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CORRELATIONS OF THE STATES OF NON-ENTRAINED OSCILLATORS IN THE KURAMOTO ENSEMBLE WITH NOISE IN THE MEAN FIELD

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We consider the dynamics of the Kuramoto ensemble oscillators not included in a common synchronized cluster, where the mean field is subject to fluctuations. The fluctuations can be either related to the finite size of the ensemble or superimposed on the mean field in the form of common noise due to the constructive features of the system. It is shown that the states of such oscillators with close natural frequencies appear correlated with each other, since the mean-field fluctuations act as common noise. We quantify the effect with the synchronization index of two oscillators, which is calculated numerically and analytically as a function of the frequency difference and noise intensity. The results are rigorous for large ensembles with additional noise superimposed on the mean field and are qualitatively true for the systems where the mean-field fluctuations are due to the finite size of the ensemble. In the latter case, the effect is found to be independent of the number of oscillators in the ensemble.

1. INTRODUCTION

Many physical, biological and even social systems can be described as complex ensembles of coupled oscillators. The synchronization phenomenon plays an important role in the behavior of such systems [1]: it is observed in electronic and radio-engineering systems, in the collective behavior of people and animals, in the neural structures of the brain, etc. This phenomenon is well studied for systems with different types of coupling or in the presence or absence of noise.

In ensembles of non-identical oscillators, a partial synchronization phenomenon is typical, in which some of the elements are synchronized and form a synchronous cluster, whereas the rest of the ensemble continues to behave asynchronously with the cluster. When considering collective phenomena, priority is given to the dynamics of synchronized elements, and the behavior of non-entrained elements is analyzed mainly in the context of their possible transition to a synchronous cluster and remains relatively poorly understood. Ensembles of finite size [2–4] are exceptions in terms of attention to dynamics of non-entrained elements. In such ensembles, non-entrained elements significantly influence the collective dynamics of the system and, in particular, the behavior of a synchronous cluster.

The subject of this paper is the collective dynamics of non-entrained elements irrespective of the problem of their transition to a synchronous cluster. Namely, for pairs of non-entrained oscillators with a small difference in the natural frequencies, mean-field fluctuations in the ensemble should act as common noise and induce some level of correlation of states [5–8]. Fluctuations can be related both with the external noise acting on the mean field and with the size of the ensemble. Consideration is carried out within the

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framework of a mathematical model that is adequate to the Kuramoto ensemble [9, 10] and some other systems. We calculate the synchronization index $|\langle\exp[i(\varphi_1-\varphi_2)]\rangle|$ (a measure of correlation of the φ_1 and φ_2 states) for non-entrained elements as a function of detuning of their natural frequencies and noise intensity in the mean field.

The structure of the paper is as follows. In Sec. 2, we describe the mathematical model under study and derive averaged evolution equations for the phase difference distribution. In Sec. 3, the synchronization index is calculated analytically. In Sec. 4, the main mathematical model is derived for ensembles of self-excited Van der Pol oscillators with a common force-type effect and chains of series-connected Josephson elements. Section 5 discusses the adequacy of the examined mathematical model to the Kuramoto ensemble. Section 6 presents the conclusions.

2. DYNAMICS OF OSCILLATORS IN A CONSTANT FIELD WITH NOISE

Consider the dynamics of N phase oscillators affected by a constant common force with small fluctuations:

