

PAPER: Disordered systems, classical and quantum

Scaling of energy spreading in a disordered Ding-Dong lattice

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Abstract. We study numerical propagation of energy in a one-dimensional Ding-Dong lattice composed of linear oscillators with elastic collisions. Wave propagation is suppressed by breaking translational symmetry, and we consider three ways to do this: position disorder, mass disorder, and a dimer lattice with alternating distances between the units. In all cases the spreading of an initially localized wavepacket is irregular, due to the appearance of chaos, and subdiffusive. For a range of energies and of weak and moderate levels of disorder, we focus on the macroscopic statistical characterization of spreading. Guided by a nonlinear diffusion equation, we establish that the mean waiting times of spreading obey a scaling law in dependence of energy. Moreover, we show that the spreading exponents very weakly depend on the level of disorder.

Keywords: connections between chaos and statistical physics, nonlinear dynamics, transport properties

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1. Introduction

Nonequilibrium processes in nonlinear Hamiltonian lattices are the subject of intensive research. Here, typically, three setups are considered. In one setting, thermal conductivity of such a lattice is of interest, thus one couples it to two thermostats with different temperatures and studies the properties of the energy flux. An early review of these studies can be found in reference [1], and for recent progress see [2, 3]. Another setting deals with properties of a relaxation toward an equilibrium chaotic state with constant energy density. Here one identifies modes which can be treated as the first and second sound [4–6]. Some of these modes can be described theoretically in the framework of fluctuating hydrodynamics [7, 8]. In this paper we follow the third widely used setup, where the spreading of an initially localized wavepacket into a vacuum is considered. In a regular lattice, such spreading is of course dominated by linear or nonlinear waves, propagating with a constant speed. However, in a disordered lattice, due to Anderson localization, there are no propagating linear waves and the spreading typically appears as a slow subdiffusion. A very popular model here is the discrete Anderson nonlinear Schroedinger lattice (DANSE) [9–11]. Another class of models are oscillator chains [12–15]; in contradistinction to Schroedinger lattices they possess only one integral of motion (energy) and are therefore simpler to treat. The spreading of energy in such systems is due to chaos, which in the course of spreading and the corresponding decrease of energy density becomes weaker [16–19]. This leads to the slowing of the spreading, and there are suggestions that the spreading may eventually stop [20]. This still unsolved puzzle makes further numerical investigations of disordered nonlinear lattices, especially at very large times, extremely relevant.

In this paper we address the problem of energy spreading for a Ding-Dong model [21, 22], which has several useful properties. The model is formulated as a chain of linear oscillators, interacting via elastic collisions (see section 2 below). Thus, calculation of its evolution in time is quite simple, because one can write an analytic solution between the collision events and calculate next collision times. This allows following the evolution until very large times without essential loss of accuracy. In our calculations below the maximal times are 10^{10} , to be compared with the characteristic period of oscillators 2π . Furthermore, the Ding-Dong model belongs to a class of strongly nonlinear lattices [23–28], which are characterized by a ‘sonic vacuum’: linear modes (even localized ones like in disordered models) do not exist. This makes the edges of the spreading wavepacket extremely (superexponentially for lattices with a smooth potential) sharp and allows for characterization of propagation via the edge velocity [28]. Below, we define the Ding-Dong model and different types of disorder in it in section 2. In section 3 the main scaling characteristics of the spreading of energy are introduced, which are numerically explored in different configurations in section 4. Finally, we discuss the results and compare them with properties of other strongly nonlinear lattices in section 5.

2. Disordered Ding-Dong lattice

The Ding-Dong lattice [21] is a chain of linear oscillators described by the Hamiltonian

$$H = \sum_k \left(\frac{p_k^2}{2M_k} + M_k \frac{q_k^2}{2} \right). \quad (1)$$

The masses of oscillators M_k are generally different, but all the oscillators have the same frequency $\omega = 1$. The oscillators are aligned along a line, with generally different distances $R_{k,k+1}$ between them. The interaction of oscillators is due to elastic collisions: when oscillators k and $k+1$ collide, i.e. when $q_k = R_{k,k+1} + q_{k+1}$, they exchange their momenta according to

$$p_k \rightarrow \frac{2M_k p_{k+1} + (M_k - M_{k+1})p_k}{M_k + M_{k+1}}, \quad p_{k+1} \rightarrow \frac{2M_{k+1} p_k - (M_k - M_{k+1})p_{k+1}}{M_k + M_{k+1}}. \quad (2)$$

