorem for a contour closing the upper halfplane in (6) and taking the only pole at $\xi = i$, one reduces (6) to $Z_n(t) = \int d\varphi e^{in\varphi} P(\varphi, t; i)$. Now, let us multiply (5) with $e^{in\varphi} g(\xi)$ and integrate in ξ and φ . The only additional integral to be calculated (again by virtue of the residue method) is

$$\int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} d\xi \xi e^{in\varphi} P|_{\xi} g(\xi) = i \int_0^{2\pi} d\varphi e^{in\varphi} P|_i = iZ_n.$$

This yields the system of equations

$$\dot{Z}_n = in(\omega + i\gamma)Z_n + nhZ_{n-1} - nh^*Z_{n+1},$$

which coincides with (4).

We end this section with two remarks. First, the validity of Eq. (4) for Cauchy independent noises is unconditional, while for the Cauchy distributed constant additions to the frequency, an extra assumption of analyticity has to be adopted; the validity of this assumption is commonly assumed in the OA theory and its applications, but one can construct distributions of the phase which, at least during some time interval, violate this assumption.³⁰

The second remark is that if one has both Cauchy-distributed constant and noisy additions to the frequency, with intensities γ_1 and γ_2 , then one can use Eq. (4) with the total intensity $\gamma = \gamma_1 + \gamma_2$.

III. GENERATING FUNCTIONS AND FINITE-DIMENSIONAL REDUCTIONS OF THE DYNAMICS

A. Ordinary and exponential generating functions

In our treatment of the infinite system (4), we will make use of generating functions, which are formal power series. We will use both ordinary generating function (OGF),³¹

$$\mathcal{F}(k) = \sum_{n=1}^{\infty} f_n k^n,$$

and the exponential generating function (EGF)

$$F(k) = \sum_{n=0}^{\infty} f_n \frac{k^n}{n!}.$$

There is no simple relation between these functions for the same sequence $\{f_n\}$, and in different situations we will use different generating functions.

B. Finite-dimensional reductions of the infinite system for circular moments

In this section, we briefly introduce the finite-dimensional reductions described in our recent letter.²² First, we characterize the state with complex order parameters Z_n by introducing the complex-valued EGF (because $|Z_n| \leq 1$, this series converges for all k),

$$Z(k, t) = \sum_{n=0}^{\infty} Z_n(t) \frac{k^n}{n!}. \quad (7)$$

Then, the dynamics (4) are recast as a single PDE (see Appendix A for the derivation), which in contrast to Eq. (2) is generally complex (prime denotes derivative with respect to k),

$$\dot{Z} = (i\omega - \gamma)kZ' + hkZ - h^*kZ''. \quad (8)$$

The normalization condition is $Z(0, t) = 1$. The structure of this equation allows for a particular solution with the exponential ansatz $Z(k, t) = e^{kQ(t)}$, revealing a single ODE for the complex variable $Q(t)$,

$$\dot{Q} = (i\omega - \gamma)Q + h - h^*Q^2. \quad (9)$$

This is commonly known as the Ott–Antonsen ansatz,¹⁶ which reveals a two-dimensional invariant manifold in the infinite system (4). In this case, higher circular moments are powers of the first one: $Z_n = Q^n$. The distribution of the phases is the wrapped Cauchy

distribution (a.k.a. Poisson kernel),

$$P(\varphi, t) = \frac{1}{2\pi} \frac{1 - |Q|^2}{|1 - Qe^{-i\varphi}|^2}. \tag{10}$$

We recently generalized this solution with an ansatz allowing for an additional function,²²

$$Z(k, t) = e^{kQ(t)} \mathbf{B}(k, t), \tag{11}$$

in which case we obtain, in addition to (9), another PDE for the newly introduced function $\mathbf{B}(k, t)$,

$$\dot{\mathbf{B}} = (i\omega - \gamma - 2h^*Q)k\mathbf{B}' - h^*k\mathbf{B}''. \tag{12}$$

Although at first glance this equation is similar to Eq. (8), it does not contain a term without k -derivative of $\mathbf{B}(k, t)$ and, thus, allows for a more general dimensionality reduction. Namely, we expand the function $\mathbf{B}(k)$ as an EGF (we will see below that coefficients β_n grow not faster than $\sim \text{const}^n$; thus, this series converges for all k),

$$\mathbf{B}(k, t) = \sum_{n=0}^{\infty} \beta_n(t) \frac{k^n}{n!}, \tag{13}$$

($\beta_0 \equiv 1$ due to normalization), thus introducing new dynamical variables $\beta_n(t)$, that describe the dynamics with an infinite set of ODEs [plus one ODE (9) for Q],

$$\dot{Q} = (i\omega - \gamma)Q + h - h^*Q^2, \tag{14a}$$

$$\frac{1}{n} \dot{\beta}_n = (i\omega - \gamma - 2h^*Q)\beta_n - h^*\beta_{n+1}, \quad n \geq 1, \tag{14b}$$

[see Appendix A for the relation of (12) to (14b)]. Notice how the right-hand side of (14b) only contains terms proportional to β_n and β_{n+1} , but no term with β_{n-1} is present. This means that if the system is truncated at a finite number N of variables β_n (i.e., assuming that all higher terms vanish identically: $\beta_{n \geq N} = 0$), the dynamics is exactly described by the first N equations of system (14b) for all times. These truncations represent dynamically invariant finite-dimensional manifolds. The β_n variables relate to the Kuramoto–Daido order parameters Z_n via a modified binomial transform (cf. Refs. 32 and 33),

$$Z_n(t) = \sum_{m=0}^n \binom{n}{m} \beta_m(t) [Q(t)]^{n-m}, \tag{15}$$

$$\beta_n(t) = \sum_{m=0}^n \binom{n}{m} Z_m(t) [-Q(t)]^{n-m}.$$

For example, the first three order parameters are expressed with the newly introduced variables as

$$Z_1 = Q + \beta_1, \quad Z_2 = Q^2 + 2Q\beta_1 + \beta_2, \\ Z_3 = Q^3 + 3Q^2\beta_1 + 3Q\beta_2 + \beta_3.$$

IV. REDUCTION OF THE DYNAMICS TO THREE COMPLEX VARIABLES

As outlined in Sec. III, there are many finite-dimensional invariant manifolds (with a finite number of additional variables β_n)

beyond the OA two-dimensional manifold (which corresponds to vanishing β_n for $n \geq 1$). However, as already mentioned in Ref. 22, it is not excluded that different β_n could be dependent. Below we show that this is indeed the case, and the dynamics of the whole (even infinite) hierarchy of variables $\beta_n(t)$ can be reduced to two complex equations.

A. Six-dimensional reduction

We now introduce two new complex variables $y(t), s(t)$ and new dynamical equations,

$$\dot{Q} = (i\omega - \gamma)Q + h - h^*Q^2, \tag{16a}$$

$$\dot{y} = (i\omega - \gamma - 2h^*Q)y, \tag{16b}$$

$$\dot{s} = h^*y. \tag{16c}$$

Our goal below is to demonstrate that these equations are equivalent to the infinite system (14) and, therefore, to the original system (4). At this point, we would like to mention that Eqs. (16) look like a skew system: variable Q appears on the r.h.s. of (16b), and variable y appears on the r.h.s. of (16c), but variables y, s do not appear on the r.h.s. of (16a). However, in most applications, one describes a population with global coupling, where ω, h depend on the order parameters Z_n and, thus, on all dynamical variables Q, y, s [cf. Eq. (23)].

To show how this system represents dynamics (14), we first introduce additional auxiliary variables $\alpha_n(t)$ by transforming $\beta_n(t)$,

$$\beta_n(t) = y^n(t)\alpha_n(t). \tag{17}$$

We take the time derivative of this relation and divide both sides by $n\beta_n$

$$\frac{1}{n} \frac{\dot{\beta}_n}{\beta_n} = \frac{\dot{y}}{y} + \frac{1}{n} \frac{\dot{\alpha}_n}{\alpha_n},$$

and then insert the dynamics of β_n (14b) and y (16b),

$$(i\omega - \gamma - 2h^*Q) - h^* \frac{\beta_{n+1}}{\beta_n} = (i\omega - \gamma - 2h^*Q) + \frac{1}{n} \frac{\dot{\alpha}_n}{\alpha_n}.$$

Notice how the majority of the terms cancel, including all the effects of frequency $\omega(t)$ and noise γ . As a result, the dynamics of the variables $\alpha_n(t)$ simplifies to

$$\frac{1}{n} \dot{\alpha}_n = -h^*y \alpha_{n+1}. \tag{18}$$

Now let us introduce the OGF of the variables $\alpha_n(t)$,

$$\mathcal{A}(k, t) = \sum_{n=1}^{\infty} \alpha_n(t) k^n, \tag{19}$$

and express the dynamics (18) in terms of this OGF (see Appendix A for the derivation),

$$\dot{\mathcal{A}} = -h^*y \left[\mathcal{A}' - \frac{1}{k} \mathcal{A} \right]. \tag{20}$$

Next, we introduce yet another set of variables μ_n . This time we launch them not directly, but via an expression of the corresponding

OGF $\mathcal{M}(k, t) = \sum_{n=1}^{\infty} \mu_n k^n$ in terms of $\mathcal{A}(k, t)$,

$$\frac{\mathcal{M}(k)}{k} = \frac{\mathcal{A}(k+s)}{k+s}, \tag{21}$$

where s is the variable in (16c). By taking the time derivative of Eq. (21), we obtain

$$\frac{\dot{\mathcal{M}}(k)}{k} = \frac{\dot{\mathcal{A}}(k+s)}{k+s} + \dot{s} \left[\frac{\mathcal{A}'(k+s)}{k+s} - \frac{\mathcal{A}(k+s)}{(k+s)^2} \right].$$

Now we insert the dynamics of s according to Eq. (16c), as well as the dynamics of \mathcal{A} according to Eq. (20) and behold, the right-hand side of this relation is zero. This means that the OGF \mathcal{M} and the corresponding variables μ_n are constant in time: $\dot{\mu}_n = 0, \dot{\mathcal{M}}(k) = 0$. In other words, the variables μ_n are integrals of motion. This completes the proof that Eqs. (16) are equivalent to Eqs. (14).

