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TWO-BUNCH SOLUTIONS FOR THE DYNAMICS OF OTT–ANTONSEN PHASE ENSEMBLES

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We have developed a method for deriving systems of closed equations for the dynamics of order parameters in the ensembles of phase oscillators. The Ott–Antonsen equation for the complex order parameter is a particular case of such equations. The simplest nontrivial extension of the Ott–Antonsen equation corresponds to two-bunch states of the ensemble. Based on the equations obtained, we study the dynamics of multi-bunch chimera states in coupled Kuramoto–Sakaguchi ensembles. We show an increase in the dimensionality of the system dynamics for two-bunch chimeras in the case of identical phase elements and a transition to one-bunch "Abrams chimeras" for imperfect identity (in the latter case, the one-bunch chimeras become attractive).

1. INTRODUCTION

The dynamics of coupled self-oscillatory systems is of interest for many applications in physics, biology, and technology [1, 2]. In the case of weak coupling, a universal approach can be developed based on the phase approximation, in which only the phase dynamics of oscillators is considered, and their amplitudes are assumed algebraically coupled with phases. The famous Kuramoto model describes a system of phase oscillators coupled through the mean field and considers analytically the transition to synchronization. For a certain class of phase systems in a common field (see a more specific definition in the next section, as well as a recent review [2]), Watanabe and Strogatz [3, 4], and Ott and Antonsen developed analytical approaches to obtain closed equations for complex order parameters of phase ensembles.

The Watanabe–Strogatz theory shows a partial integrability of ensembles of identical elements. In particular, motion constants, whose distribution determined by the initial conditions is preserved in the process of evolution, can be introduced at the thermodynamic limit of an infinite ensemble. With arbitrary distributions of these constants, the laws of motion are quite complex, while in the particular case of a uniform distribution, they are significantly simplified. It is exactly this fairly simple case for which a solution was found in [5]. It follows from the above that the Ott–Antonsen solution, in view of the Watanabe–Strogatz partial integrability, can be only neutrally stable for identical ensembles. At the same time, the Ott–Antonsen solution is apparently stable with the introduction of non-identity. Until now, however, there was no description of situations in the vicinity of the Ott–Antonsen solution which could permit one to follow the indicated properties.

Such a description proposed in this paper is based on the so-called circular cumulants. These quantities can be considered special complex order parameters for phase ensembles. For the Ott–Antonsen

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solution, only the first cumulant is non-zero. We show the existence of another invariant solutions that can be fully specified by the first and second cumulants and analyze how the approximation of the solution to the Ott–Antonsen manifold can be described within the framework of this approach in the case of non-identical ensembles.

The paper comprises six sections. Elements of the Ott–Antonsen theory, which are essential for this work, are given in Sec. 2. Formalism of circular cumulants and common equations of dynamics in terms of cumulants is introduced in Sec. 3. Section 4 shows the possibility to obtain a more general solution than the Ott–Antonsen solution and establishes the two-bunch nature of the new solution. In Sec. 5, the obtained results are used to study multi-bunch chimera states in the Abrams system [7]. Final conclusions are given in Sec. 6.

2. OTT–ANTONSEN THEORY FOR PHASE SYSTEMS

In this section, we present the elements of the Ott–Antonsen theory for phase oscillators, which will be needed for the further analysis. This theory applies to systems of identical elements described by the equations

$$\dot{\varphi}_k = \omega(t) + \mathrm{Im}[2h(t)\exp(-i\varphi_k)], \qquad k = 1, \ldots, N, \tag{1}$$

where $\omega(t)$ and $h(t)$ are the arbitrary real-valued and complex functions of time, respectively; the overdot denotes a time derivative. The theory is valid at the thermodynamic limit $N \to \infty$, where it is natural to describe the evolution of the system in terms of the probability density of the phase distribution $w(\varphi, t)$. In these terms, the system has the properties of the Watanabe–Strogatz integrability. However, it is difficult to describe these properties in terms of the density $w(\varphi, t)$ or in terms of complex modes $a_n(t)$, where $a_n(t)$ are the Fourier expansion coefficients

$$w(\varphi, t) = (2\pi)^{-1}\Big\{1 + \sum_{n=1}^{\infty}[a_n(t)\exp(-in\varphi) + a_n^*(t)\exp(in\varphi)]\Big\}.$$

Here, the asterisk denotes complex conjugation. The equation for $w(\varphi, t)$, which follows from the dynamic equation (1) and is given by