The Ding-Dong model belongs to a class of strongly nonlinear lattices: here in the linear approximation of infinitesimal oscillation amplitudes, any interaction between units is absent, so no waves are propagating (so-called sonic vacuum). In a regular Ding-Dong lattice, i.e. when $R_{k,k+1} = R = \text{const}$ and $M_k = M = \text{const}$, nonlinear waves, compactons, can propagate [21, 22]. As we want to avoid compactons in this study, we consider three types of inhomogeneous Ding-Dong lattice:

- (a) *Position-disorder lattice*: here we assume that all the masses are the same $M_k = 1$, but the distances between the oscillators are distributed randomly. Namely, the

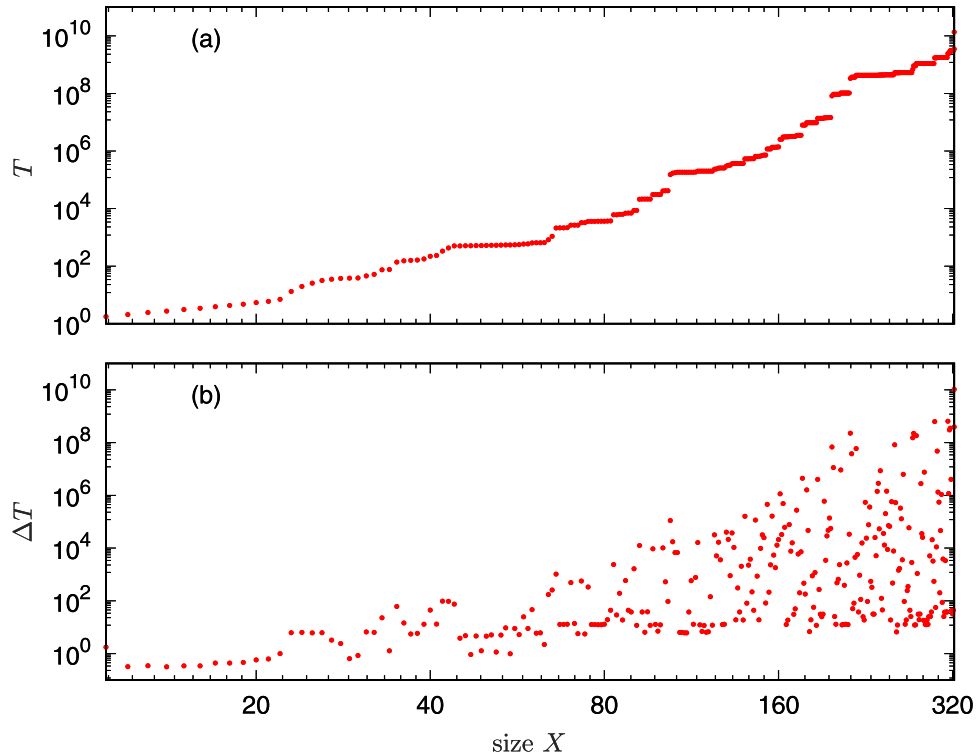


Figure 1. Example of spreading of an initially local perturbation with energy $E = 10$ in a position-disorder lattice with $r = 0.1$. Upper panel: time $T(X)$, at which the active region achieves size X . Bottom panel: waiting times $\Delta T(X)$.

distances are chosen in each realization of disorder as independent random numbers from a uniform distribution $1 - r < R < 1 + r$.

- (b) *Mass-disorder lattice*: here we assume that all the distances are the same $R_{k,k+1} = 1$, but the masses are chosen as independent random numbers with a uniform distribution $1 - m < M < 1 + m$.
- (c) *Dimer lattice*: here we break the homogeneity in the simplest way, assuming alternating distances between the oscillators: $R_{k,k+1} = 1 + d(-1)^k$, while all the masses are equal $M_k = 1$.