The radius of convergence of the constant function $\mathcal{M}(k)$ is determined by the asymptotic behavior of the coefficients μ_n . Below we show that these coefficients initially coincide with the order parameters $Z_n(0)$. Since they are bounded $|Z_n| \leq 1$, the radius of convergence of \mathcal{M} is at least one. During dynamical evolution, the relevant argument of function \mathcal{M} is $s(t)$, and in all our simulations we never observed it getting larger than one in absolute value: $|s(t)| < 1$.

It is instructive to rephrase the relation (21), formulated above in terms of OGFs, to the level of the variables (where it corresponds to a modified binomial transform, see Appendix B for the derivation),

$$\begin{aligned} \mu_n &= \sum_{m=n}^{\infty} \binom{m-1}{n-1} \alpha_m(t) [s(t)]^{m-n}, \\ \alpha_n(t) &= \sum_{m=n}^{\infty} \binom{m-1}{n-1} \mu_m [-s(t)]^{m-n}. \end{aligned} \tag{22}$$

Notice that we do not write the time argument of μ_n because these quantities are constants.

Using this relation, as well as how the variables $\alpha_n(t)$ relate to $\beta_n(t)$ (17), and then how $\beta_n(t)$ relate to the order parameters $Z_n(t)$ (15), we can express the order parameters in terms of the constant function $\mathcal{M}(k)$ and the three dynamic variables $Q(t), y(t), s(t)$ (see Appendix C for the derivation) for all times (here we omit the time dependence in notation for convenience),

$$Z_n = Q^n - \sum_{m=1}^n \binom{n}{m} Q^{n-m} y^m \sum_{d=0}^{m-1} \frac{s^{d-m}}{d!} \mathcal{M}^{(d)}(-s), \tag{23}$$

where $\mathcal{M}^{(d)}$ denotes the d th derivative of \mathcal{M} with respect to k . In particular, the first circular moment (the Kuramoto order parameter) is expressed as

$$Z_1 = Q - y \frac{\mathcal{M}(-s)}{s}. \tag{24}$$

Notice how at $s = 0$ one has to take the limit $\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{M}(-\varepsilon)}{\varepsilon} = -\mu_1$. For all moment expressions (23), we have to consider similar limits:

$\lim_{\varepsilon \rightarrow 0} \sum_{d=0}^{m-1} \frac{\varepsilon^{d-m}}{d!} \mathcal{M}^{(d)}(-\varepsilon) = -\mu_m$. When performing numerical integration, one thus requires an expansion of the above expression (23) for small Q and small s ,

$$Z_n = y^n \left[\mu_n + n\mu_{n-1} \frac{Q}{y} - n\mu_{n+1} s \right] + \mathcal{O}(Q^2, Qs, s^2), \tag{25}$$

we remind that $\mu_0 \equiv 1$ for normalization reasons. The need for expansion (25) can also be avoided by considering a different definition of the constant function (21), see Appendix E—it also simplifies the expression for moments (23).

B. Initial conditions

The new set of variables Q, y, s, μ_n is not uniquely determined by the initial order parameters Z_n , and in this section we discuss possible variants of determining them. Different choices for the initial conditions of Q, y, s, μ_n define the constant function $\mathcal{M}(k)$ differently. We first illustrate this with the simplest example of the OA manifold.

1. Different choices of variables for OA initial conditions

As discussed above, on the OA manifold the EGF reads $Z(k, t) = \exp[kZ(t)]$ and the order parameter Z obeys $\dot{Z} = (i\omega - \gamma)Z + h - h^*Z^2$. Suppose, having a set of initial moments $Z_n(0) = Z^n(0)$, we want to introduce new variables Q, β_n according to (11). One can immediately see that a set $\beta_n(0) = \beta^n(0)$ is admissible if $Q(0) + \beta(0) = Z(0)$, in this case $\mathbf{B}(k, 0) = \exp[k\beta(0)]$. Furthermore, relation (17) allows for different choices of $y(0)$ and $\alpha_n(0)$. For any choice of $y(0)$, we obtain $\alpha_n(0) = \alpha^n$, with $\alpha = \beta(0)/y(0)$. It is easy to see that independently of the choice of $\beta(0)$ and $y(0)$, the dynamics of the order parameters is the same. Indeed, in this case $\mathcal{M}(k) = \alpha k / (1 - \alpha k)$ and the main order parameter according to (24) is $Z_1 = Q + \alpha y / (1 + \alpha s)$. Calculation of the derivative \dot{Z}_1 from the general dynamical equations (16) yields the correct equation $\dot{Z}_1 = (i\omega - \gamma)Z_1 + h - h^*Z_1^2$, i.e., the system remains on the OA manifold. In this specific case of pure OA dynamics, it is natural to consider $\alpha = 0$ such that $\mathcal{M} = 0$ and the only relevant equation is the OA equation (16a).^{34,35}

2. Variant A: A simple choice of initial variables

Here, we present possibly the simplest choice of initial conditions for Q, y, s, μ_n . We initially set Q and s to zero and set y to 1 so that all the variable sets μ_n, α_n, β_n , and Z_n coincide

$$\begin{aligned} Q(0) &= 0, \\ y(0) &= 1, \\ s(0) &= 0, \\ \mu_n &= \alpha_n(0) = \beta_n(0) = Z_n(0). \end{aligned} \tag{26}$$

Then, the constant function can simply be determined by the initial order parameters,

$$\mathcal{M}(k) = \sum_{n=1}^{\infty} Z_n(t=0) k^n. \tag{27}$$

It is instructive to express this function in terms of the initial distribution of the phases

$$\mathcal{M}(k) = \int_0^{2\pi} d\varphi P(\varphi, 0) \frac{k e^{i\varphi}}{1 - k e^{i\varphi}}. \tag{28}$$

As discussed above, this is not the only possible choice of initial conditions and in some cases might not be optimal. Notice that for this choice, the OA manifold corresponds to the constants being powers of $Z_1(0)$: $\mu_n = Z_1^n(0)$, so they do not vanish, as is commonly considered.^{34,35} In terms of the dynamics, the two descriptions are equivalent.

Next, we list specific functions $\mathcal{M}(k)$ for some examples of the initial states in this variant A of the initial conditions (26). We stress here that a representation via an initial distribution density is valid for the interpretation of the system with identical oscillators under Cauchy white noise (Sec. II A). For the case of non-identical oscillators with a distribution of frequencies (Sec. II B), one should operate with the order parameters directly.

- A uniform distribution $P(\varphi, 0) = \frac{1}{2\pi}$ corresponds to $\mathcal{M}(k) = 0$. Here, all the moments vanish, which is a trivial invariant state of the dynamics (2).
- A delta distribution of the phases $P(\varphi, 0) = \delta(\varphi - \varphi_0)$ corresponds to $Z_n = \exp(in\varphi_0)$ and, thus, $\mathcal{M}(k) = \frac{e^{i\varphi_0} k}{1 - e^{i\varphi_0} k}$.
- A wrapped Cauchy distribution $P(\varphi, 0) = \frac{1}{2\pi} \frac{1 - |\mu|^2}{|1 - \mu e^{-i\varphi}|^2}$ with a complex parameter $\mu \in \mathbb{C}$ corresponds to $\mathcal{M}(k) = \frac{\mu k}{1 - \mu k}$. The moments are powers of the parameter μ , $\mu_n = \mu^n$, which means this state is on the OA manifold.¹⁶
- A Kato–Jones distribution³⁶ $P(\varphi, 0) = \frac{1}{2\pi} (1 + c \frac{\mu e^{-i\varphi}}{1 - \mu e^{-i\varphi}} + \text{c.c.})$ with complex parameters $c, \mu \in \mathbb{C}$, corresponds to $\mathcal{M}(k) = c \frac{\mu k}{1 - \mu k}$. Its moments are described by a power series multiplied with a complex constant: $\mu_n = c \mu^n$, making it a skewed/asymmetric generalization of the wrapped Cauchy distribution.
- Distributions with a finite number of moments $P(\varphi, 0) = \frac{1}{2\pi} (1 + \sum_{n=1}^N \mu_n e^{-in\varphi} + \text{c.c.})$ correspond to polynomials $\mathcal{M}(k) = \sum_{n=1}^N \mu_n k^n$.
- Distributions with binomial moments $\mu_n = \binom{n}{m} \mu^m$ for $m \geq 1$ correspond to rational functions $\mathcal{M}(k) = \frac{(\mu k)^m}{(1 - \mu k)^{m+1}}$. Their phase density reads $P(\varphi, 0) = \frac{1}{2\pi} (1 + \frac{\mu^m e^{-im\varphi}}{(1 - \mu e^{-i\varphi})^{m+1}} + \text{c.c.})$.
- A half-uniform distribution

$$P(\varphi, 0) = \begin{cases} \frac{1}{\pi} & \text{if } \varphi \in (\varphi_0, \varphi_0 + \pi), \\ 0 & \text{else,} \end{cases} \tag{29}$$

corresponds to $\mathcal{M}(k) = \frac{2i}{\pi} \operatorname{arctanh}(e^{i\varphi_0} k)$. Here, odd moments are fractions $\mu_{2n-1} = \frac{2i}{\pi} \frac{\exp(i(2n-1)\varphi_0)}{2n-1}$, and even ones are equal to zero $\mu_{2n} = 0, n \geq 1$.

- A sawtooth distribution $P(\varphi, 0) = \frac{1}{\pi} (1 - \frac{\varphi - \varphi_0}{\pi} + \lfloor \frac{\varphi - \varphi_0}{\pi} \rfloor)$ corresponds to $\mathcal{M}(k) = -\frac{i}{\pi} \log(1 - e^{i2\varphi_0} k^2)$. Here, even moments are fractions $\mu_{2n} = \frac{2i}{\pi} \frac{\exp(i2n\varphi_0)}{2n}$, and odd ones are equal to zero $\mu_{2n-1} = 0, n \geq 1$.

In practice, the phase density can only be observed empirically and, therefore, it may not be clear how to choose $\mathcal{M}(k)$. In such situations, it can always be approximated with a finite truncation of its Taylor series $\mathcal{M}(k) \approx \sum_{n=1}^N Z_n(0) k^n$, one only needs to estimate the first few initial moments $Z_n(0)$. This just corresponds to approximating the initial state with a finite Fourier representation. In Appendix F, we present a numerical example where we test the convergence of such approximations.