$$\dot{\varphi}_j=\omega_j+[h+\sigma_0\xi(t)]\sin\varphi_j\,, \tag{1}$$

where $j=1,2,\,\dots\,,N$, the overdot denotes a time derivative, h and $\sigma_0\xi(t)$ are the constant and fluctuation components of the common force, σ_0 is the fluctuation amplitude, $\xi(t)$ is assumed to be normalized, δ-correlated Gaussian noise, namely, $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=2\delta(t-t')$, and the angle brackets denote ensemble averaging. Section 4 substantiates the adequacy of Eq. (1) to a wide range of physical systems. For definiteness, in Secs. 2 and 3, we talk about dynamic system (1) in the context of an interpretation in which $h+\sigma_0\xi(t)$ is the common force. However, we mention that this equation also describes the dynamics of the Kuramoto ensemble [9] at the thermodynamic limit of a large number of oscillators, the natural frequencies of the oscillators are not identical, and noise is added to the common force (mean field) acting on the oscillators. In this interpretation, φ_j is the difference of the phase of the jth oscillator and the phase of the mean field, and ω_j is a deviation of the natural frequency of the oscillator from the frequency of the mean field (which is close to the mean natural frequency of oscillators in the ensemble, but is not equal to it). It is expedient to come back to justifying this interpretation of Eq. (1) later, after studying its properties, because understanding of the latter will permit us to discuss its adequacy to the Kuramoto ensemble with a finite number of elements, where the mean field appears to fluctuate with partial synchronization.

We focus on oscillators whose phase is not locked by the force h and would not be locked by it in the absence of noise, which within the framework of Eq. (1) means $|\omega_j|>h$. For $h\neq 0$, the instantaneous growth rates of the phases φ_j start to depend on time even in the absence of noise, while the oscillation frequencies are varied. Let us introduce the replacement of variables

$$\tan\left(\frac{\widetilde{\varphi}_j-\pi/2}{2}\right)=\sqrt{\frac{\omega_j-h}{\omega_j+h}}\tan\left(\frac{\varphi_j-\pi/2}{2}\right).$$

Then Eq. (1) in the variables of new phases $\widetilde{\varphi}_j$ takes the form

$$\dot{\widetilde{\varphi}}_j=\sqrt{\omega_j^2-h^2}+\sigma_0\xi(t)\frac{\omega_j\sin\widetilde{\varphi}_j-h}{\sqrt{\omega_j^2-h^2}}. \tag{2}$$

It can be seen that for $\sigma_0=0$ the new phases grow at constant rates

$$\widetilde{\omega}_j=\sqrt{\omega_j^2-h^2},$$

i. e., are the natural phase variables at $h\neq 0$. Since with the increment of the new phase $\widetilde{\varphi}_j$ by 2π the initial phase φ_j will also grow by 2π, the rates $\widetilde{\omega}_j$ are oscillation frequencies of the oscillators in the absence of noise.

Consider the dynamics of two oscillators with close frequencies. Denote the mean frequency of these oscillators as $\widetilde{\omega} = \sqrt{\omega^2 - h^2}$ and the frequency difference, as Δ. For the further analysis, it is convenient to introduce a new effective noise amplitude $\sigma = 2\omega(\omega^2 - h^2)^{-1/2}\sigma_0$. Then, with a small frequency difference $\Delta = \widetilde{\omega}_1 - \widetilde{\omega}_2$, from Eq. (2) one can obtain

$$\dot{\widetilde{\varphi}}_1 = \widetilde{\omega} + \frac{1}{2}\Delta + \frac{1}{2}\sigma\xi(t)\left(\sin\widetilde{\varphi}_1 - \frac{h}{\omega}\right),$$
$$\dot{\widetilde{\varphi}}_2 = \widetilde{\omega} - \frac{1}{2}\Delta + \frac{1}{2}\sigma\xi(t)\left(\sin\widetilde{\varphi}_2 - \frac{h}{\omega}\right). \tag{3}$$

We now determine the new variables $\theta = \widetilde{\varphi}_1 - \widetilde{\varphi}_2$ and $\psi = \widetilde{\varphi}_1 + \widetilde{\varphi}_2$. In these variables, Eqs. (3) take the form

$$\dot{\theta} = \Delta + \sigma\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi}{2}\right)\xi(t), \qquad \dot{\psi} = 2\widetilde{\omega} + \sigma\left[\sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\theta}{2}\right) - \frac{h}{\omega}\right]\xi(t). \tag{4}$$

The probability density of states $W(\theta, \psi, t)$ for system (4) is determined by the following Fokker–Planck equation:

$$\frac{\partial}{\partial t}W(\theta,\psi,t) + \frac{\partial}{\partial\theta}\Delta W + \frac{\partial}{\partial\psi}2\widetilde{\omega}W = \sigma^2\hat{\mathcal{L}}^2(W), \tag{5}$$

where $\hat{\mathcal{L}}(X)$ is the following differential operator:

$$\hat{\mathcal{L}}(X) = \frac{\partial}{\partial\theta}\left[\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi}{2}\right)X\right] + \frac{\partial}{\partial\psi}\left\{\left[\sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\theta}{2}\right) - \frac{h}{\omega}\right]X\right\}.$$