In all these cases we consider the problem of energy spreading from a local in space initial distribution of energy. We fix the initial energy E (which is, of course, conserved in the course of evolution), and prepare the initial conditions by setting the initial momenta of 10 neighboring particles to be nonzero, and all initial coordinates are zero:

$$q_k(0) = 0, \quad p_k(0) = \begin{cases} SN(0, 1) & \text{for } 1 \leq k \leq 10, \\ 0 & \text{else.} \end{cases} \quad (3)$$

Here $N(0, 1)$ are Gaussian random numbers, and S is a normalization factor ensuring $\sum_k \frac{p_k^2(0)}{2M_k} = E$. Then we run the evolution, and observe an irregular spreading of energy. It is illustrated in figure 1, where we show an example of this spreading. At each moment

of time, there is an ‘excited domain’ $k_- \leq k \leq k_+$, where the energies of oscillators are positive, while outside of this domain all the oscillators are at rest. The size of the excited domain $X = k_+ - k_- + 1$ extends by one at certain moments of time $T(X)$, when a neighboring oscillator at rest is hit by an active one. In figure 1 we plot these times as a function of the corresponding sizes $T(X)$, and additionally we plot ‘waiting times’ $\Delta T(X) = T(X+1) - T(X)$, which show how long it takes to extend the size of the excited domain by one unit. The calculations in figure 1 end when the waiting time exceeds 10^{10} .

3. Scaling of energy spreading

The main goal of our study is to find statistical properties of the energy spreading, qualitatively presented in figure 1. Similarly to the analysis of strongly nonlinear lattices with a smooth potential [28], we use a nonlinear diffusion equation (NDE) as a guiding phenomenological tool. The NDE equation, where the variable $\rho(x, t)$ should be interpreted as an energy density, reads

$$\frac{\partial \rho}{\partial t} = D_0 \frac{\partial}{\partial x} \left(\rho^a \frac{\partial \rho}{\partial x} \right) = \frac{D_0}{a+1} \frac{\partial^2}{\partial x^2} \rho^{a+1}, \quad \text{with} \quad \int \rho dx = E. \quad (4)$$

This equation possesses a self-similar solution, describing a spreading domain

$$\rho(x, t) = \begin{cases} (t - t_0)^{-1/(2+a)} \left(c E^{2a/(a+2)} - \frac{ax^2}{2(a+2)(t - t_0)^{2/(2+a)}} \right)^{1/a} & \text{for } |x| < x_m, \\ 0 & \text{for } |x| > x_m. \end{cases} \quad (5)$$

Here c is a normalization constant,

$$c = \sqrt{\frac{a}{2\pi(a+2)} \frac{\Gamma(3/2 + 1/a)}{\Gamma(1 + 1/a)}},$$

and x_m denotes the edge of the excited domain,

$$x_m = \sqrt{2c \frac{2+a}{a} E^{a/(2+a)} (t - t_0)^{1/(2+a)}}. \quad (6)$$

This quantity has the same meaning as X for the Ding-Dong lattice, so our aim is to use (6) to find the scaling of the spreading presented in figure 1. Here it is convenient to eliminate an additional parameter t_0 ; according to [28] this can be accomplished by calculating the time derivative of the edge propagation:

$$\frac{1}{E} \frac{dt}{dx_m} \sim \left(\frac{E}{x_m} \right)^{-a-1}. \quad (7)$$

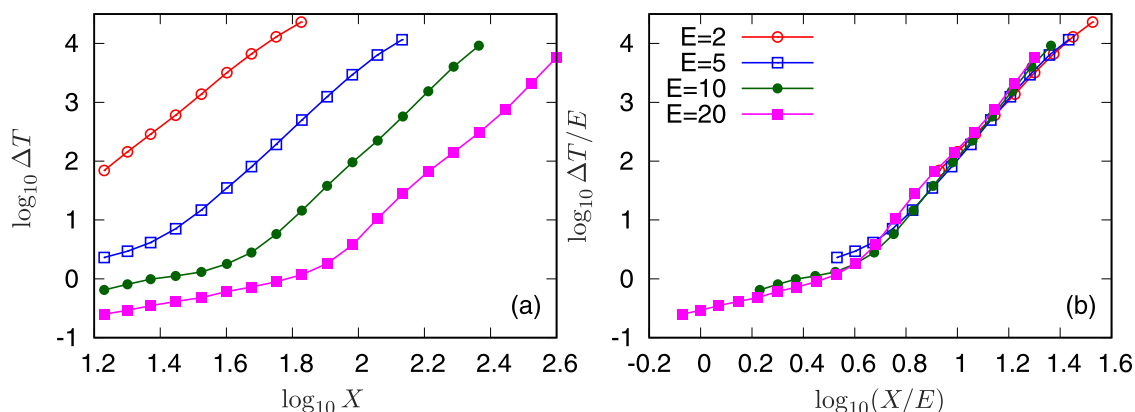


Figure 2. Panel (a): averaged waiting times in a lattice with position disorder $r = 0.15$, and four values of initial energies. (b): The same data in the scaled coordinates.