We end this subsection with the following remark: if the initial distribution of the phases is a weighted sum of “elementary” distributions $P(\varphi, 0) = \sum_m c_m P_m(\varphi, 0)$ with real weights $c_m \in \mathbb{R}$ that add up to 1: $\sum_m c_m = 1$ [additionally one should ensure that $P(\varphi, 0) \geq 0$], then the constant generating function is the weighted sum of the corresponding “elementary” generating functions $\mathcal{M}(k) = \sum_m c_m \mathcal{M}_m(k)$. In particular, in Refs. 21 and 37 a superposition of several wrapped Cauchy distributions has been considered as an initial state; in terms of the approach above this corresponds to $\mathcal{M}(k) = \sum_m c_m \frac{\varkappa_m k}{1 - \varkappa_m k}$, where complex parameters \varkappa_m characterize partial distributions.

3. Variant B: Initial conditions based on the OA manifold

Often initial states that are close to the OA manifold are of interest. Suppose that the order parameters are well described as powers of a complex constant, with minor perturbations,

$$Z_n(0) = R^n + \varepsilon_n. \tag{30}$$

In this case, a different initial condition appears natural,

$$\begin{aligned} Q(0) &= R, \\ y(0) &= 1, \\ s(0) &= 0, \end{aligned} \tag{31}$$

$$\mu_n = \alpha_n(0) = \beta_n(0) = \sum_{m=1}^n \binom{n}{m} \varepsilon_m (-R)^{n-m},$$

and the constant function $\mathcal{M}(k)$ is expressed as

$$\mathcal{M}(k) = \sum_{n=1}^{\infty} k^n \sum_{m=1}^n \binom{n}{m} \varepsilon_m (-R)^{n-m}. \tag{32}$$

This function is small if values of ε_n are small. Notice, however, that this “perturbation” approach is actually global, because smallness of ε_n is not needed, and $\mathcal{M}(k)$ need not be small for this description to be valid. Note that the definition of function $\mathcal{M}(k)$ depends on the choice of initial conditions [cf. (27) and (32)], so the list of specific $\mathcal{M}(k)$ functions in Sec. IV B 2 does not apply here. The moments are

still described with Eq. (23) but for numerical integration one needs to expand them beyond (25) for only small s (such expansions can be avoided by considering an alternative constant function to \mathcal{M} , see Appendix E).

A simple specific example where the perturbation to the OA manifold in (30) is one where only the first harmonic term is perturbed: $\varepsilon_n = 0$ for $n > 1$. In this case, $\mathcal{M}(k) = \varepsilon_1 \frac{k}{(1+Rk)^2}$ and so the first moment is given by $Z_1 = Q + \varepsilon_1 \frac{\gamma}{(1-Rs)^2}$. The dynamics follow (16).

We mention here that in some cases, an extension of the set of variables might be appropriate. As an example, we show in Appendix D the possibility to describe an initial state similar to (30) with a system of four complex variables.

C. Finite ensemble numerical comparison

Here, we numerically compare the derived low-dimensional dynamics (16), which is exact in the thermodynamic limit, with a simulation of a finite ensemble. We take the example already explored in Ref. 22: an array of overdamped noisy Josephson junctions coupled via a resistive load.¹⁵ The equations for the Josephson phases read

$$\dot{\varphi}_j = 1 + a \sin(\varphi_j) + \frac{\varepsilon}{N} \sum_{n=1}^N \sin(\varphi_n) + \gamma \xi_j(t). \quad (33)$$

In terms of the basic model (1), this corresponds to the choice of ω, h : $\omega = 1 + \varepsilon \text{Im}[Z_1]$ and $h = -\frac{a}{2}$. We consider parameters $a = -0.7$, $\varepsilon = 1.5$, and noise strength $\gamma = 0.02$. To highlight the advantage of the new derivation, we consider the initial distribution of phases to be half-uniform (29), thus starting far from the OA manifold, and also far from initial states that are easy to represent in terms of the Q, β_n hierarchy.²² For the finite ensemble, we consider 10 000 phases, which are initially randomly sampled in the interval $[0, \pi)$. For our exact reduction in the thermodynamic limit, we take initial conditions from variant A (26): $Q(0) = s(0) = 1 - \gamma(0) = 0$ and, therefore, the constant function $\mathcal{M}(k) = \frac{2i}{\pi} \arctanh(k)$. The comparison of the trajectories in Fig. 1 shows a very close match, and we expect it to be even closer for larger ensembles. The discrepancies appear to be only due to finite size effects, we stress that in the limit of an infinitely large ensemble our reduction (16) is exact.

V. GLOBAL STABILITY OF THE OA MANIFOLD

Stability of the OA manifold has been discussed in Refs. 19 and 21. Here, we demonstrate how these results are reproduced in our approach. To show the attractiveness of the OA manifold, it is enough to demonstrate that the variable y tends to zero $y \rightarrow 0$. Indeed, for $y = 0$, we have from (17) $\beta_n = 0, n \geq 1$, and from (11) it follows that the solution is on the OA manifold.

Let us introduce two new variables Y, S according to relations

$$y = (1 - |Q|^2)Y, \quad s = Q^*S. \quad (34)$$

The equations for these variables read

$$\dot{Y} = \left(i\omega + hQ^* - h^*Q - \gamma \frac{1 + |Q|^2}{1 - |Q|^2} \right) Y, \quad (35a)$$

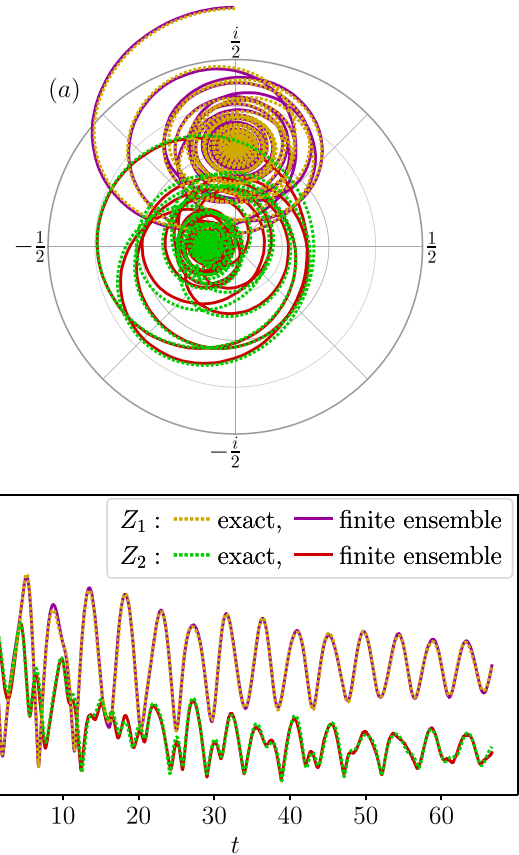


FIG. 1. Comparison of the finite ensemble dynamics of 10 000 Josephson phases (33) with the low-dimensional dynamics in the thermodynamic limit (16). In panel (a), the first two circular moments are compared in the complex plane, and in panel (b) their absolute values are compared as functions of time. Solid lines: simulation of a finite ensemble; dotted lines (which practically overlap with solid lines): solution of exact equations in the thermodynamic limit.

$$\dot{S} = \left(i\omega + hQ^* - h^*Q + \gamma \right) S + \left(h^*Q - \frac{h^*}{Q^*} \right) (S - Y). \quad (35b)$$

Here, we focus on the equation for Y and will use Eq. (35b) in Sec. VI. From (35a), we obtain the following evolution of $|Y|$:

$$\frac{d|Y|}{dt} \frac{1}{|Y|} = -\gamma \frac{1 + |Q|^2}{1 - |Q|^2} \leq -\gamma. \quad (36)$$

The latter inequality on the r.h.s. follows from the property $0 \leq |Q|^2 < 1$. Indeed, the equation for the evolution of $|Q|^2$ reads

$$\frac{d}{dt} |Q|^2 = -2\gamma |Q|^2 + (hQ^* + h^*Q)(1 - |Q|^2)$$

and at $|Q| = 1$ the derivative $\frac{d}{dt} |Q|^2 = -2\gamma$ is negative for $\gamma > 0$; thus, this boundary is not reachable.

Integrating inequality (36) yields

$$|Y(t)| \leq |Y(0)| e^{-\gamma t}, \quad (37)$$

and consequently, since $|y| \leq |Y|$,

$$|y(t)| \leq \frac{|y(0)|}{1 - |Q(0)|^2} e^{-\gamma t}, \quad (38)$$

which means that $|Y|$ and $|y|$ vanish exponentially fast in time. This proves the attractiveness of the OA manifold for system (4) for $\gamma > 0$.

A. Example

Here, we illustrate the stability of the OA manifold numerically. We take the already explored example of Josephson junctions (33), with the same parameters $a = -0.7$, $\varepsilon = 1.5$. In Ref. 22, it was demonstrated that for $\gamma = 0$, the dynamics outside of the OA manifold is chaotic. In Fig. 2, we again demonstrate chaotic behavior in this system for $\gamma = 0$, and a transition to regular dynamics for $\gamma = 10^{-4}$ and $\gamma = 2 \times 10^{-4}$. The exponential decay of $|y|$, which is bounded by (38), is evident at large times.