Using the method of multiple scales [11], averaging of Eq. (5) over a rapidly varying phase ψ can be rigorously performed (as it was done in Appendex A in [12]). For the averaged probability density $w(\theta, t) = \int_0^{2\pi} W(\theta, \psi, t)\,\mathrm{d}\psi$ we obtain the equation

$$\frac{\partial}{\partial t}w(\theta,t) + \frac{\partial}{\partial\theta}\Delta w(\theta,t) = \frac{\sigma^2}{4}\frac{\partial^2}{\partial\theta^2}\left[(1-\cos\theta)w(\theta,t)\right]. \tag{6}$$

The relative error of this equation for the averaged probability density has the order of small parameters $\Delta/\widetilde{\omega}$ and $\sigma^2/\widetilde{\omega}$.

In the case of a statistically stationary state of the system (the probability density $w(\theta)$ is time independent), Eq. (6) can be integrated over θ:

$$\Delta w(t) - \frac{\sigma^2}{4}\frac{\mathrm{d}}{\mathrm{d}\theta}[(1-\cos\theta)w(\theta)] = J = \frac{\nu}{2\pi}, \tag{7}$$

where J is the probability flux and ν is the difference of the observed mean frequencies.

For the further analysis, it is convenient to introduce new normalized variables

$$a = \frac{4\Delta}{\sigma^2} = \frac{\sqrt{\omega^2 - h^2}}{\omega\,\sigma_0^2}|\omega_1 - \omega_2|, \qquad j = \frac{4J}{\sigma^2}.$$

In these variables, Eq. (7) is given by

$$aw(\theta) - \frac{\mathrm{d}}{\mathrm{d}\theta}[(1-\cos\theta)w(\theta)] = j. \tag{8}$$

With allowance for the normalization condition $\int_0^{2\pi} w(\theta)\mathrm{d}\theta = 1$, integration of Eq. (8) over θ permits one to relate the normalized probability flux with the only essential parameter a of the problem:

$$j = a/(2\pi).$$

It can be noted that the latter relation corresponds to the equality $\nu = \Delta$, i. e., noise does not affect the difference of observed frequencies. Substituting the found value into expression (8), we obtain a final equation for the probability density in the stationary case:

$$aw(\theta) - \frac{\mathrm{d}}{\mathrm{d}\theta}[(1-\cos\theta)w(\theta)] = \frac{a}{2\pi}\,. \tag{9}$$

3. ANALYTICAL SOLUTION FOR THE SYNCHRONIZATION INDEX

For the further analysis of stationary system (9) we replace the variables

$$x = \mathrm{ctg}(\theta/2). \tag{10}$$

Denote the probability density of states in new variables by $p(x)$. Since the equality $p(x)\,|\mathrm{d}x| = w(\theta)\,|\mathrm{d}\theta|$ should be fulfilled, the probability densities in the former and new variables are related as follows:

$$p(x) = w(\theta)\frac{2}{1+x^2}\,. \tag{11}$$

To determine the boundary conditions for the function $p(x)$, we consider the limiting case where $w(\theta)$ is singular, i. e., represents a Dirac delta function. If the phase difference $\theta = \pi$, then the corresponding $x = 0$ and the singularity $w(\theta) = \delta(\theta + \pi)$ corresponds to the singularity $p(x) \propto \delta(x)$. If the oscillators tend to the synchronicity state, then the phase difference $\theta \to 0$, while $|x| \to \infty$. Replace the delta function by its finite representation

$$w(\theta) \propto \frac{1}{\epsilon}\exp\left[-\left(\frac{\theta}{\epsilon}\right)^2\right], \tag{12}$$

where ϵ is an arbitrary small number. With allowance for the smallness of θ ($\theta \to \sin\theta$), the latter expression yields the following distribution of the variable x:

$$p(x) \propto \frac{2}{\epsilon(1+x^2)}\exp\left\{-\left[\frac{2x}{(1+x^2)\epsilon}\right]^2\right\}. \tag{13}$$