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}\Big\{ [\omega(t) - ih(t)\exp(-i\varphi) + ih^*(t)\exp(i\varphi)]\, w\Big\} = 0,$$

in the Fourier space takes the form of an infinite system

$$\dot{a}_n = in\omega a_n + nha_{n-1} - nh^* a_{n+1}\,, \qquad n = 1, 2, \ldots, \tag{2}$$

where it should be assumed that $a_0 = 1$. Ott and Antonsen note that the chain of equations (2) admits a solution of the form $a_n(t) = [a_1(t)]^n$, for which one can obtain a closed dynamic equation:

$$\dot{a}_1 = i\omega a_1 + h - h^* a_1^2. \tag{3}$$

Herewith, the variable $a_1 = \langle \exp(i\varphi_k)\rangle$ is the order parameter of the system (the angle brackets denote ensemble averaging). The set of solutions, for which $a_n = (a_1)^n$, is called the Ott–Antonsen manifold [5]. It corresponds to the probability density distribution of quite a definite form, which can be found by substituting $a_n = a_1^n$ in the Fourier expansion $w(\varphi)$:

$$w_{\mathrm{OA}}(\varphi) = \frac{1 - |a_1|^2}{2\pi|1 - a_1\exp(-i\varphi)|^2}\,.$$

In the Watanabe–Strogatz variables, this manifold corresponds to a uniform distribution of constants in the Möbius transform underlying the Watanabe–Strogatz theory [4]. Note that other distributions of constants also remain invariant, but they are difficult to express through a_n modes. The Watanabe–Strogatz theory suggests that the Ott–Antonsen manifold is neutrally stable. However, in the realistic cases of imperfect identity, i. e., with spread of natural frequencies, this manifold becomes attractive [5, 6], which determines the importance of this particular solution $a_n = (a_1)^n$ and justifies reducing the description to a dynamic equation such as (3) in some problems [5, 7–11].

The Ott–Antonsen theory is extended to the case of an ensemble of oscillators with non-identical natural frequencies ω as follows. Oscillators can be bunched by natural frequencies, and Eqs. (2) should be written for each bunch of oscillators with identical frequencies ω:

$$\dot{a}_{\omega,n} = in\omega a_{\omega,n} + nha_{\omega,n-1} - nh^* a_{\omega,n+1}, \qquad n = 1, 2, \dots . \tag{4}$$

The distribution for the total ensemble is $w(\varphi, t) = \int w_\omega(\varphi, t)\, g(\omega)\, \mathrm{d}\omega$, where $g(\omega)$ is the distribution of frequencies ω, and, accordingly, $a_n = \int a_{\omega,n} g(\omega)\, \mathrm{d}\omega$. If it is assumed that at some instant of time, $a_{\omega,n}$ are smooth functions of ω, which are analytical in the upper half-plane of the complex value ω, then the increments $\mathrm{d}a_{\omega,n}$ described by Eqs. (4) will also be analytical functions of ω in the upper half-plane, and $a_{\omega,n}$ will remain analytical. In the case of analytical $a_{\omega,n}$, the integrals $\int a_{\omega,n} g(\omega)\, \mathrm{d}\omega$ for specific distributions $g(\omega)$ will be calculated by the methods of the theory of residues. In particular, for the Lorentz distribution of frequencies with the mean frequency Ω and half-width γ, which is given by the expression

$$g(\omega) = \frac{\gamma}{\pi[(\omega - \Omega)^2 + \gamma^2]} ,$$

by closing the contour through the upper complex half-plane, one can calculate

$$a_n = \int a_{\omega,n} g(\omega)\, \mathrm{d}\omega = a_{\Omega + i\gamma, n}$$

and use Eq. (4) for the complex value $\omega = \Omega + i\gamma$:

$$\dot{a}_n = n(i\Omega - \gamma) a_n + nha_{n-1} - nh^* a_{n+1} , \qquad n = 1, 2, \dots . \tag{5}$$

We then will work with this system of equations bearing in mind that the case of identical frequencies corresponds to $\gamma = 0$.

3. CIRCULAR CUMULANTS

Describing the dynamics of the system in the vicinity of the Ott–Antonsen manifold in terms of a_n is problematic for states with a high degree of synchronicity, where $|a_1|$ is close to 1 and the series $a_n \approx (a_1)^n$ has a slow convergence. In this respect, it may be efficient to pass from considering the moments $a_n = \langle [\exp(i\varphi)]^n \rangle$ to cumulants that formally correspond to them

$$K_1 = a_1, \quad K_2 = a_2 - a_1^2, \quad K_3 = a_3 - 3a_2 a_1 + 2a_1^3, \quad \dots .$$