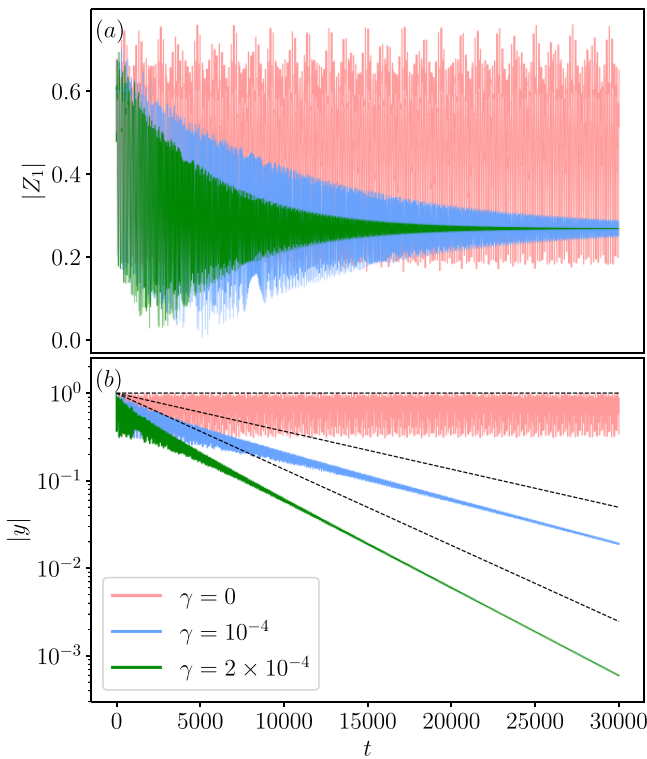


FIG. 2. Time series of the coupled Josephson junctions array (33) for three different noise strengths; $\gamma = 0$ depicted with red, $\gamma = 10^{-4}$ depicted with blue, and $\gamma = 2 \times 10^{-4}$ depicted with green. (a) the norm of the first order parameter Z_1 ; (b) the norm of variable y . The initial conditions for all three cases are the same: $Q(0) = s(0) = 1 - \gamma(0) = 0$ (variant A) and constant function $\mathcal{M}(k) = 0.4k$, which means the initial distribution has only one harmonic. In the case of no noise, chaotic dynamics is observed, while in the noisy case there is an initial chaotic stage which is followed by a clear exponential decay of the y variable. In panel (b), the upper bound (38) for all three cases is shown with a dashed black line.

VI. NOISE-FREE CASE AND A RELATION TO THE WATANABE-STROGATZ THEORY

Watanabe and Strogatz^{14,15} demonstrated that a population of identical noiseless oscillators can be reduced to three real dynamical variables plus constants of motion. To see that this case is included in our theory, let us consider identical oscillators and no noise, thus taking $\gamma = 0$.

It is instructive to start with Eq. (35). It is easy to see, that for $\gamma = 0$, the manifold $S = Y$ is invariant,

$$\frac{d}{dt}(S - Y) = \left[i\omega + hQ^* - \frac{h^*}{Q^*} \right] (S - Y).$$

Using the definition of variables Y, S (34), the manifold is described in terms of y and s as

$$Y = \frac{y}{1 - |Q|^2} = \frac{s}{Q^*} = S. \quad (39)$$

Since we initially set $Q(0) = s(0) = 1 - \gamma(0) = 0$ for arbitrary states, one can always set initial conditions on this manifold such that $Y(0) = S(0) = 1$. Thus, two equations (35) reduce to one. Moreover, because of (36), for $\gamma = 0$ the variable Y remains on the unit circle for all times, and we can introduce an angle variable $\theta(t) = \arg(Y(t))$. Its evolution follows from (35),

$$\dot{\theta} = \omega - i(hQ^* - h^*Q). \quad (40)$$

This angle variable θ just corresponds to the WS angle variable, while Q is the WS order parameter.^{14,15} The two equations (16a) (with $\gamma = 0$) and (40) then represent the exact evolution. We explicitly derive the equivalence with the WS approach in Appendix G.

In the WS theory, the relation between the original phases $\varphi_j(t)$ and constant phases ψ_j is given by the Möbius transform,^{15,38}

$$e^{i\varphi_j} = \frac{e^{i(\psi_j + \theta)} + Q}{1 + Q^* e^{i(\psi_j + \theta)}}, \quad e^{i(\psi_j + \theta)} = \frac{e^{i\varphi_j} - Q}{1 - Q^* e^{i\varphi_j}}. \quad (41)$$

Our choice of the initial conditions $Q(0) = \theta(0) = 0$ [it corresponds to the “identity conversion” in terms of WS, cf. Eq. (5.10) in Ref. 15] means that $\psi_j = \varphi_j(0)$. Thus, because the integrals μ_n are defined as $\mu_n = Z_n(0)$, these quantities are the circular moments of the transformed constant phase variables in the WS approach $\mu_n = \langle e^{in\psi} \rangle$.

Watanabe and Strogatz have shown that these transformations are also valid for a finite number of oscillators, but this case is not covered by our approach. We mention here that in the WS formalism there is also a freedom in choosing the order parameter Q and the phase variable θ ; this freedom is similar to the one discussed in Sec. IV B.

VII. LYAPUNOV SPECTRUM

Our theory describes the dynamics outside of the OA manifold and is, thus, suitable for the consideration of small perturbations transversal to this manifold. Such perturbations define the Lyapunov spectrum of the dynamics, together with the perturbations tangential to this manifold. The system of Eq. (14) is most suitable for this analysis. The OA manifold corresponds to vanishing β_n ; therefore,

Eq. (14b) defines the transversal perturbations. Since these equations are a skew system, each β_n defines two Lyapunov exponents (because β_n are complex). One can straightforwardly derive from (14b), omitting the skew term $\sim \beta_{n+1}$ on the r.h.s., the averaged evolution for the magnitude of a perturbation,

$$\frac{1}{2n} \left\langle \frac{d}{dt} \ln |\beta_n|^2 \right\rangle = (-\gamma - h^*Q - hQ^*) = \Lambda.$$

Thus, the Lyapunov spectrum consists of the exponents within the OA manifold [which are calculated using linearized equation (9)] and of doubly degenerated values $n\Lambda$, $n = 1, 2, 3, \dots$

VIII. RESPONSE OF THE OTT-ANTONSEN REGIME TO A RESETTING

As has been already discussed in the literature¹⁹⁻²¹ and in Sec. V, in the system of Eqs. (16) the OA manifold is attracting if $\gamma > 0$ (at least in the weak sense, but because we follow only the moments of the phase distribution, such an attraction is enough). In terms of variables Q, y, s with a nontrivial constant function $\mathcal{M}(k)$, this corresponds to $\gamma \rightarrow 0$ as $t \rightarrow \infty$ (see Sec. V). For the conservative case $\gamma = 0$, see Sec. VI.

The approach above allows for calculating the evolution from an arbitrary state to the OA manifold via solutions of (16). One can reformulate such a problem as a resetting one: One starts with the dynamics on the OA manifold; then an instant “resetting” to a state outside of this manifold is performed. The evolution of (16) then shows what will be the final state after re-attraction to the OA manifold. A particular question of interest depends on the type of attractors on the OA manifold. If there is only one global attractor, then the trajectory returns to it. If this attractor is periodic or quasiperiodic, the returning trajectory will be phase shifted with respect to the unperturbed one (in the quasiperiodic case, one expects phase shifts in every direction of independent oscillations). Here, one speaks about a phase resetting or a phase response curve (PRC).^{39,40} For a chaotic global attractor, generally one does not expect a resetting to have a drastic effect (although for strange attractors with well-defined phase variables a phase resetting similar to the periodic case can be defined;⁴¹ it can lead to phase synchronization of chaos if periodically repeated⁴²). In the case of multistability, the most drastic effect of resetting would be a jump to another basin of attraction so that the final state will be another attractor on the OA manifold (in case of multistable periodic attractors one can additionally follow the phase response⁴³). Below we consider several examples, for small and large resettings.

A. Perturbation theory in terms of an (infinitesimal) PRC

Suppose we have a state on the OA manifold with a complex order parameter R so that $\langle e^{im\varphi} \rangle = R^n$. Let us apply to all the phases a transformation

$$\varphi \rightarrow \varphi + \varepsilon f(\varphi), \tag{42}$$

where $f(\varphi) = \sum_m f_m e^{im\varphi}$ is a PRC function (given by its Fourier representation) and $\varepsilon \ll 1$ is assumed to be small. Let us calculate the

circular moments just after a resetting, in order ε :

$$\begin{aligned} Z_n &= \langle e^{in(\varphi + \varepsilon f(\varphi))} \rangle \approx \langle e^{in\varphi} (1 + in\varepsilon f(\varphi)) \rangle \\ &= R^n + in\varepsilon \sum_{m=-\infty}^{\infty} f_m \langle e^{i(n+m)\varphi} \rangle. \end{aligned}$$

Since m can be negative, the calculation of the latter average is not a simple expression because

$$\langle e^{i(n+m)\varphi} \rangle = \begin{cases} R^{n+m} & n+m \geq 0, \\ (R^*)^{|n+m|} & n+m < 0, \end{cases}$$

therefore we restrict ourselves to the two simplest cases.

1. First harmonics resetting

In this case, $f(\varphi) = f_1 e^{i\varphi} + f_1^* e^{-i\varphi}$. For $n \geq 1$, we have $n+m \geq 0$ and, therefore, for both $m = \pm 1$ we can write $\langle e^{i(n+m)\varphi} \rangle = R^{n+m}$. Thus,

$$Z_n = R^n (1 + in\varepsilon (f_1 R + f_1^* R^{-1})), \quad n \geq 0.$$

Calculation of the EGF yields

$$Z(k, 0) = e^{kR} (1 + i\varepsilon k (f_1 R^2 + f_1^*)),$$

where we used $\sum_n n \frac{x^n}{n!} = xe^x$.

Let us now transform to variables Q, y, s and take $Q(0) = R$ (like in variant B, Sec. IV B 3). This means that the EGF B is

$$B(k, 0) = 1 + i\varepsilon k (f_1 R^2 + f_1^*).$$

We come to the conclusion that only one variable β_1 is non-zero, and the system can be directly and exactly solved with variables Q, β_1 by virtue of Eqs. (14); there is no need to go to the full system (16). Alternatively, one can consider only the first two equations of system (16) and function $\mathcal{M}(k) = \beta_1(0)k$ and then the third variable s does not matter. If we rewrite Eqs. (14) in terms of variables (Z_1, β_1) , we obtain

$$\begin{aligned} \dot{Z}_1 &= (i\omega - \gamma)Z_1 + h - h^*Z_1^2 + h^*\beta_1^2, \\ \dot{\beta}_1 &= (i\omega - \gamma - 2h^*(Z_1 - \beta_1))\beta_1. \end{aligned}$$

One can see that the correction to the standard OA equation is $h^*\beta_1^2 \sim \varepsilon^2$. Thus, in the first order in ε , the inclusion of the additional variable β_1 is irrelevant and the resetting is well described within the OA equation.