For large $|x| \to \infty$, expression (13) can be simplified:

$$p(x) \propto \frac{1}{x^2\epsilon}\exp\left[-\frac{4}{(x\epsilon)^2}\right] \propto \begin{cases} 0, & x < 2/\epsilon; \\ (x^2\epsilon)^{-1}, & x > 2/\epsilon. \end{cases} \tag{14}$$

Thus, it is seen that even for a singular distribution of the probability density, $p(x) \to 0$ for $|x| \to \infty$. Let us come back to the Fokker–Planck equation for the probability density. With allowance for Eqs. (10) and (11), Eq. (9) for $p(x)$ takes the form

$$\frac{\mathrm{d}p}{\mathrm{d}x} + ap = \frac{a}{\pi(1+x^2)}\,. \tag{15}$$

The solution of this equation can be represented by the formula

$$p(x) = \frac{a}{\pi}\int\limits_{-\infty}^{x}\frac{1}{1+y^2}\exp[a(y-x)]\,\mathrm{d}y. \tag{16}$$

To calculate the synchronization index

$$\gamma = |\langle\exp(i\theta)\rangle| = |\langle\cos\theta\rangle + i\langle\sin\theta\rangle| \tag{17}$$

it is needed to find the mean values $S = \langle \sin\theta \rangle$ and $C = \langle \cos\theta \rangle$. In view of expression (16) for the probability density, we find

$$S = \frac{a}{\pi} \int\limits_{-\infty}^{+\infty} \frac{2x}{1+x^2}\,\mathrm{d}x \int\limits_{-\infty}^{x} \frac{\exp[a(y-x)]}{1+y^2}\,\mathrm{d}y = 2a[-\mathrm{ci}(2a)\cos(2a) - \mathrm{si}(2a)\sin(2a)], \tag{18}$$

$$C = \frac{a}{\pi} \int\limits_{-\infty}^{+\infty} \frac{x^2-1}{1+x^2}\,\mathrm{d}x \int\limits_{-\infty}^{x} \frac{\exp[a(y-x)]}{1+y^2}\,\mathrm{d}y = 1 - 2a[\mathrm{ci}(2a)\sin(2a) - \mathrm{si}(2a)\cos(2a)], \tag{19}$$

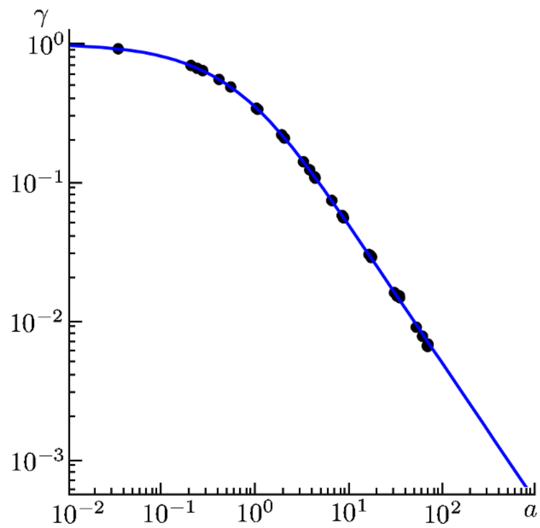

Fig. 1. The synchronization index of two oscillators in the ensemble, which is affected by a constant common field with noise. Analytical solution (17) is represented by a line. The points correspond to the results of direct numerical integration of Eq. (1).

where

$$\mathrm{ci}(x) = -\int\limits_{x}^{\infty} \frac{\cos t}{t}\,\mathrm{d}t = \mathrm{Ci}(x),$$

$$\mathrm{si}(x) = -\int\limits_{x}^{\infty} \frac{\sin t}{t}\,\mathrm{d}t = \mathrm{Si}(x) - \pi/2.$$

Substituting the found expressions into Eq. (17), one can calculate the synchronization index of two oscillators. The obtained dependence of the synchronization index on the parameter a is presented in Fig. 1. It can be seen that the analytical theory agrees well with direct numerical calculations for Eq. (1), which were performed for $\omega_0 = (\omega_1 + \omega_2)/2 = 2$, $h = 1$, and $\sigma_0 = 0.05$; the extreme right point in the diagram corresponds to $\omega_1 - \omega_2 \approx 0.1$, i. e., the frequency difference remains fairly small.

