Chaos

Some elements for a history of the dynamical systems theory

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ABSTRACT

Writing a history of a scientific theory is always difficult because it requires to focus on some key contributors and to "reconstruct" some supposed influences. In the 1970s, a new way of performing science under the name "chaos" emerged, combining the mathematics from the nonlinear dynamical systems theory and numerical simulations. To provide a direct testimony of how contributors can be influenced by other scientists or works, we here collected some writings about the early times of a few contributors to chaos theory. The purpose is to exhibit the diversity in the paths and to bring some elements—which were never published—illustrating the atmosphere of this period. Some peculiarities of chaos theory are also discussed.

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Chaos is a word that is, in science, very often quickly associated with the overgeneralized *butterfly effect*, inherited from the title of a talk given in 1972 by Edward Lorenz, one of the great contributors to the so-called *chaos theory*. This theory is a branch of the nonlinear dynamical systems (NDS) theory, which was boosted by Poincaré's works at the late 19th century. It was then further developed by many great mathematicians for a few decades. In the 1960s, with the occurrence of computers, chaos theory emerged as a new methodology that is neither "pure" mathematics nor disconnected from the strongly mathematical NDS theory. The scientists working on chaos constitute a very interdisciplinary community whose emergence is associated with a high rate of disciplinary migration. Some of its contributors describe here how this migration occurred.

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I. INTRODUCTION BY CHRISTOPHE LETELLIER

Chaos emerged in the 1970s. In their contribution to the problem of turbulence, Ruelle and Takens introduced the concept of a *strange attractor*, strange meaning neither a limit cycle nor a quasiperiodic motion.[1] They associated turbulence with a "very complicated, irregular and chaotic" motion. A few years later, the term *chaos* was used by Li and Yorke in a very suggestive title *Period-3 implies chaos*.[2] Then, Rössler used it systematically to designate the aperiodic behavior he was studying in the state space.[3–6] With the word "chaos" as a banner, scientists paid attention to aperiodic solutions that were not quasi-periodic and characterized with concepts inherited from the early works by Poincaré[7,8] and Birkhoff,[9] which were synthetized in a masterpiece by Lorenz in 1963.[10] A history of the dynamical systems theory and chaos was already provided by Aubin and Dahan-Dalmédico, focusing on three important contributors from the 1960s (Smale, Lorenz, and Ruelle).[11] This field is polymorphic and many branches emerged in the 1960s, in mathematics with Thom and Smale, in plasma physics with Chirikov, in meteorology with Lorenz, in control theory with Mira and Gumowski, and exploded in the 1970s. Depending on the field from which it emerged, the influences were not always the same. Browsing the list of quotations in pioneering papers does not always allow to reveal them as evidenced with Lorenz's paper in which the book by Nemytskii and Stepanov[12] is quoted after the suggestion from a reviewer.[13] It is, therefore, important to have access to direct recollections of contributors as published, for instance, by Abraham and Ueda.[14]

This paper is devoted to a few contributors who never wrote before about their early times in chaos. Some others were contacted but declined the invitation. All of them were asked to focus on their early times without any other indication. Various drafts were produced, inflating under some exchanges with Letellier who always asked for more details. Contributors were always left free to develop or not some points. No length limitation was imposed, and final texts extend from 2 to 22 pages. Here, only excerpts are selected by Letellier, focusing on the very early times. When possible, the context in which a notable result was obtained is exhibited. These excerpts may (should?) be considered biased by contributor's views as well as by the "selection process" from the whole text. We do not pretend to provide an objective view of the history of chaos, but rather how each one remembers his history of chaos. One of the motivations—not clearly stated to the contributors—was to exhibit the plurality of the paths followed by some scientists who contributed to chaos theory as well as the recurrence of some influential works.

The subsequent part of this paper is organized as follows. Sections II–XIV are the excerpts of the contributions, ordered according to a chronological order of the key contributions. Section XV provides a short analysis and gives some conclusion.

II. THE SMALE PROGRAM BY RALPH ABRAHAM

Steve Smale finished his Ph.D. thesis in differential topology in 1956, working with Raoul Bott at the University of Michigan. At that time, I was there in Ann Arbor, finishing my undergraduate program in Engineering Mathematics. I was introduced to differential topology in a course by Bott, on general relativity in 1960, working with Nathaniel Coburn.

Solomon Lefschetz began devoting half of every year to build up a graduate program in the mathematics department of the National Autonomous University of Mexico. He had become interested in the Russian literature on dynamical systems theory.[15] Smale attended Lefschetz's summer conference in Mexico City.[16] There, he met René Thom, Morris Hirsch, and Elon Lima. Around 1958, Lima finished his Ph.D. thesis on topology with Edwin Spanier in Chicago and introduced Smale to Mauricio Peixoto. Peixoto was a Brazilian student of Lefschetz in Princeton, 1958–1959. His theorem on the structural stability of flows in two dimensions[17] was an early breakthrough in dynamical systems theory.

In 1960, I arrived at UC Berkeley, which suddenly had a brand new staff of mathematics professors and visitors. Smale arrived along with Spanier (algebraic topology), Shing-Shen Chern (differential geometry), and Hirsch (differential topology) from Chicago, Thom (differential topology) from Paris, Chris Zeeman (topology, expositor of catastrophe theory) from Warwick, Peixoto from Rio, Bob Williams (knot theory), Dick Palais (nonlinear functional analysis), and others comprising a research group on dynamical systems theory based on differential topology. Hirsch (a student of Spanier) and I were among the newbies in this group. The Smale program was focused on the stable manifolds, structural stability, and conjugacy of diffeomorphisms. At this time, we devoted much time reading and discussing the works of Poincaré and Birkhoff, especially concerning the stable curves of surface transformations and their transversal intersections.

Smale had proved the existence of stable and unstable manifolds, his first major result in this field.[18] He developed the horseshoe map, his second major result. After this publication, Thom proved that transversal intersection of stable manifolds is a generic property of diffeomorphisms.[19]

Smale's program was boosted into orbit by his influential survey,[20] which set out its foundations: conjugacy of diffeomorphisms, fixed and periodic points, stable and generic properties, the nonwandering set, hyperbolic fixed points, stable manifolds, and so on. Already, Smale drew homoclinic intersections of stable and unstable manifolds for surface transformations, discovered by Poincaré and analyzed in detail by Birkhoff and Smith.[21] Smale's ingenious simplification of the homoclinic tangle in the two-dimensional case, the horseshoe map, is shown in Fig. 1. Smale carefully credits his predecessors—Poincaré,[22] Birkhoff,[23] Morse,[24] Andronov and Pontrjagin,[25] Thom,[26] Elsgolts,[27] Reeb,[28] and Peixoto.[29]

In 1962, I moved on to Columbia and in 1964 to Princeton where Lefschetz still had a huge influence. I was able to teach graduate courses, and, with Jerry Marsden and Joel Robbin, I rewrote much of celestial mechanics with the new language and technology of global analysis.[30] In another, I treated the transversality of stable manifolds in the global context of (infinite-dimensional) manifolds of mappings.[31]

Around 1966, I began to receive letters from René Thom in which he reported regular progress in his creation of catastrophe theory. In this simple context of gradient (non-chaotic) dynamical systems, he made crucial use of the language of *attractors*, *basins*, and *bifurcations*, which became fundamental in the further evolution of dynamical systems theory and later chaos theory. He popularized a style of application of these notions, introduced earlier

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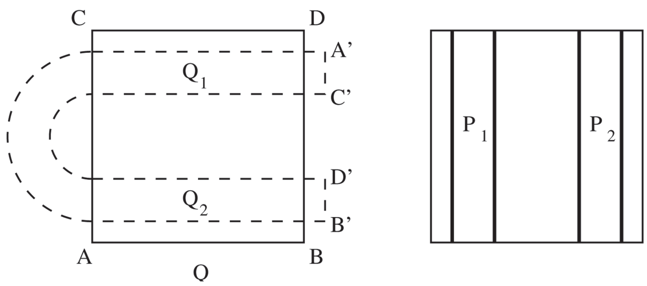

FIG. 1. Smale's horseshoe map. A global diffeomorphism maps the square Q into the region bounded by dotted lines with $G(A) = A'$, etc. Each component P_1 and P_2 of $g^{-1}(g(Q) \cap Q)$ is such that g is a linear map with $g(P_i) = Q_i$ $(i = 1, 2)$.

by Poincaré and his Russian followers.[32] The impact on the mathematical community was further facilitated by a series of exemplary articles by Zeeman.[33]

In 1968, a four-week conference on global analysis (July 1–26) was edited by Chern and Smale.[34] This was the moment, I believe, at which our group finally became aware of the experimental work and simulations on chaotic attractors. Yoshisuke Ueda, discovered the first clearly chaotic attractor in analog simulation, the *Japanese attractor*,[35] for which he accurately drew the homoclinic tangle of inset and outset curves for the forced Duffing equation,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu\left(1 - \gamma x^2\right) y - x^3 + B \cos \nu t \end{cases} \tag{1}$$

obtained at Kyoto University in November 1961 (Fig. 2),[36] Edward Lorenz, discovered his chaotic attractor, at MIT, in 1963,[10] and Christian Mira discovered in 1978 his chaotic attractor in an iterated quadratic map of the plane creating the theory of critical curves for iterated maps.[37,38] These discoveries sounded the death knell for our approach based on differential topology.

In 1971–1973, new people had arrived in my department (UC Santa Cruz), including John Guckenheimer, a recent Ph.D. with Smale, and veterans of our group, Palais and Mike Shub. Palais collaborated in creating a computational program using a digital mainframe and a primitive graphics terminal. We were able to recreate the attractors of Ueda, Lorenz, and Mira, with assistance of a talented group of undergraduates. After a couple of years, we also studied the Rössler attractor and other new developments. Guckenheimer was also active in computational dynamics at UCSC in the 1970s.