2. Second harmonics resetting

In this case, $f(\varphi) = f_2 e^{i2\varphi} + f_2^* e^{-i2\varphi}$, and we have $\sum_m f_m \langle e^{i(n+m)\varphi} \rangle = f_2 \langle e^{i(n+2)\varphi} \rangle + f_2^* \langle e^{i(n-2)\varphi} \rangle$. Thus, the term with $n = 1$ reads $f_2 R^3 + f_2^* R^*$, while all the higher-order terms $n \geq 2$ can be written in a unified way $f_2 R^{n+2} + f_2^* R^{n-2}$. Rewriting the term with $n = 1$ as

$f_2 R^3 + f_2^* R^{-1} + [f_2^* R^* - f_2^* R^{-1}]$, we obtain

$$Z_n = R^n + i\varepsilon n R^n [f_2 R^2 + f_2^* R^{-2}] + i\varepsilon \delta_{n,1} [f_2^* R^* - f_2^* R^{-1}],$$

where $\delta_{n,1}$ is the Kronecker delta. This yields the following EGF:

$$Z(k, 0) = e^{kR} (1 + i\varepsilon k (f_2 R^3 + f_2^* R^{-1})) + i\varepsilon k [f_2^* R^* - f_2^* R^{-1}].$$

Now the EGF $B(k, 0)$ is nontrivial, choosing $Q(0) = R$,

$$B(k, 0) = 1 + i\varepsilon k (f_2 R^3 + f_2^* R^{-1}) + i\varepsilon k e^{-kR} [f_2^* R^* - f_2^* R^{-1}].$$

This allows for obtaining a closed expression for the constant function [for choice $y(0) = 1, s(0) = 0$] as

$$\mathcal{M}(k) = i\varepsilon k \left[f_2 R^3 + f_2^* \frac{R^* + 2k + Rk^2}{(1 + Rk)^2} \right].$$

After this, system (16) is to be solved.

B. Large resettings

Unfortunately, a transformation of the type (42) is hardly tractable for large ε . Here, we discuss another way of resetting, which leads to closed expressions even for large changes of the phases. This approach is applicable to identical oscillators subject to Cauchy white noise, but not for the distribution of natural frequencies.

Suppose, in the OA state with order parameters $Z_n = R^n$, we randomly choose a portion ε of all oscillators and reset them completely (this means that they “forget” their old states), cf. Ref. 44. We consider two variants below.

1. Random resetting

Here, we assume that the new phases in the affected set become uniformly distributed in the interval $[0, 2\pi)$. These oscillators do not contribute to new order parameters, which, thus, take the values $Z_n = (1 - \varepsilon)R^n$. This corresponds to an offset Cauchy distribution or, specifically, a weighted superposition of the Cauchy distribution and the uniform distribution. If one uses variant A of initial conditions, then evolution starts from the initial values $Q(0) = s(0) = 0, y(0) = 1$, and $\mathcal{M}(k) = (1 - \varepsilon)Rk / (1 - Rk)$ is determined via Eq. (27). Alternatively, adopting variant B, one can start from the initial conditions $Q(0) = R, y(0) = 1, s(0) = 0$ and then $\mathcal{M}(k) = -\varepsilon Rk / (1 + Rk)$ is determined via Eq. (32). Then, the system evolves according to Eqs. (16).

However, due to the simplicity of this example, there is an even easier way of treating this situation. Notice how moments can be viewed as a superposition of two OA contributions, referred to as Poisson kernels by Ref. 21,

$$Z_n = (1 - \varepsilon)Q_1 + \varepsilon Q_2, \tag{43}$$

where initially $Q_1(0) = R$ and $Q_2(0) = 0$. In this case, therefore, one can evolve the system by considering two OA equations (9), which only interact through the forcing h , and the solution maintains the form (43) for all times.

2. Coherent resetting

Consider now that reset phases are not distributed uniformly but rather take on another distribution $P^{(\text{res})}(\varphi)$. If this distribution

is a wrapped Cauchy (which includes the uniform and the delta distribution), the setting can again be treated simply as a superposition of the OA modes (a.k.a. Poisson kernels).⁴⁵ However, the reset distribution can generally have a different form. Below we consider two cases.

Case (i): Partially coherent resetting. Here, the reset phases are distributed according to a single harmonic density: $P^{(\text{res})}(\varphi) = \frac{1}{2\pi} [1 + 2c \cos(\varphi - \varphi_0)]$, $c, \varphi_0 \in \mathbb{R}$. The reset distribution, therefore, has only one non-zero moment $Z_1^{(\text{res})} = ce^{i\varphi_0}$, and the full distribution after a portion ε of phases are reset is described by $Z_n = (1 - \varepsilon)R^n + \varepsilon ce^{i\varphi_0} \delta_{n,1}$. If using variant A initial conditions, the variables initialize as $Q(0) = s(0) = 0, y(0) = 1$, and $\mathcal{M}(k) = (1 - \varepsilon) \frac{Rk}{1 - Rk} + \varepsilon ce^{i\varphi_0} k$ according to Eq. (27). Alternatively if choosing variant B, then $Q(0) = R, y(0) = 1, s(0) = 0$, and $\mathcal{M}(k) = -\varepsilon \left[\frac{Rk}{1 + Rk} + \frac{ce^{i\varphi_0} k}{(1 + Rk)^2} \right]$ according to Eq. (32). Then, the system evolves following Eqs. (16). One could also treat this setting as one OA mode and one general contribution described by full equations (16), we provide such a description in Appendix D.

Case (ii): Fully coherent resetting. Here, the reset phases take the same value φ_0 . The new order parameters are, thus, $Z_n = (1 - \varepsilon)R^n + \varepsilon e^{in\varphi_0}$. Again both variants of initializing the variables after the reset are possible. In variant A, one sets $Q(0) = s(0) = 0, y(0) = 1$ and $\mathcal{M}(k) = (1 - \varepsilon) \frac{Rk}{1 - Rk} + \varepsilon \frac{e^{i\varphi_0} k}{1 - e^{i\varphi_0} k}$ is determined via Eq. (27), while following variant B, we can set $Q(0) = R, y(0) = 1, s(0) = 0$ and $\mathcal{M}(k) = \varepsilon \frac{(e^{i\varphi_0} - R)k}{1 - (e^{i\varphi_0} - R)k}$ is determined via Eq. (32). As mentioned before, since the delta distribution is a special case of the wrapped Cauchy, this case could also be described by just two OA modes (43).

The expressions above can be readily extended to a setup where several randomly chosen subpopulations of oscillators are reset with different distributions. Such an approach has been discussed in the context of application to de-synchronization of neurons for Parkinson patients.⁴⁶

3. Numerical example for large resettings

As an example, we consider a simple Kuramoto-type system with a synchrony-asynchrony bistability.⁴⁷ In this setup, $\omega = \text{const}$ (and one can without loss of generality set this parameter to zero), and force is

$$h = Z_1 \exp[i\theta_0 + i\theta_1 |Z_1|^2]. \tag{44}$$

We set $\gamma = 0.1, \theta_0 = 0.8\pi$, and $\theta_1 = 4$. For these parameters, the states with $Z_1 = 0$ and $|Z_1| \approx 0.948$ are both stable. We start with the latter state of a nearly synchronized ensemble and apply the three types of resetting as described above. By solving the reduced six-dimensional equations (16), we obtain the domain of parameters ε, φ_0 for which the resettings lead to a transition to the asynchronous state $Z_1 = 0$.

For the random resetting, there is no dependence on φ_0 , and the corresponding domain is $\varepsilon > 0.196$ (above red line in Fig. 3). For the coherent resetting, we consider two cases discussed above, (i) and (ii), and their corresponding basins are depicted in Fig. 3 with blue and green domains, respectively. One can see that in all three cases, a finite perturbation is needed to suppress synchrony. For the coherent resettings, there is an optimal combination of ε

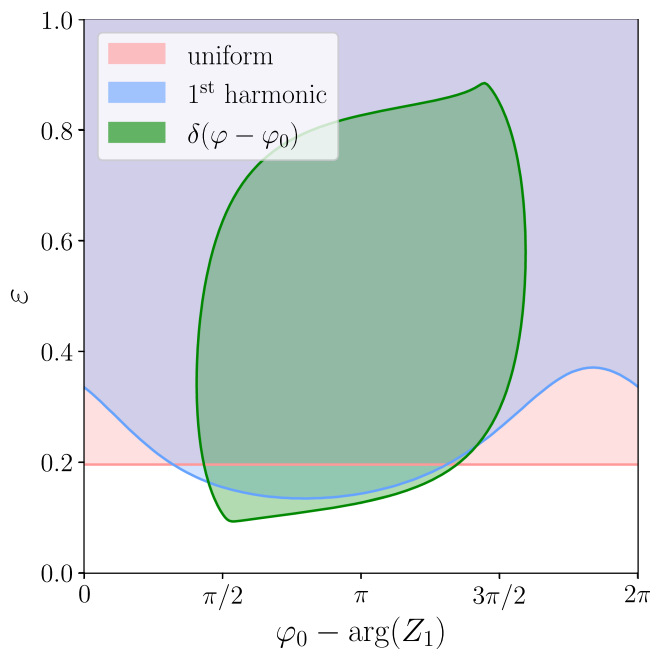


FIG. 3. Switching domains for large resetting in the bistable system (44). Both the synchronous and asynchronous regimes are stable. Starting from the synchronous regime on the OA manifold, we reset an ε portion of phases according to three example distributions: uniform (red domain), case (i) single harmonic density with amplitude $c = 0.5$ (blue domain), and case (ii) delta distribution (green domain). The shaded regions mark areas that induce a switch to asynchrony.

and φ_0 ; the coherent subpopulation should be phase shifted around π relative to the phase of the mean field of the non-reset units. For case (ii), we also see that if ε is too large, the reset units form a new cluster and the synchrony remains.

IX. CONCLUSION

First, we summarize the approach and findings of this paper. Our starting point is an infinite system of equations for the circular moments (order parameters). These equations contain damping due to either Cauchy white noise or a Cauchy distribution of natural frequencies. By virtue of several transformations, which are formulated in terms of generating functions, we reduce this system to three complex equations. Additionally, a complex-valued function of one variable is defined, which remains constant during the evolution. The order parameters at each moment of time are represented through this function and the three complex dynamical variables.