In comparing with numerics for the Ding-Dong lattice, we have to identify x_m with X , and the inverse velocity of the propagation $\frac{dt}{dx_m}$ with the waiting time ΔT . Because of strong fluctuations in particular realizations (see figure 1), we average $\log_{10} \Delta T$ in narrow ranges around exponentially spaced values of X . As a result, we obtain dependencies $\log_{10} \Delta T$ vs X for different values of the energy E and different values of disorder. The scaling relation (7) predicts that graphs of $\log_{10} \Delta T/E$ vs E/X should collapse and provide a power-law dependence with some exponent a .

4. Numerical results

4.1. Scaling of the wavepacket spreading

As outlined above, we performed a series of simulations of spreading in disordered Ding-Dong lattices. Here, we present the obtained statistical results. They were obtained by averaging over realizations where disorder and initial conditions were chosen independently from the corresponding distributions (for the dimer lattice only the initial conditions were random). The number of realizations used was ≈ 2000 for position disorder, ≈ 400 for mass disorder, and ≈ 500 for the dimer case (the exact values were slightly different for different sets of parameters, due to fluctuations in the running times). In each run, we stopped calculations when the waiting time for the propagation to the next neighbor exceeded a fixed value 10^{10} . We discuss the role of this parameter in section 5.

Figure 2 illustrates the scaling procedure we apply to characterize the spreading. Panel (a) shows spreading in a lattice with distance disorder $r = 0.15$, for four different values of the energy E . The same data are shown in panel (b) in the scaled form. One can see a rather good collapse of the data, confirming the qualitative prediction of the NDE model.

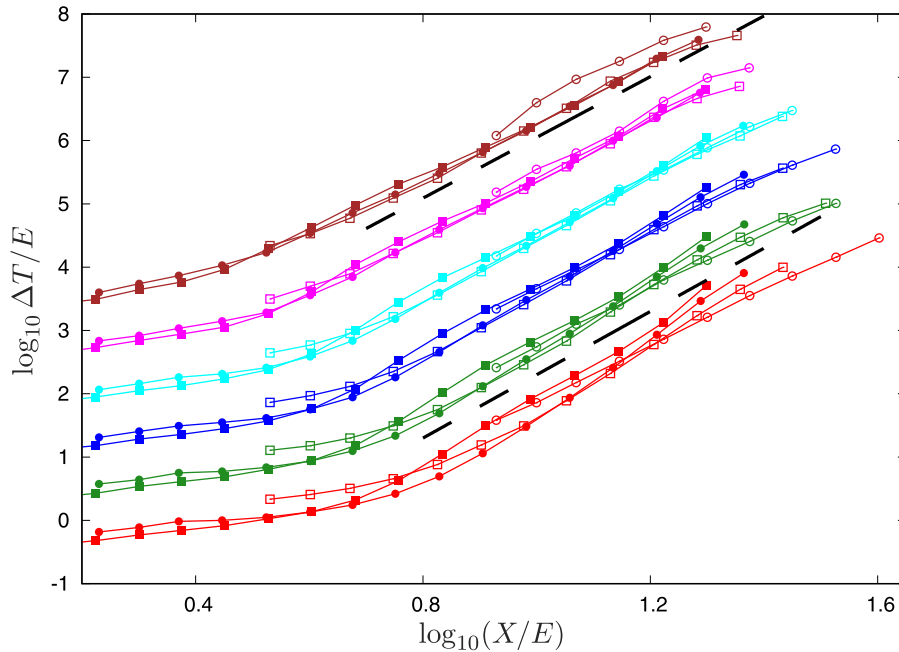


Figure 3. Spreading of states with different initial energies (markers: open circles, open squares, filled circles and filled squares correspond to energies $E = 2, 5, 10, 20$, respectively) in lattices with different position disorder. $r = 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$ correspond to colors red, green, blue, cyan, magenta and brown. The curves for $r > 0.05$ are shifted vertically for better visibility. The lower dashed black line has slope 5, and the upper dashed black line has slope 4.8.