An important meeting was jointly sponsored in 1977 by the New York Academy of Sciences and the University of Tübingen. My own contribution was the first announcement of my simulation of chaos using digital computer graphics. This work evolved into the graphic introduction for chaos theory written jointly with the artist, Christopher Shaw.[39]

Around 1978, a group of students, primarily Rob Shaw, Doyne Farmer, Norman Packard, and Jim Crutchfield, later known as the *Santa Cruz Chaos Cabal* (after Gleick's best-seller[40]) began a literature seminar and *chaos program*—the methodology to investigate chaotic attractors with the help of computers, for instance, as synthetized by Lorenz. This resulted in an audacious article in

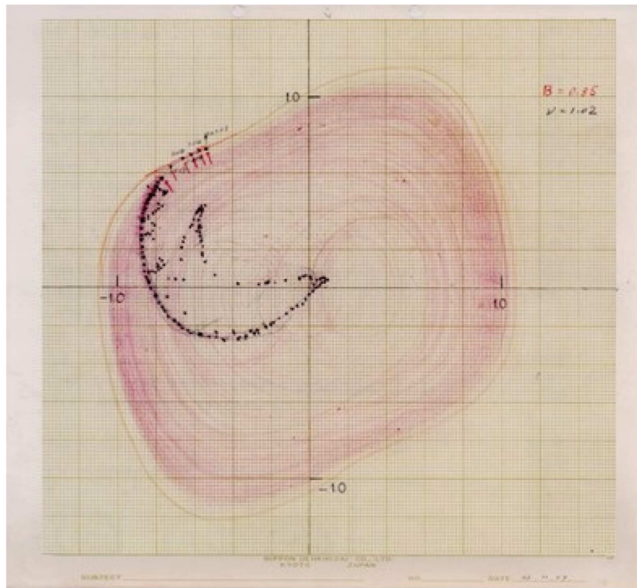

FIG. 2. Yoshisuke Ueda discovered (November 27, 1961) his chaotic attractor in a Poincaré section of the forced Duffing equation (1). Courtesy of Yoshisuke Ueda.

the *Scientific American* of December 1986 in which chaos theory reached a wide popular audience for the first time.[41]

III. BORIS CHIRIKOV—SPUTNIK OF CHAOS BY DIMA L. SHEPELYANSKY

Boris Chirikov (1928–2008) was the founder of the physical theory of Hamiltonian chaos and made pioneering contributions to the theory of quantum chaos.[42,43] In 1959, he invented a simple analytical criterion, now known as the Chirikov criterion, which determines the conditions for the emergence of deterministic chaos in dynamical Hamiltonian systems.[44,45]

There are various research directions launched by Boris Chirikov in the field of chaos. They include chaotic dynamics of particles in plasma magnetic traps and accelerators, chaos border for the Fermi acceleration model, emergence of chaos in various Hamiltonian systems,[46,47] quantum chaos,[48,49] in dissipative dynamical systems,[50,51] and many others.

I joined Chirikov's group at the Institute of Nuclear Physics (INP) in September 1976 at the beginning of my fourth year at the Novosibirsk State University. As many other students, I knew Chirikov from the course of electrodynamics given by him and Igor Meshkov at our second year. However, my choice was also significantly influenced by a recommendation of George Zaslavsky who had worked with Chirikov and gave outstanding recommendations for his research.

Chirikov was the head of a theory group composed of about ten people working on nonlinear dynamics and *stochasticity* (now, we say *chaos*); it included essentially Felix Izrailev, Vitaly Vecheslavov, and Lida Hailo, who worked as a programmer.

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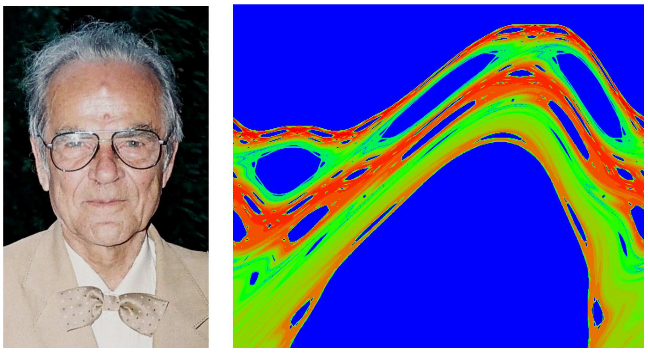

FIG. 3. Left: Boris Chirikov, Toulouse, June 6, 1998. Photo by D. L. Shepelyansky. Right: Amplitude of the eigenstate of the Ulam approximate of the Perron–Frobenius operator of the Chirikov standard map at $K = 0.971\,635\,406$; the amplitude is proportional to color with maximum for red and zero for blue; the upper part of a phase plane is shown for the range $0 < \frac{x}{2\pi} \leq 1, 0 < \frac{p}{2\pi} \leq 0.5$. Reprinted with permission from K. Frahm and D. Shepelyansky, Eur. Phys. J. B **76**, 57–68 (2010).

I remember Chirikov's office in 1976–1978. The main focus of the room was a teletype terminal directly connected to a computer BESM-6 at the Computer Center of Siberian Division of the Russian Academy of Sciences, located at about 1 km distance down along Prospect Nauka. This was the most powerful Soviet computer at that time. From the terminal, it was possible to submit short runs on BESM-6 and even to work in an interactive mode. Chirikov defined the main scientific group aim as the investigation of fundamental laws of chaos and foundations of statistical mechanics for classical and quantum systems.

In 1977, the now famous quantum kicked rotator model was invented. The model is the quantized version of the classical standard map, now known as the Chirikov standard map.[46,47] It has the form

$$\begin{cases} \bar{p} = p + K \sin x, \\ \bar{x} = x + \bar{p}, \end{cases} \tag{2}$$

where bars mark the new values of conjugated variables of momentum p and coordinate x and K is a dimensionless parameter characterizing the kick strength. An example of the Poincaré phase space is shown in Fig. 3.[52]

Back in the late spring of 1977, Chirikov suggested that I work on the kicked rotator model, starting from the improvements of the computer code. Following his suggestions, I achieved a significant reduction of the CPU time, and I am still proud that the improved figures we obtained were used in the Russian version of the kicked rotator paper published as INP preprint in 1978.[53]

At those times, even chaotic dynamics in nonlinear classical systems was a rather new and unusual subject for the world scientific community. For example, there was not any specialized journal in this field, and often, it was not easy to explain to an editor how it happens that, in spite of Laplace determinism, simple equations produce chaotic unpredictable behavior. Quite often, editors blamed errors of numerical simulations and rejected papers on chaos. The worldwide circulation of research results was initiated by Joe Ford (Georgia Tech) who, every week, patiently collected the abstracts of new preprints on chaos and nonlinearity, with young collaborator Franco Vivaldi, and send them to colleagues and friends. Chirikov knew Joe Ford from their first meeting in Kiev in 1966, where Ford came as a tourist with a group of school pupils to visit the USSR. Finally, the first specialized nonlinear journal, *Physica D*, was created in 1980. During many years, Ford and Chirikov worked in the editorial board of this journal.

IV. MY DANCE WITH CHAOS BY OTTO E. RÖSSLER

The first skill that I developed and which was significant for my contribution to chaos was related to my interest in telephones. It gradually led to repairing and then building radios and radio emitters. I got my radio amateur's license DL9KF when I was 17. After my medical studies, I got a first scientific position at the Max Planck Institute for Behavioural Physiology in Biocybernetics at Seewiesen. During that year, I developed a big friendship in very long discussions with Konrad Lorenz. I then spent one year as an intern at the University of Marburg partly under the supervision of Reimara Waible who became my wife one year later.

I hereafter obtained a one-year position for working with Robert Rosen. In the continuation of Nicholas Rashevsky, the pioneer of mathematical biology,[54–56] Rosen developed a bridge between dynamical systems theory and biology.[57] Bob and I had an immediate resonance. From his book, I discovered Andronov, Khaikin, and Vitt's textbook,[58] which later led me to building a three-variable chemical multivibrator.[59] Bob's book made me firm in dynamical systems thinking.

A few years before, I had met Friedrich-Franz Seelig. He offered me to join his new group at the University of Tübingen. In the early 1960s, Seelig had done his diploma work with Hans Kuhn and Fritz-Peter Schäfer to build an analog computer consisting of a network of electrical oscillators, connected to capacitors to solve the two-dimensional Schrödinger equation.[60] This system was triggered by means of a radio frequency generator. Sharing an interest for the origin of life, in differential equations and electronics (computers), Kuhn, Seelig, and myself agreed that nonlinear systems such as my evolutionary soup and electronic systems were virtually isomorphic. This triggered a cooperation project between Seelig and me to look for reaction-kinetic analogs to electronic circuits. I, therefore, joined Seelig in 1970. Seelig bought an analog computer—a Dornier DO 240—equipped with potentiometers, a digital clock, and two function generators. I was free in my research and started to study few-variable systems. Chaos theory is fun. With three variables, recurrent motions can fall into a dance that is beautiful and non-repeating and surprising at every round. This dance is chaos. Art Winfree stimulated and paved my way into the fascinating topic of chaos theory. In 1972, Art Winfree had invited me for a talk on chemical automata at Purdue University.[59,61] He showed me his later well known beautiful experiments with the Zhabotinsky reaction.[62,63] We started to exchange letters about interpreting chemical reactions in terms of dynamical system theory. In 1975, we met again at a *Chronobiology Meeting* held in Vienna where I gave a talk on biological clocks. Art found my talk a little bit boring and asked me whether I could do something more interesting in the context of my liquid automata?[64] I told him that I was thinking about a three-variable

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limit cycle that looks like a knot and hence cannot be flattened into a planar circle-like thing for being irreducibly three-dimensional. He replied that this sounded to him like chaos. He told me that he had just attended a conference in Aspen, Colorado, on "chaos" and that he had collected all the papers written on the subject and he would send me a folder with them. Four weeks later, I received a big folder with all the papers, mostly still in a preprint form.

The folder included Lorenz's paper of 1963[10] and more recent ones like those by Jim Yorke,[2] Bob May, and George Oster. Art wrote me explicitly that I should do him the favor of finding a chemical version of the Lorenz attractor.[65] Of course, I did not succeed. I came up with the idea of a rope wrapped about my nose several times before falling down and then coming back up again in a loop. This mental picture was the origin. Next came a characterless turning of the knobs on the analog computer. Simplifying and reducing the system then was sufficient to arrive at the desired attractor.[3] I finally got the equations[5]

$$\begin{cases} \dot{x} &= -y - 0.95z, \\ \dot{y} &= x + 0.15y, \\ \epsilon\dot{z} &= (1 - z^2)(z - 1 + x) - \delta z. \end{cases} \tag{3}$$

Describing sharp transitions at the two thresholds [Fig. 4(a)] where suddenly things switch from a one two-dimensional plane to the other is a bit demanding numerically. After simplifying the equation as much as possible by trial and error, the "miracle" happened that the two formerly overlaid simple linear two-dimensional flows gave rise to a new everywhere smooth three-dimensional flow,[3]

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases} \tag{4}$$

It is not actually the really simplest one, by the way, since I could find later the still simpler one,[66]

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x, \\ \dot{z} = b(1 - y^2) - cz. \end{cases} \tag{5}$$

I should add here that the stimulation I had obtained from Ralph Abraham and his school—the "Santa-Cruz kids"—was crucial. Norman Packard, Rob Shaw, Jim Crutchfield, and Doyne Farmer then jointly coined the name for the attractor.[67]

The movie with the "sound of chaos" (named after Simon and Garfinkel's "Sound of Silence") was obtained on the analog computer jointly with Reimara.[68] The sound produced by this simulational reality proved to be familiar to the ear. Therefore, chaos is something that is very close to everyday life.

V. HOW I BECAME INVOLVED IN CHAOS BY PHILIP HOLMES

In 1973, I was finishing a Ph.D. thesis on noise transmission in structures at Southampton University (UK) when I noticed that a course on differential topology and its applications to dynamical systems would be taught by David Chillingworth in the Mathematics Department. This introduced me to René Thom's catastrophe theory and to many analytical tools that I had not known before.