Strict relation between a_n and K_n is determined by means of generating functions:

$$F(\zeta) \equiv \big\langle \exp[\zeta \exp(i\varphi)] \big\rangle \equiv \sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!}, \qquad \Phi(\zeta) \equiv \ln[F(\zeta)] \equiv \sum_{m=1}^{\infty} K_n \frac{\zeta^n}{n!}. \tag{6}$$

In terms of K_n, Eqs. (2) take the form

$$\dot{K}_n = n(i\Omega - \gamma) K_n + h\delta_{1n} - nh^* \Big[K_{n+1} + \sum_{m=1}^{n} \frac{(n-1)!}{(m-1)!\,(m-n)!} K_{n-m+1} K_m \Big],$$

where for $n = 1$ we have $\delta_{1n} = 1$, and for $n \neq 1$ the quantity $\delta_{1n} = 0$. Strict derivation of the equations for K_n from Eqs. (2) is given in [12], while in the present paper we rely on this procedure as on the fact whose reliability can be checked for the first several n using explicit relations between a_n and K_n. For $\varkappa_n \equiv K_n/(n-1)!$, the dynamic equations are slightly simplified:

$$\dot{\varkappa}_n = n(i\Omega - \gamma)\varkappa_n + h\delta_{1n} - nh^*\Big(n\varkappa_{n+1} + \sum_{m=1}^{n} \varkappa_{n-m+1}\varkappa_m\Big)\,. \tag{7}$$

It follows from the last equation that the dynamics of $\varkappa_n$ is determined by the values of $\varkappa_1$, $\varkappa_2$, ..., $\varkappa_{n+1}$, and $\varkappa_{n+1}$ enters Eq. (7) through the term $-n^2h^*\varkappa_{n+1}$. Thus, it does not seem possible in general to single out the closed system of a finite number of equations.

Hereafter we call the quantities $\varkappa_n$ circular cumulants. Since a_n are not the moments of the phase, but the moments of $\exp(i\varphi)$, the quantities K_n are not true cumulants of the phase distribution. This permits admissible arbitrariness in the selection of circular cumulants (choosing between the variables K_n and $\varkappa_n$).

4. TWO-BUNCH INVARIANT MANIFOLD

4.1. Extension of the Ott–Antonsen manifold

In terms of circular cumulants, the Ott–Antonsen solution ($a_n = (a_1)^n$) takes a very simple form

$$\varkappa_1 = a_1\,, \qquad \varkappa_{n\geq 2} = 0. \tag{8}$$

This fact serves as a considerable argument in favor of the cumulant representation: firstly, it appears that the higher-order cumulants describe a deviation from the Ott–Antonsen manifold and, secondly, it becomes possible to avoid the problem of slow convergence of the series a_n for $|a_1|\to 1$.

For solving Eq. (8), only the first equation, which coincides with Eq. (3), remains in the chain of equations (7), while all the subsequent equations are identically satisfied. However, broader specific cases are also possible, where, instead of the infinite system (7), closed equations can also be obtained. Generally, if the higher-order cumulants are not equal to zero, then the equation for $n = 1$ has the form

$$\dot{\varkappa}_1 = (i\Omega - \gamma)\varkappa_1 + h - h^*(\varkappa_2 + \varkappa_1^2)\,. \tag{9}$$

In typical problems, $h = h(\varkappa_1, t)$, and deviation of the dynamics of the order parameter $\varkappa_1$ from the Ott–Antonsen solution requires a nonzero value of $\varkappa_2$. Therefore, the particular solutions with $\varkappa_2 \neq 0$ are of interest. The system of equations (7) is closed if for some significant cumulants $\varkappa_n$ the cumulants $\varkappa_{n+1}$ will become zero. At $\varkappa_2 \neq 0$, the solutions for which all the odd cumulants $\varkappa_n$, except for the first one, are equal to zero are the simplest variant for consideration.

Consider the case $\varkappa_{2j+1} = 0$, where $j = 1, 2, \ldots$. For $n = 1$, we have the equation (9). For $n = 2j+1$, Eqs. (7) yield

$$0 = -nh^*\big[n\varkappa_{n+1} + (\varkappa_n\varkappa_1 + \varkappa_{n-1}\varkappa_2 + \cdots + \varkappa_1\varkappa_n)\big],$$

or

$$0 = (2j+1)\,\varkappa_{2j+2} + \varkappa_{2j}\varkappa_2 + \varkappa_{2j-2}\varkappa_4 + \cdots + \varkappa_2\varkappa_{2j}\,. \tag{10}$$