The original set of equations for the order parameters have the same form in two situations: if the phase oscillators are subject to a Cauchy white noise, and if the natural frequencies are Cauchy distributed (but time-independent). Only in the former case there is a simple unique correspondence between the order parameters and the distribution of the phases. In the latter situation, one can calculate the order parameters from the distribution of the phases (under the assumption of analyticity of the density in the upper

complex plane of frequencies), but it appears impossible to reconstruct this distribution from the order parameters without further assumptions. Therefore, the results of the paper are fully applicable to noisy ensembles, but some approaches (e.g., phase resetting) are not suitable for the oscillators with distributed frequencies.

The theory includes both the WS description (noise-free identical oscillators) and the OA manifold (on which the dynamical variable y vanishes). In the framework of our approach, one can simply demonstrate that the dynamical variable $y(t)$ tends to zero, which corresponds to the weak stability of the OA manifold discussed in the literature. Therefore, our approach is an essential improvement compared to OA theory, if a transient evolution from an initial state outside of the OA manifold is important. In particular, it allows for a calculation of the full basins of different attractors lying on the OA manifold.

In this paper, we operated with the phase equations. In some cases, it is convenient to transform the phase equations to other variables (e.g., theta-neurons, equations that belong to class (1),^{6,7} can be transformed to the so-called quadratic integrate-and-fire neurons⁴⁸⁻⁵⁰). An extension of the theory to quadratic integrate-and-fire neurons will be presented elsewhere.⁵¹

ACKNOWLEDGMENTS

We thank L. Smirnov and R. Toenjes for useful discussions. The work was supported by Deutsche Forschungsgemeinschaft (DFG) (Grant No. PI 220/21-1).

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Rok Cestnik: Conceptualization (equal); Formal analysis (equal); Investigation (lead); Methodology (equal); Validation (equal); Visualization (lead); Writing – original draft (equal). **Arkady Pikovsky:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (lead); Investigation (supporting); Methodology (equal); Supervision (lead); Validation (equal); Visualization (supporting); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: FROM THE DYNAMICS OF THE MOMENTS TO THE PDES FOR THE GENERATING FUNCTIONS

Consider a sequence of variables $\{f_n\}$ (the only condition is that the generating functions below do exist; because we apply the theory to bounded circular moments, this appears to always be the case). Also, in our case $f_0 \equiv 1$.

We distinguish between two types of generating functions: exponential generating function (EGF) is defined as

$$F(k) = \sum_{n=0}^{\infty} f_n \frac{k^n}{n!}, \tag{A1}$$

and ordinary generating function (OGF) is defined as

$$\mathcal{F}(k) = \sum_{n=1}^{\infty} f_n k^n. \tag{A2}$$

Here, we show how the dynamics of the variables f_n , given by an infinite set of ODEs, can be translated to the dynamics in terms of generating functions F and \mathcal{F} , given by a single partial differential equation (PDE). Suppose the dynamics of f_n is as follows:

$$\dot{f}_n = n(a f_{n-1} + b f_n + c f_{n+1}), \quad n \geq 1, \tag{A3}$$

where a, b, c are arbitrary complex quantities. Then, the PDE for the EGF F reads

$$\dot{F} = a k F + b k F' + c k F''. \tag{A4}$$

In fact, in the case of an EGF $F(k, t)$, any term f_{n+m} for $m \geq -1$ on the right-hand side of (A3) just corresponds to a term of the form $k F^{(m+1)}$ in Eq. (A4).

To prove this, it is sufficient to express derivatives of F as formal series,

$$\begin{aligned} \dot{F} &= \sum_{n=0}^{\infty} \dot{f}_n \frac{k^n}{n!}, \\ F &= \sum_{n=0}^{\infty} f_n \frac{k^n}{n!} = \frac{1}{k} \sum_{n=0}^{\infty} (n+1) f_n \frac{k^{n+1}}{(n+1)!} \\ &= \frac{1}{k} \sum_{n=-1}^{\infty} (n+1) f_n \frac{k^{n+1}}{(n+1)!} = \frac{1}{k} \sum_{n=0}^{\infty} n f_{n-1} \frac{k^n}{n!}, \\ F' &= \sum_{n=1}^{\infty} f_n \frac{k^{n-1}}{(n-1)!} = \frac{1}{k} \sum_{n=1}^{\infty} n f_n \frac{k^n}{n!} = \frac{1}{k} \sum_{n=0}^{\infty} n f_n \frac{k^n}{n!}, \\ F'' &= \sum_{n=2}^{\infty} f_n \frac{k^{n-2}}{(n-2)!} = \frac{1}{k} \sum_{n=2}^{\infty} (n-1) f_n \frac{k^{n-1}}{(n-1)!} \\ &= \frac{1}{k} \sum_{n=1}^{\infty} n f_{n+1} \frac{k^n}{n!} = \frac{1}{k} \sum_{n=0}^{\infty} n f_{n+1} \frac{k^n}{n!}, \\ F^{(m)} &= \sum_{n=m}^{\infty} f_n \frac{k^{n-m}}{(n-m)!} = \frac{1}{k} \sum_{n=m}^{\infty} (n-m+1) f_n \frac{k^{n-m+1}}{(n-m+1)!} \\ &= \frac{1}{k} \sum_{n=1}^{\infty} n f_{n+m-1} \frac{k^n}{n!} = \frac{1}{k} \sum_{n=0}^{\infty} n f_{n+m-1} \frac{k^n}{n!}. \end{aligned} \tag{A5}$$

Notice that the sum index changes in the derivation, e.g., if we have the summed terms proportional to n and the sum starts with index “1”: $\sum_{n=1}^{\infty} n(\dots)$, we can just add the zero term (since it vanishes) and write the sum from index “0.” The way we have derived the above expressions, the right-most forms of (A5) all correspond to

a sum starting at $n = 0$ and have a factor of $\frac{k^n}{n!}$; therefore, we can just compare the terms under the sum to see that the dynamics (A3) corresponds to the PDE (A4).

More generally, a term in the dynamics of the variables f_n having the form

$$\dot{f}_n = \dots + c n f_{n+m} + \dots, \quad m \geq -1, \tag{A6}$$

in terms of the EGF $F(k, t)$, corresponds to the partial derivative in the equation for F ,

$$\dot{F} = \dots + c k F^{(m+1)} + \dots. \tag{A7}$$

Now let us consider the OGF $\mathcal{F}(k, t)$. This generating function is applicable to systems with only the homogeneous term f_n and one higher term f_{n+1} on the r.h.s., i.e., to the case $a = 0$ in (A3). Let us write the relevant sums,

$$\begin{aligned} \dot{\mathcal{F}} &= \sum_{n=1}^{\infty} \dot{f}_n k^n, \\ k \mathcal{F}' &= \sum_{n=1}^{\infty} n f_n k^n, \\ \mathcal{F}' - \frac{1}{k} \mathcal{F} &= \sum_{n=1}^{\infty} n f_n k^{n-1} - f_n k^{n-1} \\ &= \sum_{n=1}^{\infty} (n-1) f_n k^{n-1} = \sum_{n=1}^{\infty} n f_{n+1} k^n. \end{aligned} \tag{A8}$$

Again, by comparing the right-most expressions of (A8), we can see that the dynamics

$$\dot{f}_n = b n f_n + c n f_{n+1}, \tag{A9}$$

in terms of the OGF \mathcal{F} , corresponds to

$$\dot{\mathcal{F}} = b k \mathcal{F}' + c \left[\mathcal{F}' - \frac{1}{k} \mathcal{F} \right]. \tag{A10}$$

APPENDIX B: TRANSFORMATION (21) IN TERMS OF VARIABLES

We rewrite here relation (21) as

$$\frac{\mathcal{M}(k)}{k} = \frac{\mathcal{A}(k+s)}{k+s}. \tag{B1}$$

Expressed as an OGF series (with a different index for the right-hand side), it reads

$$\sum_{n=1}^{\infty} \mu_n k^{n-1} = \sum_{r=1}^{\infty} \alpha_r (k+s)^{r-1}.$$

Next, we expand the binomial term on the r.h.s.:

$$\sum_{n=1}^{\infty} \mu_n k^{n-1} = \sum_{r=1}^{\infty} \alpha_r \sum_{d=0}^{r-1} \binom{r-1}{d} k^{r-d-1} s^d,$$

gather together terms for which $r - d - 1 = n - 1$, and use the same index n as on the l.h.s.,

$$\sum_{n=1}^{\infty} \mu_n k^{n-1} = \sum_{n=1}^{\infty} k^{n-1} \sum_{m=n}^{\infty} \binom{m-1}{m-n} \alpha_m s^{m-n}.$$

Now, it is clear that this transformation corresponds to (22)

$$\mu_n = \sum_{m=n}^{\infty} \binom{m-1}{m-n} \alpha_m s^{m-n}, \tag{B2}$$

where the binomial coefficient $\binom{m-1}{m-n}$ can also be written as $\binom{m-1}{n-1}$.

APPENDIX C: EXPRESSING MOMENTS Z_n WITH THE CONSTANT FUNCTION $\mathcal{M}(k)$

The d th derivative in k of $\mathcal{M}(k)$ is expressed as

$$\mathcal{M}^{(d)}(k) = \frac{d!}{k^d} \sum_{m=1}^{\infty} \binom{m}{d} \mu_m k^m. \tag{C1}$$

Next, using the binomial relation $\binom{m-1}{n-1} = (-1)^{n-1} \sum_{d=0}^{n-1} (-1)^d \binom{m}{d}$ and the relation between α_n and μ_n variables (22), we can write

$$\alpha_n = - \sum_{d=0}^{n-1} \frac{s^{d-n}}{d!} \mathcal{M}^{(d)}(-s). \tag{C2}$$

Then, we can use relation (17) to express β_n as

$$\beta_n = -y^n \sum_{d=0}^{n-1} \frac{s^{d-n}}{d!} \mathcal{M}^{(d)}(-s), \quad n \geq 1, \tag{C3}$$

and we can use (15) to express the moments

$$Z_n = Q^n - \sum_{m=1}^n \binom{n}{m} Q^{n-m} y^m \sum_{d=0}^{m-1} \frac{s^{d-m}}{d!} \mathcal{M}^{(d)}(-s). \tag{C4}$$

For initial condition variant A: $Q(0) = s(0) = 0, \gamma(0) = 1$, to see that expression (C4) is correct initially at $t = 0$, we have to evaluate the following limits:

$$\lim_{\varepsilon \rightarrow 0} \sum_{d=0}^{m-1} \frac{\varepsilon^{d-m}}{d!} \mathcal{M}^{(d)}(-\varepsilon) = -\mu_m, \tag{C5}$$

which confirm that $Z_n(0) = \mu_n$.