In figure 3 we show the same scaling as in figure 2, but for six different levels of disorder, indicated in the caption. One can see that there are two clearly different regions of scaling, with a crossover at the energy density ≈ 0.25 (i.e. at $\log_{10}(X/E) \approx 0.6$). At larger densities the spreading is rather fast and the waiting time slowly grows as the density decreases. For smaller densities, there is a very strong power-law dependence of the waiting time on the density,

$$\Delta T \sim E \left(\frac{E}{X} \right)^{-\alpha_{\text{pd}}}, \quad \alpha_{\text{pd}} \approx 5. \quad (8)$$

Equivalently, one can express this as a spreading law,

$$X \sim T^{1/(\alpha_{\text{pd}}+1)} \approx T^{1/6}. \quad (9)$$

We mention here that one can expect violations of the scaling for very small energies and very strong disorders, as here potential waiting times could typically exceed the maximal value 10^{10} adopted in our statistical numerical study. A small deviation of the rescaled data at $E = 2$ and $r = 0.4$ (see upper open circle data in figure 3) from the other values may be an indication of this.

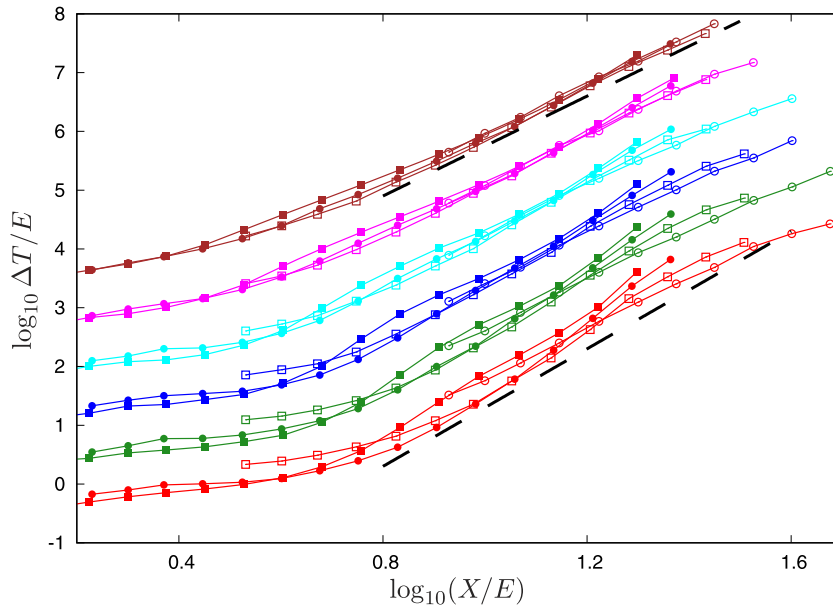


Figure 4. The same as figure 3, but for mass disorder, $m = 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$, for the corresponding colors from bottom to top (markers: open circles, open squares, filled circles and filled squares correspond to energies $E = 2, 5, 10, 20$, respectively). The curves for $r > 0.05$ are shifted vertically for better visibility. The lower dashed black line has slope 5, and the upper dashed black line has slope 4.25.

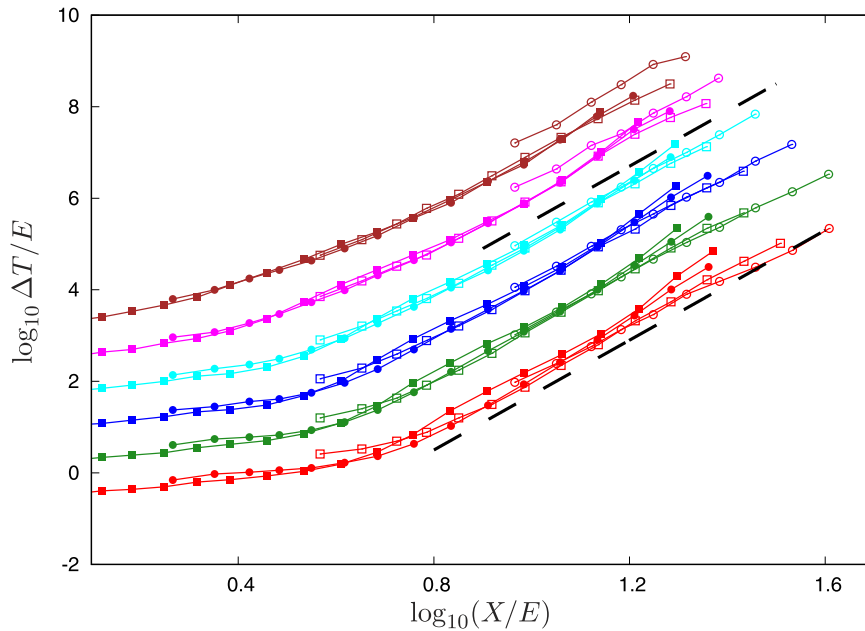


Figure 5. The same as figure 3, but for dimer lattices, with $d = 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$, for the corresponding colors from bottom to top (markers: open circles, open squares, filled circles and filled squares correspond to energies $E = 2, 5, 10, 20$, respectively). The curves for $r > 0.05$ are shifted vertically for better visibility. The dashed black lines have slope 6.