For $n = 2j$,

$$\dot{\varkappa}_n = n\,(i\Omega - \gamma)\,\varkappa_n - nh^*\big(\varkappa_n\varkappa_1 + \varkappa_{n-1}\varkappa_2 + \varkappa_{n-2}\varkappa_3 + \cdots + \varkappa_1\varkappa_n\big)\,,$$

or

$$\dot{\varkappa}_{2j} = 2j\,(i\Omega - \gamma - 2h^*\varkappa_1)\,\varkappa_{2j}. \tag{11}$$

From the latter equation it can be seen that the dynamics of all the even circular cumulants is similar to the dynamics of the second one: for some solution $\varkappa_2(t)$, we have

$$\varkappa_{2j}(t) = C_j \left[\varkappa_2(t)\right]^j. \tag{12}$$

Here, C_j are the integration constants specified by the initial conditions. By definition, $C_1 = 1$. However, it should be remembered that Eqs. (11) are valid only with the fulfilment of relations (10), which are the condition that the odd cumulants higher than the first one remain zero. Solution (12) satisfies relation (10) with the appropriate choice of constants C_j: Eq. (10) can be rewritten in the form of a recurrent relation

$$C_{j+1} = -\frac{1}{2j+1} \sum_{m=1}^{j} C_m C_{j+1-m}, \tag{13}$$

which expresses C_{j+1} through C_1, C_2, ..., C_j and for a fixed $C_1 = 1$ yields a uniquely determined set of coefficients $\{C_j\} = \{1, -1/3, 2/15, -17/315, \dots\}$.

Thus, in the system it is possible to single out the invariant manifold

$$\varkappa_{2j-1} = \varkappa_1 \delta_{1j}\,, \qquad \varkappa_{2j} = C_j \varkappa_2^j\,, \tag{14}$$

which is the expansion of the Ott–Antonsen manifold (it corresponds to $\varkappa_2 = 0$). The dynamics of the system on this manifold is determined by a closed system of two complex equations:

$$\dot{\varkappa}_1 = (i\Omega - \gamma)\,\varkappa_1 + h - h^*(\varkappa_2 + \varkappa_1^2), \tag{15}$$

$$\dot{\varkappa}_2 = 2\,(i\Omega - \gamma - 2h^*\varkappa_1)\,\varkappa_2. \tag{16}$$

If h depends only on the first cumulants, $\varkappa_1$ and $\varkappa_2$, then the system remains closed since on this set the higher-order cumulants are expressed through the second cumulant (or are equal to zero).

4.2. Two-bunch character of states of the form (14)

We will show that the constructed invariant manifold can be interpreted as a two-bunch state: the manifold comprises two subensembles, in each of which the distribution of oscillators corresponds to the Ott–Antonsen solution. Let us calculate the cumulants $\varkappa_n$ for this state. For each subensemble, we have an order parameter $\varkappa_1^{\{p\}}$ and zero higher-order circular cimulants $\varkappa_{n\geq 2}^{\{p\}}$, where $p = 1, 2$ is the subensemble number. The distribution of oscillators in the total ensemble is the superposition of distributions in the subensembles: $w(\varphi, t) = q_1 w_1(\varphi, t) + q_2 w_2(\varphi, t)$, where q_p is the fraction of elements in the subensemble p. By virtue of the linearity of the connection between the generating function of moments $F(\zeta)$ and the distribution $w(\varphi)$, one can obtain $F(\zeta) = q_1 F_1(\zeta) + q_2 F_2(\zeta) = q_1 \exp[\Phi_1(\zeta)] + q_2 \exp[\Phi_2(\zeta)]$ (see Eqs. (6)). Since for the Ott–Antonsen solution the generating function of cumulants has the form $\Phi_p(\zeta) = \varkappa_1^{\{p\}}\zeta$, we finally find

$$\Phi(\zeta) = \ln\left[q_1 \exp(\varkappa_1^{\{1\}}\zeta) + q_2 \exp(\varkappa_1^{\{2\}}\zeta)\right].$$

Expansion of the latter expression into a Taylor series yields the values of K_n and $\varkappa_n$. In the case of an equal distribution of elements between the bunches $q_1 = q_2 = 1/2$, calculations show that all the odd cumulants higher than the first one are equal to zero, while for the even cumulants, relation (12) takes place; in this case,

$$\varkappa_1 = (\varkappa_1^{\{1\}} + \varkappa_1^{\{2\}})/2\,, \qquad \varkappa_{2j} = C_j \left[\left(\varkappa_1^{\{1\}} - \varkappa_1^{\{2\}}\right)^2/4\right]^j. \tag{17}$$

For an unequal distribution between bunches, the expansion has a much more complex form and the odd cumulants are non-zero.