APPENDIX D: ALTERNATIVE PERTURBATION AROUND THE OA MANIFOLD

There are different ways one can write states close to the OA manifold. An alternative to the main text (30) is

$$Z_n = (1 - \varepsilon) Z^n + \varepsilon p_n, \quad |\varepsilon| \ll 1. \tag{D1}$$

Notice that due to the linearity of Eq. (8), we can view this simply as splitting the state into two parts: one on and one off the OA manifold: $Z_n^{(OA)} = Z^n$ and $Z_n^{(pert)} = p_n$. Each of the two parts can then be treated separately, the only thing connecting them is the force h .

We know that the part on the OA manifold requires only one complex equation (16a) by considering $\mathcal{M}^{(OA)} = 0$, while the rest can be treated as before with the full set of three equations (16) and the simplest general initial conditions (26). The complete dynamics is then written as

$$\begin{aligned} \dot{Z} &= (i\omega - \gamma)Z + h - h^* Z^2, \\ \dot{Q} &= (i\omega - \gamma)Q + h - h^* Q^2, \\ \dot{y} &= (i\omega - \gamma - 2h^* Q)y, \\ \dot{s} &= h^* y, \end{aligned} \tag{D2}$$

where the perturbations $Z_n^{(pert)} = p_n$ express with Q, y, s via Eq. (23). Global order parameters are simply the weighted sum of both contributions, e.g., the first order parameter is expressed as $Z_1 = (1 - \varepsilon)Z + \varepsilon \left(Q - y \frac{\mathcal{M}(-s)}{s} \right)$, where $\mathcal{M} = \sum_{n=1}^{\infty} p_n(t=0)k^n$. Notice that this perturbation description is actually global, ε does not need to be small.

APPENDIX E: ALTERNATIVE CONSTANT FUNCTION

$$\mathcal{N}(k) = \frac{\mathcal{M}(-k)}{-k}$$

Instead of transformation (21), we can consider the following

$$\mathcal{N}(k) = \frac{\mathcal{A}(s-k)}{s-k} = \frac{\mathcal{M}(-k)}{-k}. \tag{E1}$$

The OGF $\mathcal{N}(k)$ is also a constant function in time, its time derivative yields

$$\dot{\mathcal{N}}(k) = \frac{\dot{\mathcal{A}}(s-k)}{s-k} + \dot{s} \left[\frac{\mathcal{A}'(s-k)}{s-k} - \frac{\mathcal{A}(s-k)}{(s-k)^2} \right] = 0.$$

The benefit of this function is that it simplifies the expression for Kuramoto-Daido order parameters (23),

$$Z_n = Q^n \left[1 - \sum_{m=1}^n \binom{n}{m} \left(-\frac{y}{Q} \right)^m \frac{\mathcal{N}^{(m-1)}(s)}{(m-1)!} \right], \tag{E2}$$

the first three moments explicitly,

$$\begin{aligned} Z_1 &= Q + y\mathcal{N}(s), \\ Z_2 &= Q^2 + 2Qy\mathcal{N}(s) - y^2\mathcal{N}'(s), \\ Z_3 &= Q^3 + 3Q^2y\mathcal{N}(s) - 3Qy^2\mathcal{N}'(s) + y^3\frac{\mathcal{N}''(s)}{2}, \end{aligned}$$

and no expansions for small s like (25) are needed.

If we consider this function $\mathcal{N}(k)$ as the OGF of some variables v_n , we now have to also consider the zeroth term,

$$\mathcal{N}(k) = \sum_{n=0}^{\infty} v_n k^n, \tag{E3}$$

(notice n starting from 0). Constant quantities v_n express with μ_n as

$$v_n = (-1)^n \mu_{n+1}. \tag{E4}$$

Considering variant A initial conditions (26): $Q(0) = s(0) = 0$, $y(0) = 1$, these quantities can be determined as

$$v_n = (-1)^n Z_{n+1}(0), \tag{E5}$$

and so the constant function $\mathcal{N}(k)$ can be expressed as

$$\mathcal{N}(k) = \sum_{n=0}^{\infty} (-1)^n Z_{n+1}(0) k^n. \tag{E6}$$

APPENDIX F: APPROXIMATING $\mathcal{M}(k)$ WITH A FINITE SERIES

For empirically observed phase densities, it might not always be clear how to determine the constant function $\mathcal{M}(k)$, but one can always estimate the first few moments $Z_n(0)$ numerically and then approximate the function with a series: $\mathcal{M}(k) \approx \sum_{n=1}^N Z_n(0) k^n$. Here, in Fig. 4, we show an example of that. We consider the bistable system (44) and just like in Fig. 3 compute the switching domain for resetting with a δ -distribution. Several different truncations of the series are considered: $N = 1, 2, 3, 4$, and the corresponding domains are depicted in Fig. 4. We see a fast convergence of the approximations.

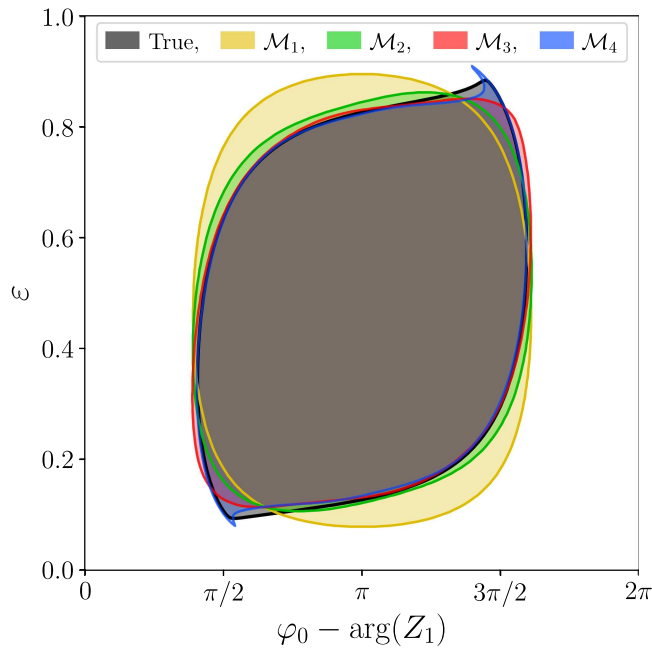


FIG. 4. The switching domains for resetting the bistable system (44) with a δ -distribution, for different approximations the constant function $\mathcal{M}(k)$ (the black region corresponds to the green region in Fig. 3). The considered approximations are truncations of its Taylor series at different numbers of terms: $\mathcal{M}_N = \sum_{n=1}^N Z_n(0) k^n$.

APPENDIX G: RELATION TO THE WATANABE-STROGATZ THEORY

In this appendix, we demonstrate explicitly that the derived equations coincide with the Watanabe–Strogatz equations for identical oscillators. Our basic equation for the oscillator dynamics (1) in the noise-free case reads

$$\dot{\varphi} = \omega + \text{Im}[2he^{-i\varphi}] = \omega + 2\text{Im}[h] \cos \varphi - 2\text{Re}[h] \sin \varphi.$$

Watanabe and Strogatz [Eq. (5.3) in Ref. 15] write this equation in slightly different notations

$$\dot{\varphi} = F + G \cos \varphi + H \sin \varphi,$$

so that $F = \omega$, $G = 2\text{Im}[h]$, $H = -2\text{Re}[h]$. After a transformation [Eq. (5.4) in Ref. 15]

$$\tan\left(\frac{\varphi - \Phi}{2}\right) = \sqrt{\frac{1 + \eta}{1 - \eta}} \tan\left(\frac{\psi - \Psi}{2}\right),$$

they obtain a set of equations for the variables η, Φ, Ψ [Eq. (5.9) in Ref. 15]

$$\begin{aligned} \dot{\eta} &= -(1 - \eta^2)(G \sin \Phi - H \cos \Phi), \\ \dot{\eta}\dot{\Phi} &= \eta F - G \cos \Phi - H \sin \Phi, \\ \dot{\eta}\dot{\Psi} &= -\sqrt{1 - \eta^2}(G \cos \Phi + H \sin \Phi). \end{aligned} \tag{G1}$$

The variables ψ are constants of motion.

Let us introduce a new variable ρ according to

$$\frac{1 - \rho}{1 + \rho} = \sqrt{\frac{1 + \eta}{1 - \eta}}.$$

Then, $\eta = -\frac{2\rho}{1 + \rho^2}$, $1 - \eta^2 = \frac{(1 - \rho^2)^2}{(1 + \rho^2)^2}$, $\sqrt{1 - \eta^2} = \frac{1 - \rho^2}{1 + \rho^2}$, and $\dot{\eta} = -2\dot{\rho} \frac{1 - \rho^2}{(1 + \rho^2)^2}$. We substitute these relations in (G1) and obtain

$$\begin{aligned} \dot{\rho} &= \frac{1 - \rho^2}{2}(G \sin \Phi - H \cos \Phi), \\ \dot{\Phi} &= F + \frac{1 + \rho^2}{2\rho}(G \cos \Phi + H \sin \Phi), \\ \dot{\Psi} &= \frac{1 - \rho^2}{2\rho}(G \cos \Phi + H \sin \Phi). \end{aligned} \tag{G2}$$

Let us now introduce variables $Q = \rho e^{i\Phi}$ and $\theta = \Phi - \Psi$ and assume that these variables fulfill Eq. (16a) (with $\gamma = 0$), (40),

$$\begin{aligned} \dot{Q} &= i\omega Q + h - h^* Q^2, \\ \dot{\theta} &= \omega - i(hQ^* - h^* Q). \end{aligned} \tag{G3}$$

The equations for ρ, Φ, Ψ following from (G3) read

$$\begin{aligned} \dot{\rho} &= (1 - \rho^2)(\text{Re}[h] \cos \Phi + \text{Im}[h] \sin \Phi), \\ \dot{\Phi} &= \omega + \frac{1 + \rho^2}{\rho}(\text{Im}[h] \cos \Phi - \text{Re}[h] \sin \Phi), \\ \dot{\Psi} &= \frac{1 - \rho^2}{\rho}(\text{Im}[h] \cos \Phi - \text{Re}[h] \sin \Phi). \end{aligned} \tag{G4}$$

Substituting here $\omega = F$, $\text{Re}[h] = -\frac{H}{2}$, and $\text{Im}[h] = \frac{G}{2}$, we obtain exactly system (G2), which proves the equivalence of our equation (16a) (with $\gamma = 0$), (40), and the WS equation (G2).