Speaking about the correlation of states, it is important to bear in mind that the mean differences of observed frequencies ν in the considered system are noise independent. The high correlation of states and the absence of "entrainment" of the mean frequency can be combined only with an essentially intermitted character of synchronization: the periods of synchronous behavior alternate with the phase slip periods in such a way that the mean resulting phase difference over a fixed interval of observation time does not depend on the noise intensity.

4. VAN DER POL OSCILLATORS AND JOSEPHSON CONTACTS

We now show how phase equations (1) correspond to some particular systems of oscillators.

As the first example, consider the ensemble of Van der Pol oscillators in the self-excited oscillation mode with an identical force-type effect $f(t)$ on each oscillator:

$$\dot{x}_j = y_j\,, \qquad \dot{y}_j = 2\varepsilon(1 - x_j^2)y_j - \Omega_j^2 x_j + f(t)\,, \tag{20}$$

where $\varepsilon > 0$, Ω_j is the frequency of small oscillations of the jth oscillator. Since harmonic oscillations with amplitude 2 are established in the Van der Pol oscillator for $\varepsilon \ll 1$ and $f = 0$, we introduce the following variable amplitudes and phase: $x_j = 2A_j \sin\phi_j$ and $y_j = 2\Omega_j A_j \cos\phi_j$. By direct substitution into Eq. (20) it can be found that

$$\dot{A}_j = \varepsilon A_j \left\{1 + \cos 2\phi_j - A_j^2[1 - \cos(4\phi_j)]\right\} + \frac{f}{2\Omega_j}\cos\phi_j, \tag{21}$$

$$\dot{\phi}_j = \Omega_j - \varepsilon\left[(1-2A_j^2)\sin 2\phi_j + A_j^2 \sin 4\phi_j\right] - \frac{f}{2A_j\Omega_j}\sin\phi_j. \tag{22}$$

For small ε and f, the Krylov–Bogolyubov averaging method [11] for system (21)–(22) in the leading order yields

$$\dot{A}_j = \varepsilon(A_j - A_j^3),$$

i. e., $A_j \to 1$, and for $A_j = 1$, with allowance for the first order of smallness, we have

$$\dot{\phi}_j = \Omega_j + \varepsilon[\sin(2\phi_j) - \sin(4\phi_j)] - \frac{f}{2\Omega_j}\sin\phi_j\,. \tag{23}$$

Transformation of the phase variables $\varphi_j = \phi_j + [\varepsilon/(2\Omega_j)]\cos(2\phi_j) - [\varepsilon/(4\Omega_j)]\cos(4\phi_j) + O(\varepsilon^2/\Omega_j^2)$ brings Eq. (23) to the form

$$\dot{\varphi}_j = \Omega_j - \frac{f}{2\Omega_j}\left[\sin\varphi_j + O\left(\frac{\varepsilon}{\Omega_j}\right)\right]. \tag{24}$$

For $\varepsilon \ll \Omega_j$, $|f| \ll \Omega_j$, and a small relative detuning of the oscillator frequencies Ω_j, Eq. (24) is equivalent to Eq. (1) with $f/(2\Omega_0) = -[h + \sigma_0\xi(t)]$, where Ω_0 is the mean value of Ω_j. The error of correspondence between these equations is of the second order of smallness. Thus, h and $\sigma_0\xi(t)$ specify the constant and variable components of the common force, respectively.