Thus, the states with an equal distribution of elements between two bunches of the form corresponding to the Ott–Antonsen solution correspond to the invariant manifold we found. Since the total set of cumulants uniquely determines the probability density distribution, the found correspondence between the two-bunch states and partial solutions of the form (14) is one-to-one. The Ott–Antonsen solution itself is a specific case of coincidence of the bunches: $\varkappa_1^{\{1\}} = \varkappa_1^{\{2\}}$.

5. DYNAMICS OF MULTI-BUNCH CHIMERAS IN COUPLED KURAMOTO–SAKAGUCHI ENSEMBLES

As an example, we consider the dynamics of two symmetric coupled ensembles of Kuramoto–Sakaguchi oscillators (generalization of the Kuramoto model):

$$\begin{aligned} \dot{\varphi}_k &= \omega_k + \frac{\mu}{N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_k - \alpha) + \frac{\nu}{N}\sum_{j=1}^{N}\sin(\psi_j - \varphi_k - \alpha)\,, \\ \dot{\psi}_k &= \omega_k + \frac{\mu}{N}\sum_{j=1}^{N}\sin(\psi_j - \psi_k - \alpha) + \frac{\nu}{N}\sum_{j=1}^{N}\sin(\varphi_j - \psi_k - \alpha)\,. \end{aligned} \tag{18}$$

Here, φ and ψ are the phases in ensembles having a size N, μ and $\nu = 1 - \mu$ are the coupling parameters within the ensemble and with another ensemble, respectively, α is the phase shift of coupling (the coupling is attractive for $\cos\alpha > 0$, repulsive for $\cos\alpha < 0$, and conservative for $\cos\alpha = 0$; for the Kuramoto model, $\alpha = 0$). In [7], the dynamics of this system was studied for the case of identical frequencies $\omega_k = \Omega$ at the thermodynamic limit $N \to \infty$. We found the chimera states in which the first ensemble is completely synchronized, while the oscillators of the second ensemble form a bunch whose width is periodically varied with time (see Fig. 1*a*). This bunch corresponds to the Ott–Antonsen solution, and the described regime on the corresponding manifold is attractive. Subsequently [13], it was noticed that this regime should not be the only attractive mode since with identical frequencies the Ott–Antonsen manifold, on which the initial conditions were specified, is not a transversally attractive one. In particular, it was demonstrated that if the initial state of the system is specified as different from the Ott–Antonsen distribution, then the system does not transform to the modes represented in [7], and the dynamics of the order parameters has a higher embedding dimension (see Fig. 1*b*).

We now use the result given by Eqs. (15) and (16) to study analytically the dynamics of system (18) outside the Ott–Antonsen manifold and establish the effect of the detuning of natural frequencies γ on the collective dynamics of the system. To simplify the equations, we assume that the first ensemble (of the φ phase) represents a single bunch with the first cumulant κ, while the first and the second cumulants in an ensemble of ψ phases, composed of two bunches, will be denoted $\varkappa_1$ and $\varkappa_2$.

Based on Eqs. (15) and (16) for system (18), it can be written

$$\dot{\kappa} = (i\Omega - \gamma)\,\kappa + h_1 - h_1^{*}\kappa^2; \tag{19}$$

$$\dot{\varkappa}_1 = (i\Omega - \gamma)\,\varkappa_1 + h_2 - h_2^{*}\,(\varkappa_2 + \varkappa_1^2); \tag{20}$$

$$\dot{\varkappa}_2 = 2\,(i\Omega - \gamma - 2h_2^{*}\varkappa_1)\,\varkappa_2, \tag{21}$$

where $h_1 = (\mu\kappa + \nu\varkappa_1)\exp(-i\alpha)/2$ and $h_2 = (\mu\varkappa_1 + \nu\kappa)\exp(-i\alpha)/2$.

5.1. Stability of the one-bunch mode to the second-bunch excitation

Consider stability of the second ensemble to the $\varkappa_2$ excitation, i. e., the behavior of the system of (20) and (21) for a small $\varkappa_2$. Equation (21) yields

$$\mathrm{Re}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\varkappa_2\right) = -2\gamma - 2\Big\{\mu\,|\varkappa_1|^2\cos\alpha + \nu\mathrm{Re}[\kappa^{*}\varkappa_1\exp(i\alpha)]\Big\}. \tag{22}$$

From Eq. (20) for $|\varkappa_2| \ll |\varkappa_1|$ one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\varkappa_1|^2 = -2\gamma\,|\varkappa_1|^2 + (1-|\varkappa_1|^2)\Big\{\mu\,|\varkappa_1|^2\cos\alpha + \nu \mathrm{Re}[\kappa^*\varkappa_1\exp(i\alpha)]\Big\}. \tag{23}$$