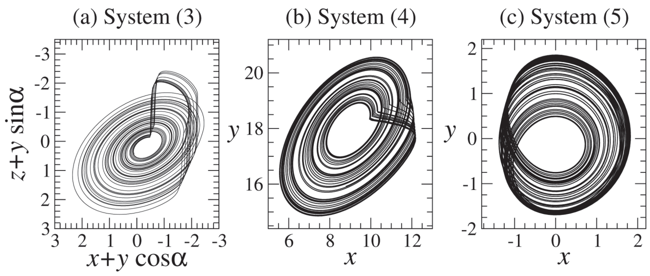

FIG. 4. The chaotic attractor obtained after some simplifications of the original equations. Parameter values for system (3): $\delta = \epsilon = 0.03$. For system (4): $a = 0.2$, $b = 0.2$, $c = 5.7$. For system (5), $b = 0.275$ and $c = 0.2$, and initial conditions are $x_0 = 1$, $y_0 = -1.4$, and $z_0 = -0.4$.

While auditing the course, I met David Rand who was also completing his thesis and who wanted to move toward applied mathematics. We began working together and found an interesting mistake in the interpretation of the amplitude response function for periodic solutions in a preprint of Christopher Zeeman's on Duffing's equation with a stiffening spring ($\alpha > 0$),

$$\ddot{x} + 2\zeta\dot{x} + k(x + \alpha x^3) = f\cos(\omega t). \tag{6}$$

The mistake involved a misinterpretation of the cusp catastrophe, which occurs as a limiting case when two curves of saddle-node bifurcations collide in the (f, ω) parameter space. Zeeman's paper finally appeared in 1976.[69] We corrected the mistake and sent our preprint to Christopher. He encouraged us to submit and publish, and in due course, our paper came out,[70] followed by further papers on van der Pol's equation[71] and on a combined Duffing–van der Pol oscillator.[72] We did not learn of Yoshisuke Ueda's analog simulations of chaos in a similar equation with a cubic damping term until I moved to Cornell University in 1977. Shortly thereafter, Ueda visited Cornell and described some of his results.

However, while still in Southampton, David Rand and I found two preprints on nonlinear oscillations by Floris Takens and things began to become chaotic. Takens' papers appeared in 1974.[73,74] I believe that they established fundamental examples for the classification of bifurcations in dynamical systems.

Cartwright had found a curve of homoclinic bifurcations in a two-dimensional averaged van der Pol equation.[75] David Rand and I were able to relate it to the homoclinic bifurcations studied by Takens and thus assemble the correct codimension-2 bifurcation set.[71] Motivated by Takens' work, I borrowed an analog computer from the undergraduate laboratory and (physically) programmed it to simulate Duffing's equation in the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3 - \delta y + \gamma\cos(\omega t). \end{cases} \tag{7}$$

This system has a potential energy function with two wells separated by a peak. With damping $\delta > 0$, but without periodic forcing ($\gamma = 0$), almost all solutions approach one or the other of the two stable equilibria $(x, y) = (\pm 1, 0)$ except for the stable manifold of the saddle point $(x, y) = (0, 0)$. More significantly for chaos, when

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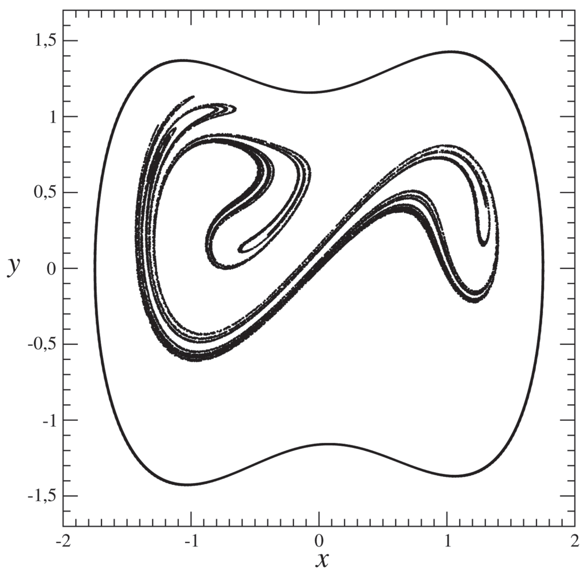

FIG. 5. Coexistence of a chaotic attractor and a large stable period-1 limit cycle. Parameter values: $\delta = 0.15$, $\gamma = 0.30$, and $\omega = 1$. Initial conditions: $x_0 = 1$ for the chaotic solution and $x_0 = 1.8$ for the period-1 limit cycle and $y_0 = 0$.

damping $\delta = 0$, there are two homoclinic orbits beginning and ending at the saddle point. I had read and understood relevant parts of Smale's paper[20] and realized that this implied that, with sufficiently large periodic forcing $\gamma \neq 0$, the Poincaré map of Eq. (7) has infinitely many transverse homoclinic points and, therefore, contains Smale horseshoes (see Fig. 5 for an example).

Two conferences sponsored by the *New York Academy of Sciences* in 1977 and 1979 focused on bifurcation theory and nonlinear dynamics[76,77] and did much to stimulate the field by bringing diverse researchers together. I chaired a related conference in 1979, which both engineers and mathematicians attended.[78] Since I had trained as an engineer and at that time was still trying to become an applied mathematician, this meeting was important for my future.

Throughout my early work and collaborations, I was most excited by the combination of computer simulations, rigorous mathematical theorems, and physical experiments that, together, could create new models of dynamical processes. The classification of bifurcations played a key role in this.

VI. HOW CHAOS SHAPED MY ACADEMIC LIFE BY RENÉ LOZI

Strangely enough, although I was very interested in the first examples of chaotic attractors from the end of the 1970s, I never paid attention too much to the Rössler attractor. However, the qualitative procedure of this method was strongly inspiring me during several years, allowing to propose a geometric model of slow–fast Lorenz-like attractor[79] and the Alpazur oscillator with Hiroshi Kawakami.[80,81] Moreover, I was interested in his research studies on hyperchaos and his prototypic models,[82–84] map (although non-continuous) three years later.[85]

I started my studies at the University of Nice in October 1967 in mathematics and physics. I had been taught that there was a list of ODEs written by Bernoulli, Lagrange, Clairaut, Riccati, etc., and a list of solving methods. No physical sense, in fact no meaning at all, was attached to these academic exercises. No numerical method was taught. Moreover, between professors, there was a strict separation between "pure mathematician" and the few "applied mathematician" who were able to use a computer. At the university, I took my first programming course about FORTRAN IV in 1968 using punched cards. I discovered with fascination the methods of numerical integration of ordinary differential equations (ODEs). During 1970–1971, I was following my bachelor's degree under the supervision of Professor Martin Zerner (1932–2017). Martin was the first guy who was able to use a computer that I ever met.

During his lectures, my mind knew a breakthrough that changed the paradigm: the set of all the equations I was taught were of zero measure in the set of all ones existing. No closed formula of solution can be found for most of them. Only numerical methods were able to provide approximate solution. Of course, in this scope, computer was essential. Moreover, ODEs were useful to model physical, chemical, or even biological situations. This new paradigm has guided my research career throughout my whole life.

While preparing my Ph.D.,[86] the name "bifurcation" was largely unknown in the communities of mathematical and numerical analysis in France. Of course, the term bifurcation was introduced 90 years before by Henri Poincaré,[87] but we must consider that the decade 1960–1970 was the golden age of the Bourbaki group[88] whose philosophy was drastically opposed to Poincaré's way of thinking. Moreover, Jean Alexandre Dieudonné, one of the founders of the Bourbaki group, arrived at Nice in 1964. He was the most prominent professor from the department of mathematics. Poincaré's works were, therefore, not at all in my mind.

With Gérard Iooss,[89] I worked on the famous dynamo problem explaining the origin of the magnetic earth field.[90] We both attended a conference in Roma (1977). The opening talk was given by David Ruelle.[91] In his talk, Ruelle conjectured that, for the Hénon attractor, the theoretical entropy should be equal to the characteristic exponent. This is how I discovered the first example of chaotic and strange attractors [Fig. 6(a)].[92] At that time, the term "strange" was used, referring to Ruelle and Takens' paper. Today, we would use "chaotic" rather than strange, which now refers to the fractal properties of the invariant set. Nevertheless, this is rather the fractal properties of this attractor, which were highlighted by Michel Hénon and astonished the research community. Hénon who explored numerically the Lorenz map using the IBM-7040 found it difficult to highlight its inner nature due to its very strong dissipativity. Hénon built the metaphoric model

$$\begin{cases} x_{n+1} = 1 - a x_n^2 + y_n, \\ y_{n+1} = b x_n. \end{cases} \tag{8}$$

With $b = 0.3$, the contraction in one iteration is mild enough that the sheaves of the attractors are visible [Fig. 6(a)].

Beyond bifurcation problems, my main interest was focused on discretization problems and the finite element method in which nonlinear functions are approximated by piecewise linear ones. During the Roma conference, I tried to apply the spirit of the method of

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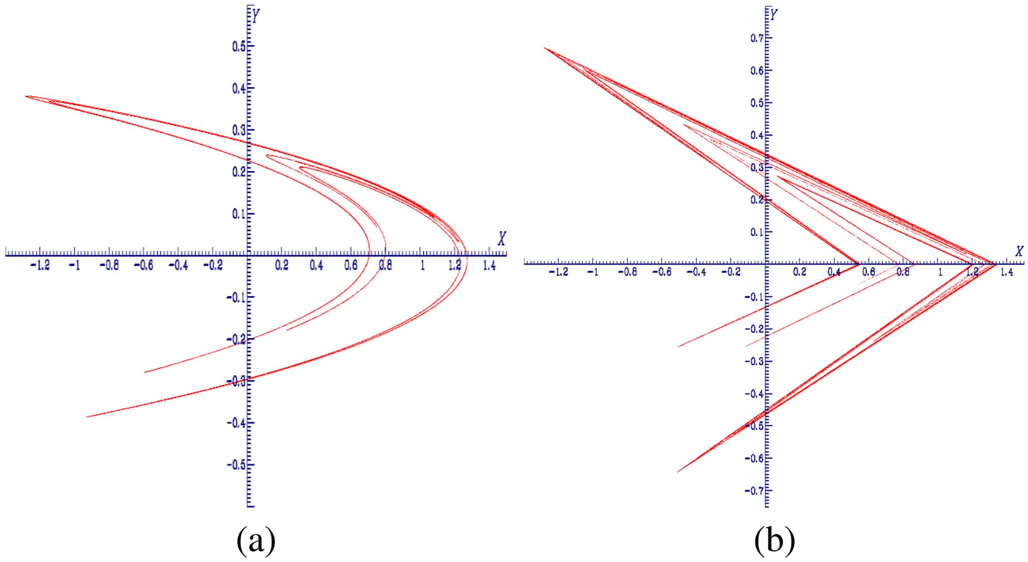

FIG. 6. Some simple mappings: (a) The Hénon attractor and (b) the Lozi attractor.

finite elements to the Hénon attractor. Back to Nice on June 15 in the morning, I eventually decided to change the square function of the Hénon attractor, which is U shaped, into the absolute value function, which has a V shape, implying a folding property. I tested this modification on my small desktop computer HP 9820. I shifted the parameter value a from 1.4 to 1.7 and b from 0.3 to 0.5 (why? I do not remember!) and plotted what is known today as the "Lozi map" [Fig. 6(b)].[93] Iooss and Chenciner encouraged me later to publish the formula[94]

$$\begin{cases} x_{n+1} = 1 - a|x_n| + y_n, \\ y_{n+1} = bx_n. \end{cases} \tag{9}$$

This was for me the very beginning of my career in chaotic dynamical systems.