As the second example, we consider a series connection of Josephson contacts. Voltage u_j at the jth contact is due to the phase jump φ_j at this contact: $u_j = \hbar\dot{\varphi}_j/(2e)$, where $\hbar$ is a Planck constant and e is an elementary charge. With a significant ohmic current of leakage through the contact, the dynamics of the phase jump is determined by the equation [13]

$$\frac{\hbar}{2er_j}\dot{\varphi}_j + I_{0j}\sin\varphi_j = I,$$

where r_j is the ohmic resistance of the contact, $I_{0j}\sin\varphi_j$ is the Josephson current, I_{0j} is the constitutive parameter of the contact, and I is the current flowing through the circuit. The latter equation can be rewritten in the form

$$\dot{\varphi}_j = \frac{2e\,r_j I}{\hbar} - \frac{2e\,r_j}{\hbar}I_{0j}\sin\varphi_j. \tag{25}$$

For metamaterials based on Josephson contacts [14–16], situations are relevant in which I_{0j} changes in the same way for all elements due to a mechanical load or exposure to external fields of a different nature. If the parameters of the elements differ only weakly (which is natural for a metamaterial) and the Josephson current is small compared to the ohmic current, it is possible to neglect the nonidentity of the coefficients before the term $\sin\varphi_j$ and obtain an equation such as (1) for all the elements.

5. KURAMOTO ENSEMBLE WITH MEAN-FIELD FLUCTUATIONS

Consider an ensemble of N oscillators described by the Kuramoto model:

$$\dot{\phi}_j = \Omega_j + \frac{\mu}{N}\sum_{k=1}^{N}\sin(\phi_k - \phi_j). \tag{26}$$

Here, ϕ_j and Ω_j are the phase and natural frequency of the jth oscillator, respectively, and μ is the global coupling coefficient in the ensemble.

All the oscillators interact with each other with identical force; therefore, the collective dynamics of the ensemble can be described using a mean field with amplitude R and phase Φ:

$$Z = R\exp(i\Phi) = \frac{1}{N}\sum_{j=1}^{N}\exp(i\phi_j). \tag{27}$$

In this case, Eq. (26) can be rewritten in the form

$$\dot{\phi}_j = \Omega_j + \mu R\sin(\Phi - \phi_j). \tag{28}$$

We introduce the phase deviation $\varphi_j = \phi_j - \Phi$ and obtain the equation of its dynamics. We assume that the coupling coefficient μ exceeds the excitation threshold of the mean field Z (or, in other words, the synchronous mode) [9, 10] well enough, so that a significant part of elements in the ensemble are synchronized, but oscillators with a large deviation of the natural frequency from the mean value remain non-entrained. Let us differentiate the mean field (27) with respect to time taking into account expression (28):

$$\dot{R} = -\frac{1}{N}\sum_{j=1}^{N}\omega_j\sin\varphi_j + \frac{\mu R}{N}\sum_{j=1}^{N}\sin^2\varphi_j, \tag{29}$$

$$\dot{\Phi} = \Omega_0 + \frac{1}{R}\frac{1}{N}\sum_{j=1}^{N}\omega_j\cos\varphi_j - \frac{\mu}{2N}\sum_{j=1}^{N}\sin(2\varphi_j). \tag{30}$$

Here, we introduced the natural-frequency detuning $\omega_j = \Omega_j - \Omega_0$. Consider the phase dynamics of the mean field in more detail. For a large, but finite number of ensemble elements, fluctuations of the amplitude and rotation velocity of the mean field Z turn out to be small, of the order of $\sqrt{N_\mathrm{a}}/N = \sqrt{N_\mathrm{a}/N}/\sqrt{N}$, where N_a is the number of non-entrained oscillators. When the oscillator is included in a synchronous cluster, its effect on the phase rotation of the mean field becomes almost stationary; therefore, only non-entrained oscillators will give a significant contribution to the fluctuating part of the second and third terms on the right-hand side of Eq. (30). It can be noted that if the greater part of the elements are synchronized, the mean field will rotate with an almost constant amplitude $\langle R\rangle$ [4, 9] and angular frequency fluctuating near the mean value of Ω_0. In this case, the phase of the common field can be represented as the sum of constant and small fluctuation components. Let us estimate the magnitude of the fluctuating component of the second and third terms in Eq. (30):

$$\frac{1}{R}\frac{1}{N}\sum_{j=1}^{N}\omega_j\cos\varphi_j \sim \frac{\delta\omega\sqrt{N_a}}{N}, \qquad \frac{\mu}{N}\sum_{j=1}^{N}\sin(2\varphi_j) \sim \frac{\mu\sqrt{N_a}}{N},$$