REFERENCES

- ¹M. Nixon, E. Ronen, A. A. Friesem, and N. Davidson, "Observing geometric frustration with thousands of coupled lasers," *Phys. Rev. Lett.* **110**, 184102 (2013).
- ²K. Wiesenfeld and J. W. Swift, "Averaged equations for Josephson junction series arrays," *Phys. Rev. E* **51**, 1020–1025 (1995).
- ³K. Wiesenfeld, P. Colet, and S. Strogatz, "Frequency locking in Josephson arrays: Connection with the Kuramoto model," *Phys. Rev. E* **57**, 1563–1569 (1998).
- ⁴J. F. Totz, J. Rode, M. R. Tinsley, K. Showalter, and H. Engel, "Spiral wave chimera states in large populations of coupled chemical oscillators," *Nat. Phys.* **14**, 282 (2018).
- ⁵B. Eckhardt, E. Ott, S. H. Strogatz, D. M. Abrams, and A. McRobie, "Modeling walker synchronization on the millennium bridge," *Phys. Rev. E* **75**, 021110 (2007).
- ⁶T. B. Luke, E. Barreto, and P. So, "Complete classification of the macroscopic behavior of a heterogeneous network of theta neurons," *Neural Comput.* **25**, 3207–3234 (2013).
- ⁷C. R. Laing, "Derivation of a neural field model from a network of theta neurons," *Phys. Rev. E* **90**, 010901(R) (2014).
- ⁸N. H. Holstein-Rathlou, K. P. Yip, O. V. Sosnovtseva, and E. Mosekilde, "Synchronization phenomena in nephron-nephron interaction," *Chaos* **11**, 417–426 (2001).
- ⁹A. Prindle, P. Samayoa, I. Razinkov, T. Danino, L. S. Tsimring, and J. Hasty, "A sensing array of radically coupled genetic 'biopixels,'" *Nature* **481**, 39–44 (2012).
- ¹⁰Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).
- ¹¹Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki (Springer, New York, 1975), Vol. 39, p. 420.
- ¹²H. Sakaguchi and Y. Kuramoto, "A soluble active rotator model showing phase transition via mutual entrainment," *Prog. Theor. Phys.* **76**, 576–581 (1986).
- ¹³J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, "The Kuramoto model: A simple paradigm for synchronization phenomena," *Rev. Mod. Phys.* **77**, 137–175 (2005).
- ¹⁴S. Watanabe and S. H. Strogatz, "Integrability of a globally coupled oscillator array," *Phys. Rev. Lett.* **70**, 2391–2394 (1993).
- ¹⁵S. Watanabe and S. H. Strogatz, "Constants of motion for superconducting Josephson arrays," *Physica D* **74**, 197–253 (1994).
- ¹⁶E. Ott and T. M. Antonsen, "Low dimensional behavior of large systems of globally coupled oscillators," *Chaos* **18**, 037113 (2008).
- ¹⁷T. Tanaka, "Low-dimensional dynamics of phase oscillators driven by cauchy noise," *Phys. Rev. E* **102**, 042220 (2020).
- ¹⁸R. Tönjes and A. Pikovsky, "Low-dimensional description for ensembles of identical phase oscillators subject to Cauchy noise," *Phys. Rev. E* **102**, 052315 (2020).
- ¹⁹E. Ott and T. M. Antonsen, "Long time evolution of phase oscillator systems," *Chaos* **19**, 023117 (2009).
- ²⁰B. Pietras and A. Daffertshofer, "Ott-Antonsen attractiveness for parameter-dependent oscillatory systems," *Chaos* **26**, 103101 (2016).
- ²¹J. R. Engelbrecht and R. Mirrollo, "Is the Ott-Antonsen manifold attracting?," *Phys. Rev. Res.* **2**, 023057 (2020).
- ²²R. Cestnik and A. Pikovsky, "Hierarchy of exact low-dimensional reductions for populations of coupled oscillators," *Phys. Rev. Lett.* **128**, 054101 (2022).
- ²³W. Braun, A. Pikovsky, M. A. Matias, and P. Colet, "Global dynamics of oscillator populations under common noise," *EPL* **99**, 20006 (2012).
- ²⁴C. C. Gong, C. Zheng, R. Toenjes, and A. Pikovsky, "Repulsively coupled Kuramoto-Sakaguchi phase oscillators ensemble subject to common noise," *Chaos* **29**, 033127 (2019).
- ²⁵A. V. Chechkin, J. Klafter, V. Y. Gonchar, R. Metzler, and L. V. Tanatarov, "Bifurcation, bimodality, and finite variance in confined Lévy flights," *Phys. Rev. E* **67**, 010102 (2003).
- ²⁶R. Toenjes, I. M. Sokolov, and E. B. Postnikov, "Nonspectral relaxation in one dimensional Ornstein-Uhlenbeck processes," *Phys. Rev. Lett.* **110**, 150602 (2013).
- ²⁷R. Toenjes, I. M. Sokolov, and E. B. Postnikov, "Spectral properties of the fractional Fokker-Planck operator for the Lévy flight in a harmonic potential," *Eur. Phys. J. B* **87**, 1–11 (2014).
- ²⁸H. Daido, "Onset of cooperative entrainment in limit-cycle oscillators with uniform all-to-all interactions: Bifurcation of the order function," *Physica D* **91**, 24–66 (1996).
- ²⁹Y. Kuramoto, "Self-entrainment of a population of coupled nonlinear oscillators," in *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics, edited by H. Araki (Springer, New York, 1975), Vol. 39, p. 420.
- ³⁰A. Pikovsky and M. Rosenblum, "Dynamics of heterogeneous oscillator ensembles in terms of collective variables," *Physica D* **240**, 872–881 (2011).
- ³¹The sum defining ordinary generating functions is typically considered from index $n = 0$, but in our context it is convenient to start from $n = 1$; note that we always have $f_0 = 1$ for normalization reasons.
- ³²P. Haukkanen, "Formal power series for binomial sums of sequences of numbers," *Fibonacci Quart.* **31**, 28–31 (1993).
- ³³H. Prodinger, "Some information about the binomial transform," *Fibonacci Quart.* **32**, 412–415 (1994).
- ³⁴A. Pikovsky and M. Rosenblum, "Partially integrable dynamics of hierarchical populations of coupled oscillators," *Phys. Rev. Lett.* **101**, 264103 (2008).
- ³⁵A. Pikovsky and M. Rosenblum, "Dynamics of heterogeneous oscillator ensembles in terms of collective variables," *Physica D* **240**, 872–881 (2011).
- ³⁶S. Kato and M. C. Jones, "A tractable and interpretable four-parameter family of unimodal distributions on the circle," *Biometrika* **102**, 181–190 (2015).
- ³⁷A. Ichiki and K. Okumura, "Diversity of dynamical behaviors due to initial conditions: Extension of the Ott-Antonsen ansatz for identical Kuramoto-Sakaguchi phase oscillators," *Phys. Rev. E* **101**, 022211 (2020).
- ³⁸S. A. Marvel, R. E. Mirrollo, and S. H. Strogatz, "Identical phase oscillators with global sinusoidal coupling evolve by Möbius group action," *Chaos* **19**, 043104 (2009).
- ³⁹C. C. Canavier, "Phase response curve," *Scholarpedia* **1**, 1332 (2006).
- ⁴⁰R. M. Smeal, G. B. Ermentrout, and J. A. White, "Phase-response curves and synchronized neural networks," *Philos. Trans. R. Soc. B: Biol. Sci.* **365**, 2407–2422 (2010).
- ⁴¹J. Schwabedal, A. Pikovsky, B. Kralemann, and M. Rosenblum, "Optimal phase description of chaotic oscillators," *Phys. Rev. E* **85**, 026216 (2012).
- ⁴²A. Pikovsky, M. Rosenblum, G. Osipov, and J. Kurths, "Phase synchronization of chaotic oscillators by external driving," *Physica D* **104**, 219–238 (1997).
- ⁴³E. Grines, G. Osipov, and A. Pikovsky, "Describing dynamics of driven multi-stable oscillators with phase transfer curves," *Chaos* **28**, 106323 (2018).
- ⁴⁴M. Sarkar and S. Gupta, "Synchronization in the Kuramoto model in presence of stochastic resetting," *arXiv:2203.00339* (2022).
- ⁴⁵Resetting with a Kato-Jones³⁶ distribution (asymmetric generalization of the wrapped Cauchy) can be accomplished with two additional OA modes.²²
- ⁴⁶P. A. Tass, *Phase Resetting in Medicine and Biology. Stochastic Modelling and Data Analysis* (Springer-Verlag, Berlin, 1999).
- ⁴⁷A. Pikovsky and M. Rosenblum, "Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators," *Physica D* **238**(1), 27–37 (2009).
- ⁴⁸C. R. Laing, "Exact neural fields incorporating gap junctions," *SIAM J. Appl. Dyn. Syst.* **14**, 1899–1929 (2015).
- ⁴⁹E. Montbrío, D. Pazó, and A. Roxin, "Macroscopic description for networks of spiking neurons," *Phys. Rev. X* **5**, 021028 (2015).
- ⁵⁰C. Bick, M. Goodfellow, C. R. Laing, and E. A. Martens, "Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: A review," *J. Math. Neurosci.* **10**, 9 (2020).
- ⁵¹B. Pietras, R. Cestnik, and A. Pikovsky, "Exact finite-dimensional description for networks of globally coupled spiking neurons," *arXiv:2209.00922* (2022).