I was convinced that few weeks would be enough to explain and give a proof of the structure of a so simple attractor, but I failed. In the next two years, I attended a workshop on iteration theory at La Garde-Freinet (1979) where Michal Misiurewicz, after some questions at the end of my talk, jumped on the stage. On the blackboard, he gave some clues of his forthcoming results presented at the famous New York conference, seven months later[95] where I am proud to have shook the hand of Edward Lorenz. There, I listened with a mix of anxiety and curiosity the first proof by Misiurewicz for the existence of a chaotic attractor for the map I discovered two and half years before.[95] I was interested in the session devoted to turbulence due to the concept of a strange attractor developed by Ruelle and Takens.[96] The talk by Vidal[97] on the Belousov–Zhabotinsky reaction was of a so great interest for me. Of course, the talk by Misiurewicz[95] was a kind of ecstasy for the young researcher that I was.

VII. MY ROAD TO CHAOS BY LEON GLASS

For my Ph.D. at the University of Chicago, I studied dynamics of molecules in liquid argon. For postdoctoral studies, I was interested in going back to my original fascination with medicine and psychology. I received a postdoctoral fellowship to study the brain at the newly formed Department of Machine Intelligence and Perception at the University of Edinburgh in 1968.

I returned to Chicago to a Postdoc. Jack Cowan had hired two remarkable young scientists, Art Winfree and Stuart Kauffman for their first faculty positions. Although my initial plan was to continue working on vision, I became intrigued by Kauffman's studies. Kauffman had constructed random Boolean switching networks and found that for networks in which each element only had a couple of inputs, the dynamics was amazingly orderly.[98] I rejected the Kauffman's notion of discrete states and discrete times but embedded the switching network logic in differential equations.[99,100] This was really my first research that involved nonlinear dynamics. I learned about some of the basic notions including bifurcation and stability theory—topics that were not considered appropriate to include in graduate physical science programs at the time. This was immediately before the explosion of interest in chaos.

Michael Mackey, who had training in Biophysics and Mathematics, was a young faculty member at McGill University in Montreal. I had met Mackey at Gordon Conferences in Theoretical Biology in the early 1970s, and I was delighted when the opportunity came to apply to McGill. I moved to Montreal in March 1975 and a few months later went out west to spend a month at the Aspen Center for Theoretical Physics. A talk by Stephen Smale about the period-doubling route to chaos was intriguing. Mitchell Feigenbaum was also there, and he attributed that meeting also to piquing his interest in chaos.[101]

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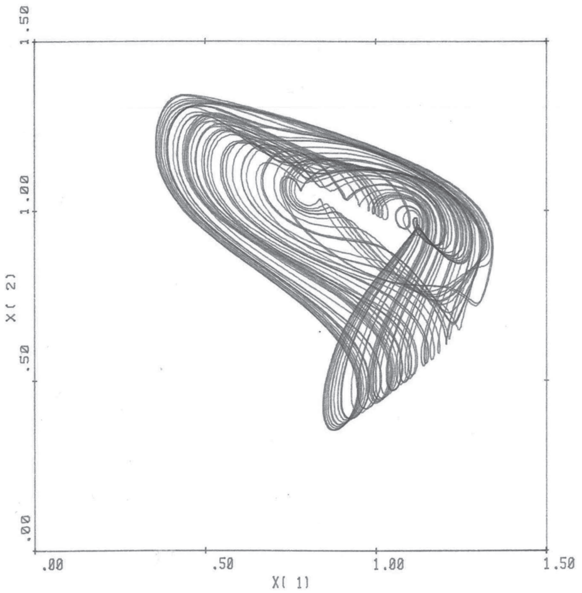

FIG. 7. Time delay embedding of a chaotic time series from the model for blood cell production using the time delay equation (10) that we proposed in 1977.[102] The trace is similar to the time delay embeddings presented at the New York 1977 conference organized by Gurel and Rössler. Image from the original tracings made in 1976 or 1977 where $x(2)$ is the delayed variable and $x(1)$ is the current variable.[103]

When I got back to Montreal, I was excited to discuss chaos with Mackey. I asked him if physiological systems could display chaos. He said he did not know. We decided to write a team grant application on the theme "Oscillation and Chaos in Physiological Systems." We certainly proposed to study both difference and delay differential equation models for two physiological systems: Mackey would look at hematopoiesis, and I would look at respiration. In one of the models for hematopoiesis, there were several novel features. There was just one variable—the blood cell concentration. Instead of just using a negative feedback, the equation

$$\dot{x} = \beta \frac{x_\tau}{1 + x_\tau^n} - \gamma x \quad \gamma, \beta, n > 0 \tag{10}$$

had a non-monotonic feedback term. Since it took some time to produce red blood cells once the signal was received, the production incorporated a time delay term. We searched for chaos in the model. Most exciting was the day when Mackey and I both sat in front of a primitive computer screen and watched the trajectory. Since there was only one variable, we plotted two coordinates, the current value and the value in the past. To the best of my knowledge, this was the first use of time delay embedding to examine complex dynamics and bifurcations. We tweaked parameters and eventually found what we were seeking.[102,103] It was chaotic (Fig. 7), and the route to chaos seemed similar to what had been observed in simple quadratic maps.[2,104]

General interest in chaos had been piqued by Robert May's 1976 review.[105] We submitted our findings on chaos in simple mathematical models of physiological systems to *Science* and were delighted when it appeared.[102] We emphasized the concept that diseases could be characterized by abnormal dynamics that might be associated with bifurcations in nonlinear equations. In November 1977, Okan Gurel and Otto Rössler organized a meeting on *Bifurcation Theory and Applications*.[106] I was invited to speak and ran through a sequence showing the various dynamics in the chaotic time delay equation as a parameter changes using the time delay embedding.[103]

There were many people at the 1977 meeting. One was Robert Shaw, spending significant time to develop a way to beat roulette by entering data from the roulette wheel into a computer program in a shoe.[107] The Dynamical Systems Collective wrote an influential paper in 1980 showing how you could get a two dimensional portrait of a time series by plotting the value of one variable on one axis and its derivative, or as suggested in a footnote its value at an earlier time, on the other axis.[108] One member of the group, Farmer, went on to study the time delay equation modeling blood cell dynamics for his doctorate, referring to it as the Mackey–Glass equation![109] Another person at the meeting was David Ruelle. Ruelle suggested to me that I could look at the return map to a cross section on the time delay embedding. He correctly thought that it would be parabolic. As far as I know, the only published return plot for this equation appeared in a *Scholarpedia* review article that Mackey and I presented many years later when we finally took Ruelle's suggestion.[110]

VIII. FIRST CHAOTIC STEPS BY ARKADY PIKOVSKY

As a second-year physics student at the Department of Radiophysics at the University of Gorky, I had to decide the direction of my studies. In early 1974, I approached Michael Rabinovich, that time reader at the Theory of Oscillations chair, and asked if I can do specialization under his supervision, and he agreed. He gave me some review articles[111,112] to read. I understood very little of them. Nevertheless, when he formulated a first research project—deriving a kinetic equation for modes for the Rayleigh–Bénard convection problem, I started to read books and articles and almost the whole third year in the University struggled with nonlinear equations for convection. Therefore, I read what was relevant to this field in the literature, and at the beginning of 1975, I read a paper by J. McLaughlin and P. Martin.[113] They wrote about a strange attractor in convection, and I understood nothing.

This paper contained only the 13th citation of the famous 1963 Lorenz paper[10] and only the 11th citation of the equally important 1971 paper by Ruelle and Takens,[1] and it was the first publication that cited both. McLaughlin and Martin matched Lorenz's nonperiodic flow with the strange attractor concept. As a matter of fact, stochastic dynamics (that is how *deterministic chaos* was called in the Russian literature that days) was a known concept due to works of Boris Chirikov and his group,[114] but a general belief was that conservative Hamiltonian chaos does not survive dissipation, and in dissipative systems, only limit cycles can be stable (robust) attractors. Lorenz's model and the theoretical concept of Ruelle and Takens demonstrated that dissipative chaos could be permanent, and more and more examples of it appeared in 1975. To us, these novel ideas came not in a direct way of reading the McLaughlin–Martin paper,

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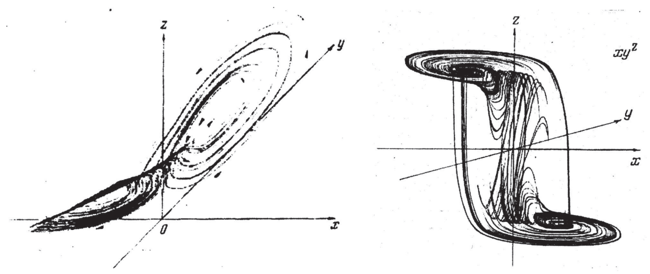

(a) (b)

FIG. 8. Phase portraits of two strange attractors created on an analog computer produced by (a) the Rabinovich system (11) and (b) the electronic chaotic generator. (a) Reproduced with permission from Pikovski *et al.*, Zh. Eksp. Teor. Fiz. **74**, 1366–1374 (1978). Copyright 1978 Russian Academy of Sciences. (b) Reproduced with permission from Pikovski and Rabinovich, Dokl. Akad. Nauk SSSR **239**, 301–304 (1978).

but through the mathematical group on a dynamical system theory around Ya Sinai in Moscow.[115,116] M. Rabinovich, together with Svetlana Vyshkind, studied low-dimensional models of nonlinearly coupled modes and observed irregular dynamics there.[117] He met Ya Sinai and from him got to know about the concept of strange attractors. M. Rabinovich returned from Moscow very enthusiastic about this, and I was eager to learn more in this direction (at this point, I also realized that I had already read about this in the McLaughlin–Martin paper). M. Rabinovich brought from Moscow several preprints and lecture notes that Sinai gave him (I remember it was a text by O. Lanford III among them), which I tried to read. However, this was rather hard for a non-mathematician, with some objects like "Axiom A" that I could not identify. Thus, I took a step back and started from more basic texts such as translated to Russian Smale's review[20] and lectures by A. Katok and others at Russian mathematical schools.

Around the middle of 1976, M. Rabinovich first realized that there is a close analog of the Lorenz system in the realm of coupled oscillators (or oscillatory modes). The linear terms in the Lorenz model can be interpreted as a combination of dissipation and parametric excitation, while the nonlinear terms correspond to a "classical" three-mode resonant interaction. This model is known as the Rabinovich system,[118]

$$\begin{cases} \dot{a}_1 = -\nu_1 a_1 & +ha_2^* & -a_2 a_3, \\ \dot{a}_2 = -\nu_2 a_2 & +ha_1^* & +a_1 a_3^*, \\ \dot{a}_3 = -a_3 & & +a_1 a_2^*, \end{cases} \tag{11}$$

where the system is written in three columns, with dissipative, excitation, and nonlinear coupling terms, correspondingly. Remarkably, the complex solutions of the Rabinovich system appeared to lie on the real-valued three-dimensional manifold [Fig. 8(a)], which made the analogy with the Lorenz system nearly perfect.