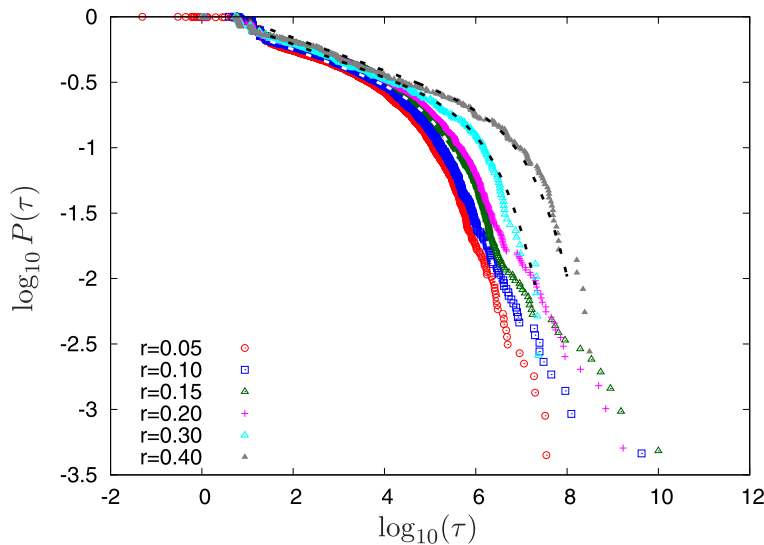


Figure 6. Distribution of waiting times for the lattice with different values of position disorder, at a prescribed level of the average density. The curves for $r = 0.05, 0.1, 0.15, 0.2$ overlap relatively well and can be approximated as $P(\tau) \sim \tau^{-c} 10^{-a_1 \tau^{b_1}}$ with $c = 1/8$, $a_1 = 4 \times 10^{-7}$, $b_1 = 8$ (white dashed line which can be seen on top of data), which indicates that the dependence on disorder is very small for weak disorder. The curves for moderate disorder $r = 0.3$ and $r = 0.4$ deviate at the cutoffs, and they are approximated with the same power-law $c = 1/8$ with stretched-exponential cutoffs $a_2 = 2 \times 10^{-8}$, $b_2 = 9$ and $a_3 = 10^{-9}$, $b_3 = 10$, respectively (black dashed curves).

Next, we present a similar scaling analysis for a Ding-Dong lattice with mass disorder (figure 4) and for a dimer lattice (figure 5). Qualitatively, all the results are similar, but there are small quantitative differences. For the dimer lattice (figure 5), for all values of the modulation level d the scaling exponent in (8) is $\alpha_{\text{dim}} \approx 6$. For the random masses lattice (figure 4), for small disorders $m = 0.05, 0.1$, the best fit is with exponent $\alpha_m \approx 5$, while for larger disorder $m = 0.15, 0.2$, the value of the exponent is different $\alpha_m \approx 4.5$, and for the strongest explored disorder $m = 0.3, 0.4$, the value of the exponent is close to $\alpha_m \approx 4.25$.

4.2. Distribution of waiting times

Here we present the results of the distribution of the waiting times. It is convenient to calculate the cumulative distribution $P(\tau) = \text{prob}(\Delta T > \tau)$. According to the discussion above, it makes sense to compare these distributions at a fixed value of energy density. In figure 6 we show these distributions for the case of position disorder. We fix here $E = 10$ and calculate the distribution of waiting times at the wavepacket size $X = 150$ (the curves for $E = 20$ and $X = 300$ nearly overlap with the presented ones). One can see a good overlap of curves at small disorders, which means that waiting times only weakly depend on the level of disorder if the latter is small. The dashed lines show empirical fits of the distributions as a power-law with a stretched-exponential cutoff, as explained