For analysis of the stability properties described by Eq. (22), it is convenient to rewrite the latter equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{1}{1-|\varkappa_1|^2} = -\frac{2\gamma\,|\varkappa_1|^2}{1-|\varkappa_1|^2} + \Big\{\mu\,|\varkappa_1|^2\cos\alpha + \nu \mathrm{Re}[\kappa^*\varkappa_1\exp(i\alpha)]\Big\}.$$

Comparing the right-hand sides of this equation and Eq. (22) and averaging over time, it can be found that

$$\mathrm{Re}\overline{\frac{\mathrm{d}}{\mathrm{d}t}\ln\varkappa_2} = -2\gamma\overline{\frac{1+|\varkappa_1|^2}{1-|\varkappa_1|^2}} - 2\overline{\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{1}{1-|\varkappa_1|^2}}, \tag{24}$$

where the overbar denotes time averaging. The quantity $\mathrm{Re}\,\overline{\mathrm{d}\ln\varkappa_2/\mathrm{d}t}$ determines the $|\varkappa_2|$ growth rate and, thus, characterizes stability of the system to perturbations leading it away from the Ott–Antonsen manifold.

It can be seen from Eq. (24) that there can be two types of trajectories, for which the properties of stability to the $\varkappa_2$ excitation can differ, namely, a limited trajectory with imperfect synchronicity ($|\varkappa_1| < 1$) and trajectories for which $|\varkappa_1| \to 1$. In the first case, the second term on the right-hand side of Eqs. (24) is equal to zero and the stability is determined by the first term, in which the expression for γ is sign-definite. At $\gamma = 0$, deviations from the Ott–Antonsen manifold are not attenuated with time, while at $\gamma \neq 0$ they start to decay. In the second case, at $|\varkappa_1| \to 1$, the argument of the logarithm $(1-|\varkappa_1|^2)^{-1}$ tends to $+\infty$ and

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{1}{1-|\varkappa_1|^2}} > 0.$$

Thus, when passing to complete synchronization, the trajectories merge to the Ott–Antonsen manifold: $\varkappa_2$ decays independently of the presence of the frequency detuning γ. During the frequency detuning ($\gamma \neq 0$), the state of complete synchronization becomes impossible, as follows from Eq. (23), which admits the solution $|\varkappa_1| = 1$ only at $\gamma = 0$. Averaging Eq. (23) over time, it can be seen that for a small γ the mode with $|\varkappa_1| = 1$ changes to some new mode, for which $\overline{1-|\varkappa_1|^2} \approx \gamma$. Thus, the stability of the complete synchronization state successor is high even for a small γ, since

$$\mathrm{Re}\overline{\frac{\mathrm{d}}{\mathrm{d}t}\ln\varkappa_2} \approx -\frac{\gamma}{1-|\varkappa_1|^2} \approx -1.$$

5.2. Dynamics of two-bunch chimeras

We now describe in detail the dynamics of two-bunch chimeras in the considered system (19)–(21). It has already been noted that for the first ensemble, the second cumulant decays either because of the complete synchronization or the γ effect. The decay rate is of the order of unity even for $\gamma \to 0$; therefore, it stands to reason to limit oneself to the case where only the first argument is not equal to zero. Since Eqs. (19)–(21) have symmetry about the complex amplitude rotations $\kappa \to \kappa\exp(i\beta)$ and $\varkappa_n \to \varkappa_n\exp(in\beta)$, the nontrivial part of the dynamics is due not to the full phases of the cumulants, but their differences. We introduce for ensemble 1 the variables $\rho \equiv |\kappa|$ and $\xi \equiv \kappa/|\kappa|$: $\kappa = \rho\xi$. The dynamics of ensemble 2 will be described in the reference frame co-rotated with κ, in terms of the variables $\chi_1 = \varkappa_1\xi^*$ and $\chi_2 = \varkappa_2(\xi^*)^2$. Then from system (19)–(21) it can be found that

$$\dot\rho = -\gamma\rho + \frac{1-\rho^2}{2}\left\{\mu\rho\cos\alpha + \nu\mathrm{Re}[\chi_1\exp(-i\alpha)]\right\}; \tag{25}$$

$$\dot\xi = i\Big(\Omega - \frac{1+\rho^2}{2\rho}\left\{\mu\rho\sin\alpha + \nu\mathrm{Im}[\chi_1\exp(-i\alpha)\right\}\Big)\xi; \tag{26}$$