The second idea came after a paper by Rössler.[3] There, he argued that in a three-dimensional slow–fast system, with a two-dimensional S-shaped slow manifold, one can reduce the dynamics to a one-dimensional non-invertible map and thus get chaos. Slow–fast systems were a popular object at the Theory of Oscillations chair in the context of electronic circuit dynamics. M. Rabinovich decided to construct a chaotic electronic generator with slow–fast dynamics.

Working on these two problems was an exciting time for me. The computations' results had to be put to the graphs (on a graph paper) by hands. We used an analog computer with a plotter. One could easily arrange a simple set of equations on this analog computer, but accuracy was miserable. Therefore, one just adjusted parameters (through rotation of a potentiometer) to obtain a beautiful plot. Moreover, while plotting a long trajectory, parameters could deviate, and it suddenly exploded. In Fig. 8, I present analog computer phase portraits of the Rabinovich system and of the slow–fast dynamics in an electronic circuit from papers.[118,119]

IX. TÜBINGEN BLUES BY LARS FOLKE OLSEN

My own introduction into the field came when I first went to the department of biochemistry, Odense University (now University of Southern Denmark) as a graduate student in 1975 to study bistability and oscillations in a single enzyme reaction known as the peroxidase–oxidase reaction. My supervisor was Professor Hans Degn who had studied this and other oscillating chemical reactions since the early 1960s.[120,121] In those days, the typical project for a biochemistry student was to purify a new enzyme (or a known enzyme in a new organism), establish an assay to measure its activity, and finally determine its K_M and turnover. However, since my high-school days, I had a crush for mathematics and physics.

In the fall of 1975, Hans Degn urged me to attend a meeting on "Rhythmic Functions in Biological Systems" in Vienna (September 8–12). The meeting was mostly on circadian rhythms, and I did not know any of the participants and also had nothing to contribute. However, I had the pleasure of meeting two scientists who have had a great influence on my later career. One was Arthur Winfree and the other was Otto E. Rössler. Back in Odense, Degn informed me that some unspent money could be used for a month visit to a lab of my own choice. I asked Otto if he would be willing to have me around for a few weeks.

When I arrived, Otto had just submitted his first paper on chaos in a (bio)chemical system.[3] My plan was to make a model of the PO reaction that could unify its ability to show both bistability and oscillatory behavior based on some enzyme-kinetic measurements done in the lab. Otto helped me with the model, and in fact, we did get it to work. The equations are

$$\begin{cases} \dot{a} = K(a_0 - a) - \dfrac{V\left(a + \kappa a^2\right) b}{\lambda b + a + \mu a^2}, \\ \dot{b} = \sigma - 2\dfrac{V\left(a + \kappa a^2\right) b}{\lambda b + a + \mu a^2}, \end{cases} \tag{12}$$

where a represents O_2 and b represents NADH. K is a constant that determines the rate of diffusion into the reaction mixture and a_0 represents the O_2 concentration at equilibrium. V, κ, λ, and μ are enzyme kinetic parameters and σ is the inflow rate of NADH. The model showed coexistence of steady state and limit cycle oscillations with an unstable periodic orbit separating the steady state and the limit cycle. It was never published in full, but the experimental data

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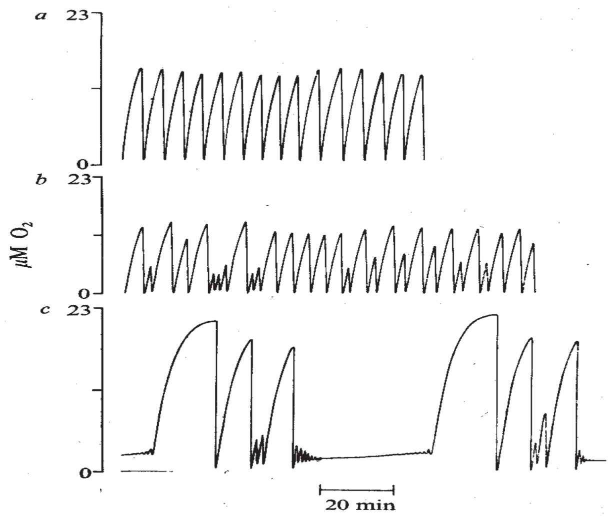

FIG. 9. Simple periodic (a), chaotic (b), and bursting (c) oscillations of O_2 in the peroxidase–oxidase reaction. Reproduced with permission from L. F. Olsen and H. Degn, Nature **267**, 177–178 (1977). Copyright 1977 Springer Nature.

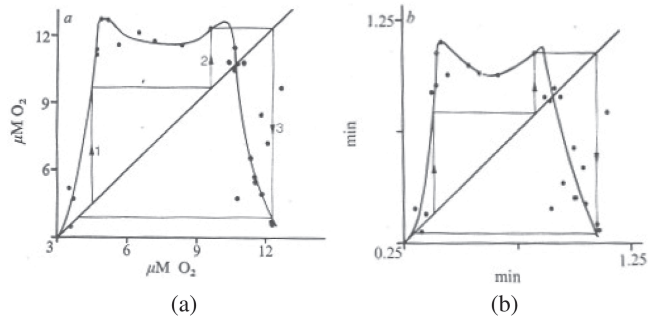

FIG. 10. (a) Next-amplitude map of oscillations of O_2 from Fig. 9(b) and (b) the same type of plot for the oscillation periods. Trajectories with arrows were drawn to show that the transition function allows period 3. Reproduced with permission from L. F. Olsen and H. Degn, Nature **267**, 177–178 (1977). Copyright 1977 Springer Nature.

underlying the model appeared in a later publication.[122] When discussing this model, Otto was also telling me about an interesting new kinetic model he had made, which could show non-periodic oscillations, sensitive to initial conditions, which he referred to as chaos.

Following my short stay in Tübingen, I started a new series of experiments with an open system where both substrates NADH and O_2 were supplied continuously to the reaction mixture containing the enzyme. Much to our surprise, the resulting oscillations were different and far more complex than we had anticipated. We observed mixed-mode oscillations and bursting oscillations, none of which could be explained by my simple two-variable model. Sometimes, we also observed non-periodic oscillations as mixtures of small and large-amplitude oscillations in a seemingly random order (Fig. 9). Initially, Degn dismissed these oscillations as artifacts generated by small random fluctuations in the pumping rate of NADH inflow, but they appeared consistently when repeating the experiments with the same experimental settings. I had an idea that these non-periodic oscillations could be chaos, and therefore, I wrote a letter to Otto with an extensive description of what I had done with free-hand drawings of the data. Otto's reply was: "Read Lorenz!"[10]

I read Lorenz's fascinating paper[10] and more recent papers by Robert May[104] and Li and Yorke,[2] and suddenly I understood. Following Lorenz's instructions, I constructed a return map by plotting each amplitude from our irregular PO oscillations against the preceding amplitude (Fig. 10) and applied the Li and Yorke theorem.[2] I showed the plot together with the papers by Lorenz and Li and Yorke to Degn who immediately changed his opinion on the results. Within a week, we had written the manuscript and submitted it as a letter to *Nature* by the end of October 1976. We also sent copies of the manuscript to Otto and to Art Winfree, from whom we received very enthusiastic responses.[123] A few months later, the paper by Schmitz, Graziani, and Hudson on chaos in the Belousov–Zhabotinskii (BZ) reaction appeared.[124] In 1978, Otto and Klaus Wegmann also published a paper on chaos in the BZ reaction.[125] It is important to note that in those days, Takens' embedding method[126] had not yet been published.

X. FROM CHEMICAL CHAOS TO CHAOTIC BRAIN BY ICHIRO TSUDA

As everyone does, I enthusiastically studied Otto E. Rössler's pioneering works on chaos[5,6,82,84] in my graduate student days in the late 1970s. I tried to understand the mathematical structure of Lorenz chaos[10] in relation to Smale's horseshoe map,[20] the relationship between Lorenz chaos and a strange attractor[1] in a sense of a mathematical representation of hydrodynamic turbulence, and also the relationship between such chaos and chaos in a sense of Li–Yorke,[2] while I kept thinking of "real" chaos observed in the Belousov–Zhabotinsky (BZ) reaction system.[124,127–130] Otto's contribution[131] to real chaos in that system with Klaus Wegmann encouraged me to pursue this direction of research.

Here, let me add some more comments about my early research with the late Kazuhisa Tomita, concerning chaos in the BZ reaction. We noticed early reports of chaotic behaviors in this chemical reaction system. One report was published by Wegmann and Rössler[131] mentioned above, while the other was by Schmitz *et al.*[124] Although they reported "chaotic" behaviors in laboratory experiments, it was not clarified that those behaviors can be characterized by a mathematical structure recognized as deterministic chaos or strange attractors. We wanted to show a definite evidence for the presence of chaos in the BZ reaction. We thought that finding evidence was easy. The reason was that Otto already showed the presence of chaos in three-dimensional continuous chemical reaction systems even with one quadratic nonlinear term, and furthermore, the BZ reaction system should include more than one quadratic nonlinear terms due to molecular collisions of two different chemical substances.

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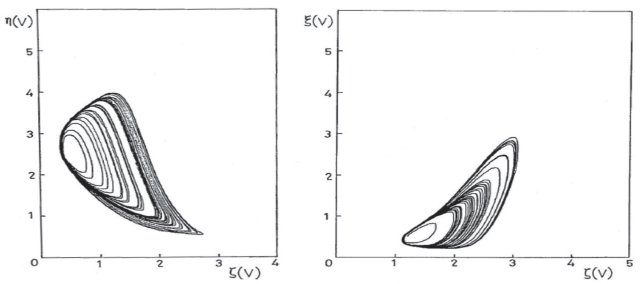

FIG. 11. "Chaotic" orbits yielded by the analog computer at Yoshisuke Ueda's laboratory in 1978 by using the model (13). Parameter values: $p \approx 90$, $\phi \approx 0.048$, and $m \approx 3.44$. Figures reproduced from I. Tsuda, master's thesis, Kyoto University, 1979.[203]

For the first time, we made a three-dimensional continuous model for the BZ reaction based on an original Oregonator proposed by Field and Noyes.[132] In 1978, Kazuhisa Tomita asked Yoshisuke Ueda to allow us to use analog computers in his laboratory. I finally found "chaotic" behaviors in our model,

$$\begin{cases} \dot{\xi} = (1-\phi)\xi + \eta - \xi\eta - \xi\zeta, \\ \dot{\eta} = -(1+\phi)\eta + \zeta - \xi\eta + m, \\ p\dot{\zeta} = \xi - (1+p\phi)\zeta - \xi\zeta \end{cases} \tag{13}$$

using analog computers (Fig. 11).[129] Christian Vidal's Bordeaux group found very similar chaotic behaviors to our findings in their laboratory experiment.[128] We were excited by this experimental finding. Unfortunately, however, I could not find any chaotic behaviors by digital computations with our model. The equations used in analog and digital computations were the same, but the computation results were different: one showed chaotic behaviors and the other showed simply periodic ones. I guessed that the chaotic behaviors found in the analog computer could be a kind of noise-induced chaos caused by the weak stability of limit cycles. Our model does not have either an additional geometric structure producing chaos as in the Rössler's system or an additional dynamical rule for changing the bifurcation parameter. One more variable was necessary for yielding deterministic BZ chaos. Therefore, I guessed the findings to be noise-induced chaos. Finally, I gave up making a continuous model for deterministic chaos of the BZ reaction. Alternatively, I concentrated on finding mathematical structures showing the existence of deterministic chaos in the experimental data.[130]

XI. REMINISCENCE OF CHAOS BY CELSO GREBOGI

In the mid-1970s, while working on my Ph.D. thesis in thermonuclear fusion, I took a course in the qualitative theory of differential equations with a visiting mathematician. It was a cautious, abstract four-month long course on specific differential equations. As I was about to discover soon after, in that course, there was none of the bold, intuitive philosophical generalizations that James Clerk Maxwell,[133] a physicist, and Henri Poincaré,[134,135] a mathematician, felt to be justified. Both understood the importance of systems having sensitive dependence on initial data, the kind of dynamics that is vibrant, compelling, and exciting.