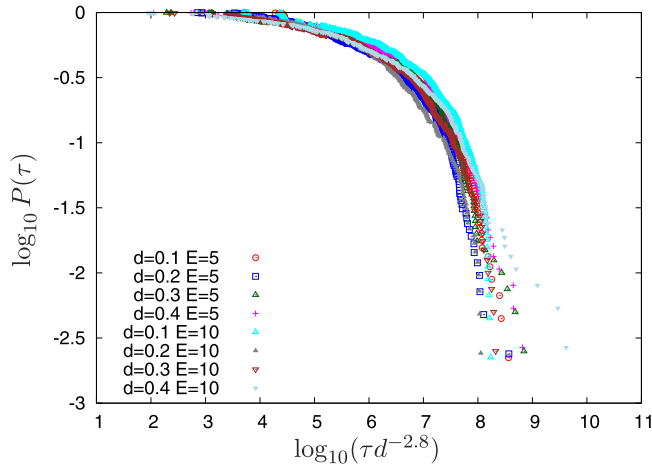


Figure 7. Distribution of waiting times for a lattice with dimer-type heterogeneity. The times are calculated at the energy density level 0.1, i.e. at $X = 50$ for $E = 5$, and at $X = 100$ for $E = 10$. Here, a good overlap is obtained when the time is rescaled with parameter d , as indicated.

in the caption. The distributions at stronger disorder have cutoffs at larger values of waiting time.

The dimer lattice shows a different property: here, the average waiting time at a certain density significantly depends on the modulation level d . We illustrate this in figure 7, which is quite similar to figure 6, but here to achieve an overlap of different distributions, a rescaled time is used. We have found that the characteristic waiting time scales with disorder roughly as $\tau \sim d^{2.8}$, although the range of disorders is too small to validate such a power law. Furthermore, in the dependence $P(\tau)$ one cannot recognize a region with a near power-law profile, contrary to the case of distance disorder in figure 6.

5. Discussion

In this paper we performed a comprehensive numerical exploration of energy spreading in a disordered Ding-Dong model for different types and strengths of disorder, and for different initial energies. First, we demonstrated that for all parameters the scaling (7), corresponding to an NDE, works well. Moreover, the characteristic exponent a in (7) appears to have very similar values for all cases with mass and position disorder, and a larger value for the dimer lattice. This power-law dependence of waiting times means that the spreading at large times is subdiffusive, $X \sim T^{1/6}$ for distance and mass disorder, and $X \sim T^{1/7}$ for the dimer lattice.

We stress here that our numerical analysis is based on finite-time calculations: each run was stopped when the waiting time exceeded 10^{10} . In our previous study of the Ding-Dong model [22], such a long waiting time for one particular run was interpreted as an effective stop of spreading. From our present statistical study we cannot draw this

conclusion, as we do not see any statistically significant ‘slowing down’ of the spreading at large wavepacket sizes. We stress here that because of a finite-time horizon in the present work, the question about eventual continuation or stop of spreading cannot be answered.

It is instructive to compare these findings with the results for lattices composed of *linear* oscillators coupled by higher-order smooth potentials [28]. In the latter case the scaling (7) works as well; however, the resulting curve is not a straight line in log–log coordinates, which means that a single parameter a does not exist, rather it increases in the course of spreading. Correspondingly, the spreading is slower than subdiffusion. In reference [28] also another class of strongly nonlinear lattices has been considered—lattices of *locally nonlinear oscillators*, coupled with higher-order smooth potentials. In this case the scaling of the spreading does not belong to the class predicted by equation (7) but requires an additional nontrivial parameter (in [28] a fractional nonlinear diffusion equation was adopted to fit this scaling). It would be interesting to study a nonlinear version of the Ding-Dong model, i.e. of locally nonlinear oscillators coupled via collisions, but this is a subject of future research.

We have also found that disorder has only a relatively small effect on the propagation speed for small disorder levels. Only for the dimer lattice does the value of position modulation significantly influence the characteristic waiting time. On the other hand, we have not studied lattices with very strong (beyond 40% modulation) disorder. It appears that in lattices with very strong disorder a different approach focused on the bottlenecks (e.g. on situations of exceptionally large distances between the oscillators, for which extremely long waiting times, larger than those used in the present study, can be expected) should be developed. The macroscopic averaging method adopted in this paper does not allow focusing on bottlenecks.

Finally, we would like to mention that a two-dimensional generalization of the Ding-Dong model is possible; this problem is under investigation now.

Acknowledgments

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