where $\delta\omega$ is the half-width of the distribution of natural frequencies ω_j. Thus, it can be written

$$\frac{1}{R}\frac{1}{N}\sum_{j=1}^{N}\omega_j\cos\varphi_j \equiv G_{\Phi,\omega}\frac{\gamma\sqrt{N_a}}{N}\xi_{\Phi,\omega}(t), \qquad \frac{\mu}{2N}\sum_{j=1}^{N}\sin 2\varphi_j \equiv G_{\Phi,\varphi}\frac{\mu\sqrt{N_a}}{2N}\xi_{\Phi,\varphi}(t),$$

where $G_{\Phi,\omega}$ and $G_{\Phi,\varphi}$ are multipliers of order 1, which remain indefinite in the analysis presented here. We introduced pseudo-stochastic processes $\xi_{\Phi,\omega}(t)$ and $\xi_{\Phi,\varphi}(t)$ normalized to unity: $\langle\xi^2_{\Phi,\omega}\rangle = \langle\xi^2_{\Phi,\varphi}\rangle = 1$. These processes cannot be truly stochastic, since they occur as a result of the dynamics of the determinate system. At the same time, they are related to the superposition and nonlinear interaction of N_a oscillatory processes, which in the absence of interaction would have incommensurate frequencies $\widetilde{\omega}_j$, so that the set $\widetilde{\omega}_j$ is not regular. Proceeding from Eq. (28) and (30), one can write the equation for the phase deviation φ_j:

$$\dot{\varphi}_j = \omega_j + \frac{\sqrt{N_a}}{N}\left[G_{\Phi,\omega}\gamma\xi_{\Phi,\omega}(t) + G_{\Phi,\varphi}\frac{\mu}{2}\xi_{\Phi,\varphi}(t)\right] + \mu R\sin\varphi_j. \tag{31}$$

In Eq. (31), the fluctuations R with respect to $\langle R\rangle$ remain undescribed. With the superposition of $N_{\rm a}$ oscillatory processes with a random set of incommensurate frequencies, it should be expected that $R - \langle R\rangle \sim \sqrt{N_{\rm a}}/N$. Let us introduce a quantity $\xi_R(t)$:

$$R - \langle R\rangle \equiv G_R \frac{\sqrt{N_{\rm a}}}{N}\xi_R(t),$$

where G_R is a multiplier of order unity and $\langle \xi_R^2\rangle = 1$. Then Eq. (31) will take the form

$$\dot{\varphi}_j = \omega_j + \frac{\sqrt{N_a}}{N}\left[G_{\Phi,\omega}\gamma\xi_{\Phi,\omega}(t) + G_{\Phi,\varphi}\frac{\mu}{2}\xi_{\Phi,\varphi}(t)\right] + \mu\left[\langle R\rangle + G_R\frac{\sqrt{N_a}}{N}\xi_R(t)\right]\sin\varphi_j. \tag{32}$$

Statistical properties of the $\xi_{\Phi,\omega}(t)$, $\xi_{\Phi,\varphi}(t)$, and $\xi_R(t)$ processes, except for the normalization, remain indefinite.

Since additive noise in a system of this type contributes to synchronization less efficiently than multiplicative noise [12], the noise terms $\xi_{\Phi,\omega}$ and $\xi_{\Phi,\varphi}$ will be neglected. In addition, in Eqs. (3), one can extract the additive component of noise (this term is proportional to h/ω), which upon averaging over phase ψ does not contribute to Eq. (7) in the leading order for evolution of the phase difference distribution $w(\theta,t)$. Since the results of the analytical theory strictly corresponding to Eq. (7) are in good agreement with direct numerical calculations for Eq. (1) (see Fig. 1), the approximation, within which the contribution of additive noise is omitted, can be considered valid. Finally, for description of the oscillator phase deviations from the mean-field phase, we have an equation of the form (1)

$$\dot{\varphi}_j \approx \omega_j + \mu\left[\langle R\rangle + G_R\frac{\sqrt{N_a/N}}{\sqrt{N}}\xi_R(t)\right]\sin\varphi_j. \tag{33}$$

At the thermodynamic limit $N\to\infty$, the noise term related to the dynamics of non-entrained oscillators disappears; however, there exist physical systems where the mean field can be disturbed by common noise for constructive reasons. The presented analysis of Eq. (1) for such systems can be considered rigorous, and the described correlations of the states of non-entrained oscillators with close natural frequencies will be observed. Synchronization by common noise [5–8, 17] will be a mechanism for the occurrence of these correlations.