$$\dot{\chi}_1 = -\gamma\chi_1 + \frac{\mu}{2}\Big[\chi_1\exp(-i\alpha) - \chi_1^*\exp(i\alpha)(\chi_1^2+\chi_2) + i(1+\rho^2)\chi_1\sin\alpha\Big] + \frac{\nu}{2}\Big\{\rho\exp(-i\alpha) - \rho\exp(i\alpha)(\chi_1^2+\chi_2) + i\frac{1+\rho^2}{\rho}\chi_1\mathrm{Im}[\chi_1\exp(-i\alpha)]\Big\}; \quad (27)$$

$$\dot{\chi}_2 = 2\Big(-\gamma + \mu\Big[-|\chi_1|^2\exp(i\alpha) + i\frac{1+\rho^2}{2}\sin\alpha\Big] + \nu\Big\{-\rho\chi_1\exp(i\alpha) + i\frac{1+\rho^2}{2\rho}\mathrm{Im}[\chi_1\exp(-i\alpha)]\Big\}\Big)\chi_2. \quad (28)$$

For $\gamma = 0$, the first ensemble can be completely synchronized ($\rho = 1$), and then Eqs. (27) and (28) take the form

$$\dot{\chi}_1 = \mu\left[\frac{\chi_1\exp(-i\alpha) - \chi_1|\chi_1|^2\exp(i\alpha) - \chi_1^*\chi_2\exp(i\alpha)}{2} + i\chi_1\sin\alpha\right] + \nu\left\{\frac{\exp(-i\alpha) - \exp(i\alpha)(\chi_1^2+\chi_2)}{2} + i\chi_1\mathrm{Im}[\chi_1\exp(-i\alpha)]\right\}, \quad (29)$$

$$\dot{\chi}_2 = 2\Big(\mu\big[-|\chi_1|^2\exp(i\alpha) + i\sin\alpha\big] + \nu\big\{-\chi_1\exp(i\alpha) + i\mathrm{Im}[\chi_1\exp(-i\alpha)]\big\}\Big)\chi_2. \quad (30)$$

The system of equations (29) and (30) specifies an autonomous dynamic system in four-dimensional phase space. It was shown that at $\gamma = 0$ small values of the cumulant χ_2 neither decay nor rise with time on the average (within the linear theory). Thus, with the same parameter values where the one-bunch dynamics demonstrated tending towards a stable limit cycle on the plane χ_1, attraction to a family of tori parameterized by a characteristic amplitude of oscillations χ_2 will be observed for two-bunch states (qualitatively, $|\chi_2|$ determines the “thickness” of the torus, see Fig. 1*b*).

In accordance with the obtained analytical results, numerical calculus shows quasiperiodic dynamics on the $(\varkappa_1/\kappa)$ plane at $\gamma = 0$ (Fig. 1*b*) and attraction of the trajectories to the limit cycle corresponding to one-bunch states at $\gamma \neq 0$ (Fig. 1*c*).

Finally, Eqs. (20) and (21) provide an explicit analytical description of the system dynamics beyond the Ott–Antonsen manifold and give the analytical proof that in the important paradigmatic problem [7] this manifold is neutrally stable (i. e., becomes attractive only with the introduction of the frequency detuning γ). Numerical calculations within the framework of a cumulant representation permit one to avoid the problems with slow divergence of the series a_n for $|a_1| \to 1$: numerical integration with several first cumulants ensures the same level of accuracy as a similar integration of Eq. (2) with hundreds of harmonics a_n.

6. CONCLUSIONS

The paper shows the possibility and prospects for using the formalism of circular cumulants for an alternative description of the dynamics of phase element ensembles at the thermodynamic limit (i. e., for a large ensemble size). For systems of the Ott–Antonsen type, this formalism allows one to obtain exact equations for the dynamics of order parameters. A more general particular solution than the Ott–Antonsen solution was found; a new solution permits one to describe the dynamics of two-bunch states of the ensemble in the finite vicinity of the Ott–Antonsen solution, which is one-bunch. In the Abrams problem [7, 13], this solution makes it possible to observe the conservative dynamics of multi-bunch states for ensembles of identical elements and describes the transition of multi-bunch to one-bunch states, which become attractive in the case of imperfect identity.