In 1978–1981, I became a postdoc in Berkeley under Allan Kaufman. During that time, a few markedly important events occurred related to chaos. Still during the Soviet times, Boris Chirikov came from Novosibirsk to visit Kaufman in the autumn of 1978. He brought and left with us a preprint copy of his seminal work.[47] With Chirikov preprint on hand, Kaufman organized a discussion group, three hours every Thursday afternoon, initially to study Chirikov's paper, later to go over V. I. Arnold's recently published book.[136] In the discussion group, there were we—Kaufman's group, his former students, and some mathematicians. The latter ones were really important because we learned from them the fundaments of the theory of dynamical systems and ergodic theory, necessary to embark in this new science. The learning of a new science, chaotic dynamics, supported by both the ergodic theory and the theory of dynamical systems, was the most exciting aspect of the multiple-year discussions.

Motivated by the studying of the Chirikov preprint, Kaufman asked the student Steven McDonald to solve the Helmholtz equation in the chaotic Bunimovich stadium. Their seminal work was on quantum chaos.[137] About the same time, Sir Michael Berry, came to Berkeley to deliver the physics colloquium. He spoke about his work on the swimming pool hot spots and on the twinkling of the stars,[138] perhaps the two most important examples that can be understood by employing the catastrophe theory of Thom.[139] It was a fascinating talk that stimulated me to read Thom's catastrophe work. We tried to apply the theory to particle and wave propagation but without much success.

In the autumn of 1981, I moved back to the University of Maryland, where another chapter in chaotic dynamics was about to take place. Upon arriving in Maryland, I delivered a course on symplectic dynamics and Lie transforms at a Navy lab. That invitation came from Robert Cawley who felt that the theory of dynamical systems was the way of the future. There I met Louis Pecora. As part of that course, I invited James Yorke to deliver a seminar as a guest. I have never met him before, though I saw him walking on campus around 1977. I was slightly aware of his work while at Kaufman's group. After his talk, we sat on the stairs in front of the building, chatting about his mathematical work on dynamical systems, and about the naming of "chaos."[2]

That initial conversation with Yorke was the beginning of a two-decade long collaboration, involving the renowned physicist, Edward Ott, resulting in well over 100 papers on the fundaments of chaotic dynamics in such a collaboration. Our work, grounded on the theory of dynamical systems and ergodic theory, and often argued in terms of point set topology, was developed with the use of mathematical maps and differential equations. The latter, typically the pendulum equation[140]

$$\ddot{x} + \nu\dot{x} + \omega^2 \sin x = f \cos t, \tag{14}$$

was usually employed to argue that the phenomenon we were addressing was not particular to a mathematical framework, but it was pervasive in science and technology. In fact, the objective of the research was to establish basic mathematical principles so that

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researchers could then apply those principles to understand and analyze the systems they were investigating in their own fields.

Visualization was a major component in the early scientific developments of chaotic dynamics. It was essential to be able to draw pictures of attractors, basins of attraction, and other invariant sets on a sheet of paper or project them on a screen. In the late 1970s and early 1980s, it was difficult, or often not doable, to carry out more intricate calculations in order to help visualization and understanding and to validate the theories and predictions.[141] We hired a technician from NASA to programme and to deal with the idiosyncrasies of computer arrays. Figure 12, showing the fractal basin boundary and the basins of attraction of a forced damped pendulum equation, is the result of such computations in that computer array.[142] Our pictures were exhibited at the *National Academy of Sciences*, in a museum in New York, and were part of a traveling exhibition throughout the United States. They were also the covers of a dozen of mathematical, scientific, and technical publications.

XII. HOW I BECAME A NONLINEAR DYNAMICIST BY ULRICH PARLITZ

In 1978, I started studying physics at the University of Göttingen, and in 1982, I was looking for an interesting topic for my diploma thesis. In those days, I read popular science books on synergertics, self-organization, and evolution theory written by Haken,[143] Prigogine and Stengers,[144] and Eigen and Winkler,[145,146] and I was fascinated by these new emerging fields because they addressed very fundamental questions of human lives and existence, on how structure comes into being, how units of increasing complexity and functionality arise as a consequence of natural dynamical laws, and in which sense all these processes can be predicted (or not). When talking to fellow students, I got the hint that Werner Lauterborn works on nonlinear dynamics and chaos, and I should talk to him about a diploma thesis in his group. I did so, and soon after, he suggested me to investigate the dynamics of a periodically driven

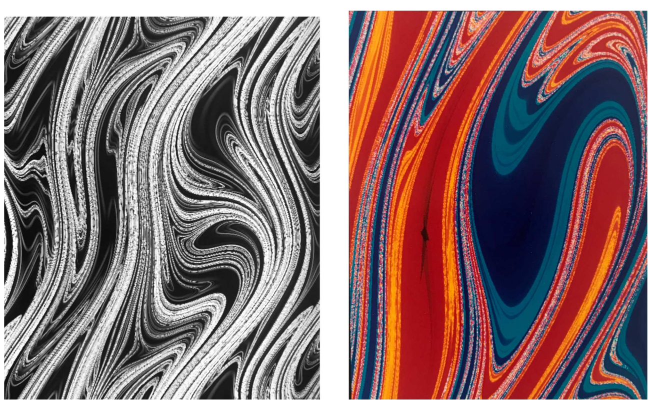

FIG. 12. Fractal basin boundary and the basins of attraction of the forced damped pendulum equation (14). Parameter values: $\nu = 0.1$, $\omega = 1$, and $f = 2$. Reproduced with permission from Grebogi *et al.*, Science **238**, 632–638 (1987). Copyright 1987 American Association for the Advancement of Science.

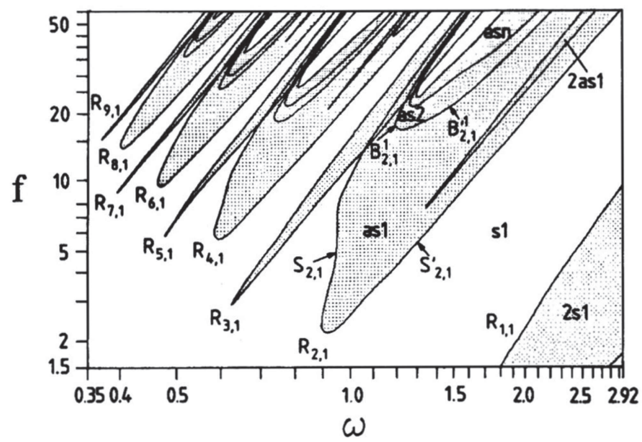

FIG. 13. Parameter space of the Duffing system (15). In the differently dotted domains, there are symmetrical period-1 limit cycle (s1), two coexisting symmetry-related period-1 limit cycle (as1), two pairs of asymmetry related period-1 limit cycles (2as1), a pair of period-2 symmetry-related limit cycles (as2), and pairs of period-n symmetry-related limit cycles (as$n$) with $n = 4, 8, 16, \ldots$. Reproduced with permission from U. Parlitz and W. Lauterborn, Phys. Lett. A **107**, 351–355 (1985). Copyright 1985 Elsevier.

Duffing oscillator

$$\ddot{x} + d\dot{x} + x + x^3 = f\cos(\omega t). \tag{15}$$

Werner was primarily interested in nonlinear resonances he found in his pioneering work on acoustically driven gas (cavitation) bubbles in a liquid in the 1970s,[147] and one of my first tasks was to look for such phenomena in the parameter space of the Duffing oscillator. Experimentally, it was shown by him and Cramer in 1981[148] that bubble dynamics can exhibit period-doubling cascades to chaos, in this context also called *acoustic cavitation noise*. Therefore, searching for chaos was also on my to-do list. It was known that chaotic attractors exist for the Duffing equation with a double-well potential.[149] For the single well oscillator (15), such theoretical results did (to our knowledge) not exist.

Therefore, I started to work, most of the time in the university computer center filling the queue of their main computer, a Sperry UNIVAC 1100/82, and using a VAX-11/780 for interactive exploration of the Duffing oscillator to learn more about its dynamics and how it changes when varying the driving frequency ω or the driving amplitude f. A surprisingly rich, complex but ordered structure of resonances, bifurcations, and coexisting periodic and chaotic attractors emerged, which fascinated both of us, Werner and me (Fig. 13).[150] If this nonlinear system, which looks so simple, already produces such a wide variety of dynamic behavior, what would happen with more complex, higher dimensional systems?

This was also the time when I listened for the first time a seminar talk given by Otto Rössler at the University of Göttingen. It was so impressive that I still remember the situation in the seminar room when Otto Rössler showed slides with the taffy puller to explain the mechanism of stretching and folding underlying chaotic dynamics in the state space.