It was mentioned above that for a finite-size Kuramoto ensemble, the fluctuations $\xi_R(t)$ will not be quite stochastic and, moreover, depend on the signals $\varphi_j(t)$. For such a system, without additional analysis, one can speak only about the qualitative picture of the occurrence of correlations. Let us estimate how the magnitude of correlations depends on the number of oscillators. It was found that for self-excited oscillatory systems with weak common noise $\xi_R(t)$ the synchronizing effect of noise, as well as its impact on the coherence of oscillations of an individual system, is qualitatively determined by the integral of the noise autocorrelation function $\int_{-\infty}^{+\infty}\langle\xi_R(t)\,\xi_R(t+\tau)\rangle\,{\rm d}\tau$ [18, 19], and for a fixed value of this integral it does not depend on other statistical properties of noise. Consequently, comparing Eqs. (1) and (33), as well as collating the normalization of noises $\xi(t)$ and $\xi_R(t)$, one can obtain the effective value σ_0 for Eq. (33): $2\sigma_0^2 = \mu^2 G_R^2 (N_{\rm a}/N^2)\int_{-\infty}^{+\infty}\langle\xi_R(t)\,\xi_R(t+\tau)\rangle\,{\rm d}\tau$. The characteristic decay time of autocorrelations of the order parameter R can be estimated by the time of the mismatch of phases growing at different rates $\widetilde{\omega}_j$: $\tau_{{\rm corr},R}\sim\delta\omega^{-1}$, where $\delta\omega$ is the half-width of the natural-frequency distribution. Hence, $\int_{-\infty}^{+\infty}\langle\xi_R(t)\,\xi_R(t+\tau)\rangle\,{\rm d}\tau \approx 2\langle\xi_R^2\rangle\tau_{{\rm corr},R}\approx 2/\gamma$. For a characteristic difference of natural frequencies, we have $|\omega_1-\omega_2|\approx 2\delta\omega/(N/2)$. Consequently, for the parameter a determining the magnitude of correlations, we obtain the formula

$$a \approx \frac{4(\delta\omega)^2}{\mu^2 G_R^2(N_a/N)}\,\frac{\sqrt{\omega^2-\mu^2\langle R\rangle^2}}{\omega}.$$

The characteristic value of a appears to be independent on the number of oscillators in the ensemble and has the order 1. Thus, for most oscillators in the ensemble, the correlation occurrence effect does not depend on the ensemble size and is relatively weak. At the same time, the frequency difference distribution is random, and there can be pairs of oscillators with a small frequency difference, for which the value of a is small. For a given a_*, the fraction of oscillators with $a < a_*$ is also independent of the ensemble size.

6. CONCLUSIONS

In this paper, we have considered the dynamics of oscillators in the Kuramoto ensemble with noise in the mean field. The focus is on oscillators that are not entrained by a synchronous cluster. It is shown that the noise of the mean field induces correlations of oscillator states with close frequencies. The mechanism of occurrence of correlations is related to the common-noise synchronization phenomenon.

We obtained an analytical expression $\gamma = |\langle \exp[i(\phi_1 - \phi_2)] \rangle|$ for the synchronization index of two oscillators as a function of the difference of their natural frequencies. This expression agrees well with numerical calculations (see Fig. 1). The problem of adequacy of the studied mathematical model of finite-size Kuramoto ensembles, where non-periodic oscillations of non-entrained oscillators create mean-field fluctuations, has been considered. However, the dependence of characteristic differences of the nearest frequencies and the intensity of effective noise on the ensemble size is such that the magnitude of correlations is independent of the ensemble size.

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