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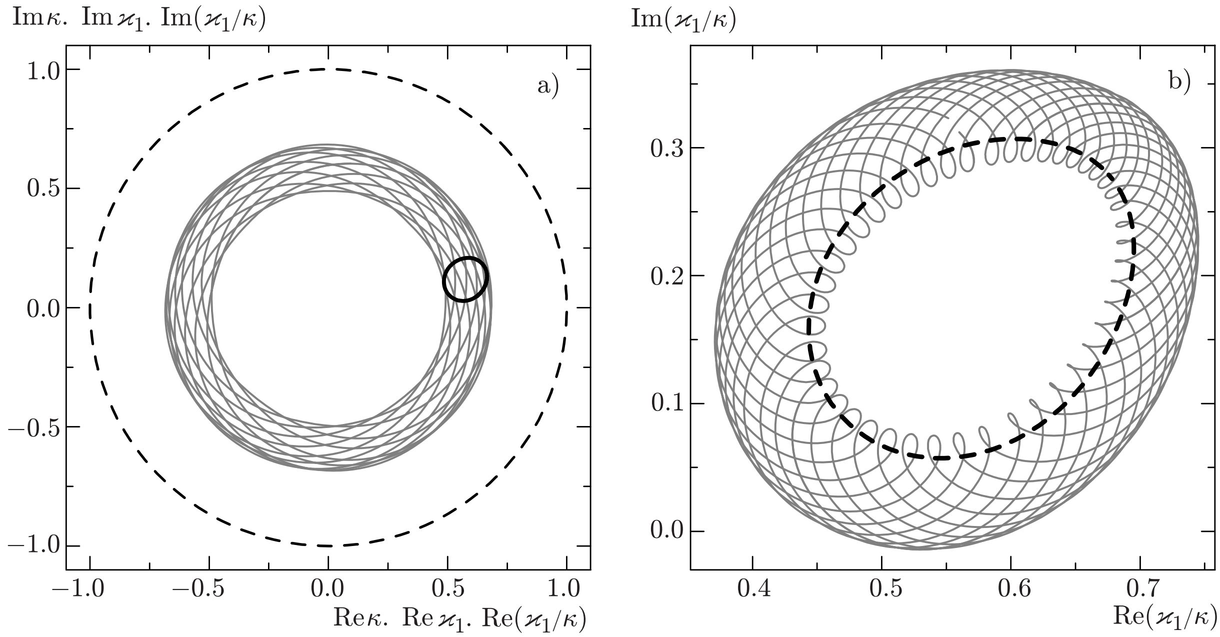

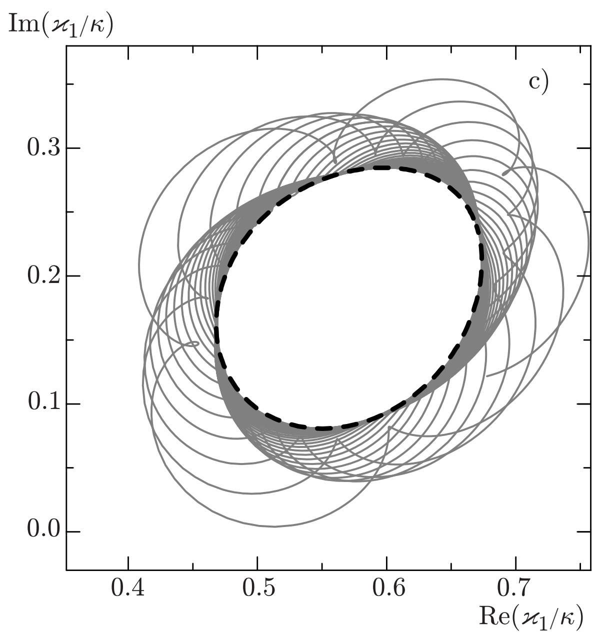

Fig. 1. Dynamics of the complex order parameters in coupled Kuramoto–Sakaguchi ensembles (18). Panel *a*: the state of ensemble 2 is one-bunch, $\gamma = 0$ [7], $\mu = 0.64$, and $\alpha = \pi/2 - 0.1$; dynamics of the order parameter κ for synchronized ensemble 1 (dashed line), dynamics of the order parameter $\varkappa_1$ for ensemble 2 (gray thin line) and the order parameter of ensemble 2 on the $(\varkappa_1/\kappa)$ plane (black line). Panel *b*: dynamics of the two-bunch state at $\gamma = 0$ (gray line) as compared with the one-bunch solution (dashed line, which corresponds to the black line on panel *a*). Panel *c*: the same as panel *b*, but for $\gamma = 5 \cdot 10^{-4}$ (the gray trajectory tends to the cycle). For panels *b* and *c*, $\mu = 0.65$ and $\alpha = \pi/2 - 0.15$; initial conditions for the second ensemble are given by expressions (17) and $\varkappa_1^{\{1,2\}} = (0.8 \pm 0.1) \exp[(0.30 \pm 0.15)\, i]$.

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