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After I had finished the diploma thesis, I continued with my Ph.D. studies in Werner's group and delved deeper into the dynamics of periodically driven oscillators. How could one characterize and label the nonlinear resonances and bifurcation curves seen in the Duffing equation (15), the periodically driven bubbles,[147] and other nonlinear oscillators? We knew the dynamics of the circle map and the theory of Arnold's tongues, but this was applicable only to self-sustained (or self-excited) systems such as the van der Pol oscillator. Therefore, the question was where can we find a second "rotational motion" in addition to the periodic driving in order to compute and analyze their frequency ratio? The solution was to consider the winding of neighboring trajectories around the periodic orbits and to quantify this motion by torsion numbers.[151,152] The (average) local torsion frequency Ω can also be used to define a winding number $w = \Omega/\omega$ that fulfills particular recursive schemes in period-doubling cascades and is also defined for chaotic attractors of this class of systems. This approach was not only applied to the Duffing oscillator, a model for a periodically driven gas bubble,[153] and many other passive nonlinear oscillators[154] but also to the periodically forced van der Pol oscillator,[155]

$$\ddot{x} + d(x^2 - 1)\dot{x} + x = f\cos(\omega t). \tag{16}$$

The main motivation for our study of the van der Pol oscillator was, however, the fact that it was known since the seminal analytical work of Cartwright and Littlewood[156] that this system may exhibit aperiodic oscillations, but we could not find in the literature any numerically computed example of a chaotic attractor for Eq. (16). In fact, it took some detailed numerical simulations until we found a complete period-doubling cascade to chaos.[155]

XIII. A KNOTTED ROAD TO CHAOS BY ROBERT GILMORE

My trajectory as a physicist was strongly perturbed by Fortunato Tito Arecchi. He is a world-class laser physicist who visited MIT for a year around 1970. Learning how the operating state changed as the parameters changed was a bifurcation theory problem.[157] We did a lot of useful work in this field; besides it was fun. At this time, a strong connection between the laser physics community and the nonlinear dynamics community was established by Haken. He showed[158] that there was a deep connection between one of the standard laser physics models in a certain limit and the behavior of fluids as described by the Lorenz equations.[10] This connection provoked a number of experimental searches for Lorenz-like output behavior of various types of lasers.[159,160]

During this period, I encountered a reference to "catastrophe theory." Somebody pointed me to Thom's book,[32] not yet translated into English. After reading it, I understood nothing and put this down to my halting French. By perseverance and luck, I was directed to Tim Poston and then in the process of writing his book on the subject with Ian Stewart.[161] Tim gave me copies of several important draft chapters. They were so well-written and straightforward that the concepts were easily assimilable. Tim also directed me to a forthcoming work of Erik Christopher Zeeman,[33] which put the subject to work through many imaginative examples—many too imaginative for the said physics community. The important takeaways from this diversion were[162] (i) the most visible singularities are the stable nodes, but the most important are the unstable saddles because their eigendirections help define basin boundaries, (ii) bifurcations on manifolds could have canonical forms, (iii) all the important ones were discretely classifiable, and (iv) the classification overlapped enormously with the classification of simple Lie groups. This was my introduction to chaos, both of maps and flows, and the Lorenz attractor.[10]

The study of chaos changed dramatically around this time. The enormously powerful tools of renormalization group theory[163,164] were applied to iterative maps in the late 1970s on both sides of the Atlantic.[165–168] These results rapidly lead to several new invariant quantities, such as the scaling ratio $\delta = 4.669\ldots$. Once "universality" was claimed, a sea change occurred. With the universality claim, "It was a very happy and shocking discovery that there were structures in nonlinear systems that were always the same if you looked at them the right way."[40] The community of experimental scientists took this as a challenge, and the race was on. Some of the early experimental tests of the universality prediction are reprinted in the excellent collection by Cvitanović.[169]

One set of experiments was carried out by Arecchi *et al.*[170] This experiment confirmed universality within an experimental error. Tredicce moved to Drexel University to work with Lorenzo M. Narducci in 1985. I moved to Drexel somewhat earlier (1981). My reasons were in part: to continue working on laser problems with Narducci.

In doing these experiments, the Arecchi/Tredicce group had collected a great deal of data, and Tredicce wondered how he could understand them. This was an exciting challenge that I eventually turned my complete attention to. The data showed, among other things, multiple coexisting basins of attraction surrounding orbits of various low periods that sometimes came into existence or winked out of existence with a small change of parameter.[170–172] Experience with catastrophes indicated the presence of saddle-node bifurcations.

The observables were the periodic orbits, and the most important ones were the unstable periodic orbits—again, a lesson from catastrophe theory. We defined the *relative rotation rate*.[173] The ensemble of these fractions for any pair of orbits had very restrictive and informative properties. Furthermore, they could be extracted from experimental data and compared with models of the system. In this way, we were able to show that the dynamics of the periodically driven laser were those of a suspension of the Smale horseshoe.[20] Not surprisingly, there was a simple relation between the linking numbers of two orbits and the sum of their relative rotation rates. This method was applied to other periodically driven systems.[174] Then, it was extended to autonomous three dimensional dynamical systems as the Rössler system[3,5] with the standard parameters.

At this point, we became aware of the Birman–Williams theorem.[175,176] This became a key tool for us. We used it as follows. We could extract a set of low-period unstable orbits from a chaotic attractor and then pairwise compute their "experimental" linking numbers. Then, we could propose a branched manifold that might be the projected limit of the attractor (Fig. 14). A following comparison of the experimental linking numbers with those derived from the branched manifold would show either that we "nailed" the analysis or had to go back to the drawing board. The net result was that we were able to classify the topological structure of chaotic attractors

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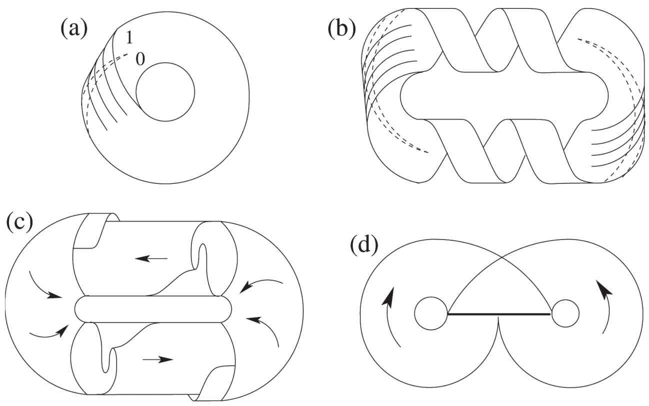

FIG. 14. Branched manifolds describing (a) Rössler, (b) periodically driven Duffing, (c) van der Pol attractors, and (d) Lorenz attractors and for particular parameter values. From Gilmore and Lefranc, *The Topology of Chaos: Alice in Stretch and Squeezeland*, p. 23. Copyright 2003 Wiley. Redrawn with permission of the authors.[204]

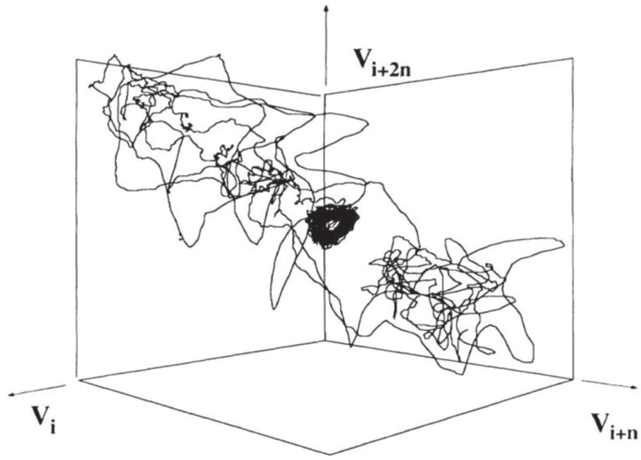

FIG. 15. Trajectory of a chaotic transient ending in a periodic attractor. Reproduced with permission from Carroll *et al.*, Phys. Rev. Lett. **59**, 2891–2894 (1987). Copyright 1987 American Physical Society.

by integers.[177] We were then able to analyze chaotic time series and search for the integer representation of the dynamics that generated these data.[178–180]

XIV. CHAOS RESEARCH AT THE NAVAL RESEARCH LABORATORY BY LOUIS M. PECORA AND THOMAS L. CARROLL

Chaos research at the U.S. Naval Research Laboratory (NRL) was started in the 1980s by Lou Pecora and Tom Carroll, physicists in the former Metals Physics Branch of the Materials Science and Technology Division (MSTD), and by Ira Schwartz originally in the Optical Sciences division of NRL.

Lou wanted to study nonlinear phenomena, especially chaotic motion, in a real system, one that would have interest to the Navy. The material yttrium iron garnet (YIG), a ferrimagnetic compound used in many radiofrequency applications, appeared to be a good candidate material for experimentation. Work by Professor Carson Jeffries at the University of California, Berkeley, showed that the magnetic spin waves in YIG were a nonlinear dynamical system and could display chaotic behavior.[181] Lou got Fred Rachford, also in our Metals Physics Branch, interested in the experiments.

Because Rachford had the experiment ready to go, Tom Carroll was able to jump right in and start taking data on chaos in spin wave interactions in YIG spheres in 1987. The results of the experiment did at first appear "chaotic" but not in the sense Tom was looking for. He did notice some strange transients, where the output from the experiment would at first appear chaotic but then suddenly become periodic (Fig. 15). Lou and Tom traveled to the University of Maryland to consult with Celso Grebogi and Ed Ott who showed them that these transients were something that Grebogi, Ott, and Jim Yorke had actually predicted theoretically.[182] Ott explained the scaling of the transient times with applied power. Our data showed that the average length of these chaotic transients as micro-wave power was increased fit the predicted theoretical power law. By March 1987, Rachford and Tom had accumulated enough data to allow Lou to present the results at the American Physical Society that month.[183,184]

Tom and Lou talked about synchronizing chaotic systems, but they could not come up with any clear way to do that. In January 1988, Lou came home tired from the trip, and after dealing with his young daughter late at night, he went to sleep thinking that somehow, they could drive a chaotic system with a signal from an identical system and maybe the two would synchronize. He managed to remember that idea the next day, and in the next weeks, some simple numerical experiments with iterated maps seemed to confirm that chaotic driving of identical nonlinear systems could cause them to synchronize. However, they needed more than just numerical examples featuring simple maps.

Lou and Tom wanted an experimental example of synchronous chaos. Tom remembered an analog computer circuit that a professor had demonstrated when he was an undergraduate. The circuit used operational amplifiers, capacitors, and resistors to simulate the equations for a bouncing ball with damping, and the circuit output was displayed on an oscilloscope. He wanted to build an analog computer circuit, and chaotic synchronization gave me a reason. He found in the literature a report of a chaotic circuit developed by Professor Robert Newcomb of the University of Maryland.[185] He built a pair of similar circuits: a drive circuit to hide a chaotic signal and a response circuit that synchronized to the drive circuit in order to extract a message signal that had been added to the driving signal.[186] He even used this pair of circuits, along with a digital spectrum analyzer to demonstrate chaotic masking of an information signal in front of NRL's director of research, Dr. Timothy Coffey.

The work on the synchronization of chaotic circuits was still unpublished but caught the interest of some people from the Space and Naval Warfare Systems Command (SPAWAR) through Dr. Mike Melich of the Naval Postgraduate School in Monterey,

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California. Tom and Lou were given some funding to pursue our idea of communicating with chaotic signals.

XV. DISCUSSION AND CONCLUSION BY CHRISTOPHE LETELLIER

From the testimonies in Secs. II–XIV, it might be relevant to ask whether the *chaos program* constitutes a revolution,[187] a new science,[40] or a new paradigm in an already established science. A science commonly designates a branch of science as astronomy, physics, chemistry, or mathematics. Chaos is a branch of mathematics with some overlap with nearly all the other sciences: it is, therefore, not a new science as supported by Holmes.[188] A revolution, as it is meant today, designates a *radical and sudden change*.[189] Nevertheless, the sudden character is questioned,[190] mostly because a scientific revolution is the result of a process that is developed over decades if not centuries. It took nearly 20 centuries to switch from the Aristotelician physics to what is called today the classical mechanics. Chaos has clearly not this stature.

More important is the concept of paradigm as promoted by Kuhn[189] and which is deliberately defined in an open way. In applied science, a paradigm is based on (i) some accepted principles, concepts, and rules (invariant set, attractor, Poincaré–Bendixson theorem, Takens theorem, Smale horseshoe, period-doubling cascade, bifurcation, sensitivity to initial conditions, etc.), which provide some permanent solution to a group of outstanding problems, (ii) a shared methodology (working in the state space, using numerical simulations, sharing some markers, etc.), and (iii) a metaphysics (universality, relationships between mind and matter, etc.). Although with a more or less conscious common ground, beliefs, and sharing applications to some concrete natural phenomena, scientists working with the chaos paradigm can belong to different schools (statistics vs topology, for instance). Indeed, chaos is a paradigm with its "tacit knowledge" acquired through practice and which is not debated. In their history, Aubin and Dahan-Dalmédico oscillate between continuity and rupture to describe the emergence of the nonlinear dynamical systems theory.[11] To us, "epistemological break" as introduced by Bachelard[191]—that we could define as a new way requiring a new concept or approach to solve a given problem but which is still mainly understood with old concepts—seems more appropriate than "rupture" or revolution as promoted by Kuhn[189] because all of the current contributors refer to a background, to a scientific heritage; in fact, they are quite sensitive to be correctly categorized regarding their academic background. For instance, some of physicists expressed their exact field (radio-physics for A.P., condensed matter for L.M.P and T.L.C.). All of the contributors reveal that they are highly influenced by someone—either by his scientific corpus or by his way of thinking (R.A. inspired by Smale and Thom, D.L.S. by Chirikov, R.L. by Thom and Hénon, L.G. by Mackey, A.P. by Rabinovich, L.F.O. by Degn and Rössler, C.G. by Kaufman and Chirikov, U.P. by Lauterborn, R.G. by Thom, L.M.P. and T.L.C. by Grebogi, Yorke, and Ott)—or by a key contribution (Lorenz,[10] Ruelle-Takens,[1] among others). Most of them experienced a change of categorization in their activity from a well-defined field (plasma physics, chemistry, radiophysics, condensed matter, engineering) to a field not so clearly identified and recognized by the academic institutions, as already presented in Aubin and Dahan-Dalmedico: nonlinear dynamical systems (NDS) theory or chaos? Very often, the two terms are combined as if it is necessary to clarify some implicit restriction.

Indeed, the NDS theory is characterized by the lack of existence of analytical solutions, and, consequently, a qualitative approach is required. This denotes a specific methodology whose foundations date back to Poincaré's works: stability analysis, phase portrait, surface of section, Poincaré map, periodic orbits, etc. Poincaré, who was deeply immersed in the history of his fields, would not deny to qualify his contribution as a epistemological break rather than a rupture or a revolution. He was clearly one of those who are producing better while standing on the shoulders of giants.

From this aspect, all the contributions from the pre-computer ages worked within their original scientific discipline: Poincaré, Birkhoff, Lefschetz, Chern, Spanier, and Thom were mostly acting as mathematicians and were recognized as such. Andronov was working in engineering (anticipating control theory). Nevertheless, a few cases deserve some specific comments. Poincaré is commonly recognized as one of the last "universalists," able to address various problems whose nature was very different.[192] His epistemological break was to switch from analytical investigations of approximated solutions to differential equations to the qualitative properties of a set of solutions in the state space. René Thom, once he received his Field Medal, felt free to promote a holistic approach of dynamical processes, evolving strong influences from Thompson[193] and Dirac[194] for developing his catastrophe theory: it would be hard to categorize this contribution. Although clearly connected to the NDS theory, as clearly testified in most of the recollections. Edward Lorenz, who was a meteorologist with a strong background in mathematics (his former academic background), switched from mathematics to meteorology during his military service.[195] Nevertheless, his 1963 contribution[10] is neither a rupture with other meteorological papers nor in continuation of them since he addressed an old problem—accuracy in weather forecasting—which dates back to Bjerknes[196] and Richardson,[197] although the way he treated it was particularly new. It was a more important epistemological break for the NDS theory than for a meteorologist (its impact on the field from where it is issued can be compared with Poincaré's *méthodes nouvelles* in celestial mechanics:[8] strong in NDS theory but less significant in astronomy). The epistemological break is this new combination of mathematical analysis (stability analysis, boundedness, periodic and aperiodic orbits, symbolic analysis of simple maps) with some numerical simulations (state portraits, isopleths, first-return map to a Poincaré section).[10] The impact of the contributions by Poincaré and Lorenz is more important at the interpretation level than for constructing predictive models.

In our view, Lorenz's 1963 contribution is a clear synthesis of the so-called "chaos program," which complements Smale's program as mentioned by R.A. Smale's program belongs to the field of mathematics where contributors most often did not use numerical simulations: they propose *theorems* that they prove analytically. The chaos program is built mostly on numerical data for validating heuristic theory (based on some presupposed assumptions) as experimentalists use their measurements for validating a theory based on presupposed assumptions. The chaos program still belongs to the NDS theory but cannot be considered, *stricto sensu*, mathematics. It can also not be considered physics, engineering,

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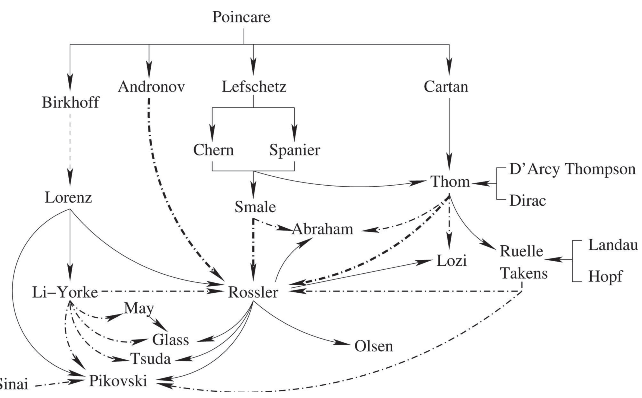

FIG. 16. Some influences between contributors based on some testimonies from Secs. II–XIV, The graph is organized for the sake of simplicity. The arrows (— · —·) correspond to the introduction of numerical simulations. The three bold arrows to Rössler are influences—through direct meetings—before 1975.

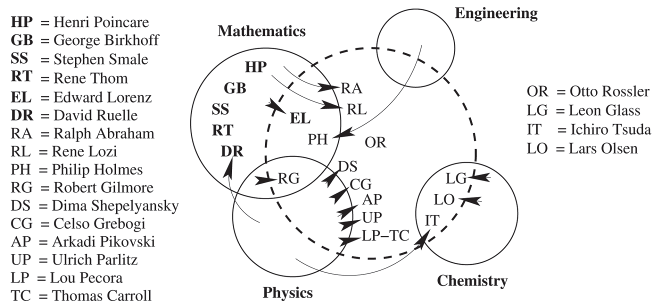

FIG. 17. Distribution of the scientists by their scientific disciplines (mathematics, physics, chemistry, and physiology). The central dashed circle represents the branch "chaos" issued from the NDS theory. In bold are the contributors of the first wave who influenced Rössler. Arrows indicate a change in their scientific disciplines.

or computer science. This automatically means that the chaos paradigm is part of the NDS theory, issued from mathematics, but it is not recognized as part of mathematics. This explains why, still today, more than 50 years after its birth, "chaos" is a field of scientific activity that is, most often, combined with a classical field (fluid mechanics, radiophysics, plasma, optics, chemistry, ecology, physiology, economy, etc.). As an example, a section in *Nonlinear Physics* from the French Physical Society (SFP) was established as late as in 2021, but, from its name, it is restricted to physics; *nonlinear dynamics* would have been a far better name.

The first wave of key contributors to the NDS theory—from the pre-computer ages—is mostly from mathematics, although some of them also contributed to other fields (Poincaré, Thom, Lorenz, Ruelle), and they continued to contribute in mathematics. In the second wave, here associated with those who introduced numerical simulations in their methodology (identified by dashed–dotted lines in Fig. 16) are from various fields (physics, chemistry, engineering, but also mathematics) as shown in Fig. 17, and their works would be more easily categorized in "chaos" rather than in the field of their initial academic background. Some of them even changed the field of their activities (I.T. from physics to chemistry, L.G. from chemistry to physiology, P.H. from engineering to math). Most of them may eventually publish time to time outside the scientific disciplines. This lack of specific discipline is in fact another source of confusion: the techniques developed can be applied to any system of differential equations: it is irrelevant to know from where the equations are coming (until an interpretation is needed). A given scientist can equally contribute to understand the dynamics of an ecosystem and of a pulsating star.

Among the contributors to the present paper, there is an atypical case: Rössler. Graduated in medicine, he never became a physician and switched first to behavioral biology (Konrad Lorenz) and to theoretical biology (Robert Rosen).[65] With his background in electronics as a radio-amateur during his teenage years, he started his career in teaching numerical simulations in chemistry (Tübingen). Before 1976, in the continuation of Rosen's approach, he met Thom, Smale, read Andronov's book, and exchanged ideas with Art Winfree and R.A. within the paradigm of the NDS theory. He was already publishing in the "chaos" paradigm before 1976 since already using numerical simulations, even though the behaviors investigated were exclusively periodic or quasiperiodic.[61,198] Earlier contributors did that too as, for instance, Bonhöffer,[199] FitzHugh,[200] and Hayashi.[201] Rössler used only rarely his medical background during the completion of his chaos program between 1976 and 1983: it is thus absent from his "book" written in the early 1980s and only recently published.[202] In that way, Rössler could be hardly associated with one of the classical scientific disciplines. He was influenced from different scientific disciplines, and, in turn, he influenced many people working in chemistry, biology, and physiology.

In conclusion, chaos is a branch of nonlinear dynamical systems theory that relies on numerical simulations for validating developed rationales (note that we do use neither "theorem" nor "proof"). This is an orphan branch of scientific research in the sense that it can hardly be associated with any classical field. This is eventually due to its abstract nature as well illustrated by Pecora's words,

"Of course, this brings a problem at dinner parties when people ask you what you do. Answering Nuclear Physics, Plant Biology, Chemistry, Astronomy, Ecology, will bring at least nods of (a dim) understanding of what you do. But saying *Nonlinear Dynamics* or, equally the exotic sounding *Chaos Theory* will bring blank stares and a rush to have the others refresh their drinks."

In spite of these humorous words, there is clearly a corpus of contributions (Poincaré, Smale, Thom, Lorenz, Chirikov, Ruelle-Takens, etc.) from which it emerged. As clearly seen by the different recollections provided, the path to be "initiated" is not unique, and, as for every scientific field, there are many ways to contribute to the development of chaos and, more widely, to the nonlinear dynamical systems theory. Each of these paths is based on the interactions and influences of scientists while drinking a coffee or a beer, eating a pizza, or trekking in mountains. This is what web-mediated interactions do not allow and why in-person conferences are absolutely needed.

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DATA AVAILABILITY

This manuscript is a historical review. Data that are quoted in this paper are fully described in the quoted literature.

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