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Phase demodulation with iterative Hilbert transform embeddings

Erik Gengel^a, Arkady Pikovsky^{a,b}

^aInstitute for Physics and Astronomy, University of Potsdam, Karl-Liebknecht Str. 24/25, 14476 Potsdam, Germany ^bDepartment of Control Theory, Institute of Information Technologies, Mathematics and Mechanics, Lobachevsky University Nizhny Novgorod, Russia

Abstract

We propose an efficient method for demodulation of phase modulated signals via iterated Hilbert transform embeddings. We show that while a usual approach based on one application of the Hilbert transform provides only an approximation to a proper phase, with iterations the accuracy is essentially improved, up to precision limited mainly by discretization effects. We demonstrate that the method is applicable to arbitrarily complex waveforms, and to modulations fast compared to the basic frequency. Furthermore, we develop a perturbative theory applicable to a simple cosine waveform, showing convergence of the technique. *Keywords:* Phase modulation, Hilbert transform, Embedding

1. Introduction

Demodulation of nearly oscillatory signals is a standard problem in signal processing. In classical communication schemes, one uses amplitude, phase, and frequency modulations to store information in an analog signal. At the receiving side, demodulation is needed to extract this information. In a more general context, when one tries to characterize the dynamics of an oscillatory system based on some observable, modulation carries important information about the system operation. For example, the electrocardiogram is an important signal characterizing the human cardio-vascular system. One of the widely used tools is demodulation of this signal via extracting information about the instantaneous periods (RR intervals), with further analysis of demodulated heart rate variability [1]. In many situations one considers a demodulation task for harmonic carrying waveforms only (see, e.g., [2]), but in fact one quite often needs to demodulate signals with unknown, potentially rather complex waveform (like the electrocardiogram signal).

In this paper we focus on the problem of demodulation of phase modulated signals with arbitrary waveforms and arbitrary phase modulations. This task is essential for characterization of oscillatory processes resulting from forced and/or coupled self-sustained oscillators, e.g. for synchronization analysis [3–5] and for reconstruction of the phase dynamics [6–8]. In particular, in Refs. [9, 10] a continuous phase was ex-

Email addresses: egengel@uni-potsdam.de (Erik Gengel), pikovsky@uni-potsdam.de (Arkady Pikovsky)

tracted from the electrocardiogram signal, and further used to build a phase model for the cardio-respiratory interaction.

The Hilbert transform (HT) is a widely used tool in oscillatory data analysis and signal processing. It lies at the heart of the analytic signal approach [11–15], which provides an effective processing of nearly harmonic signals. The Hilbert transform is also extensively used for the phase demodulation in synchronization studies [3, 16], see also recent review [17]. One uses it to perform a two-dimensional embedding of the observed signal; this embedding then is usually treated as an analytic signal and the phase is extracted as the argument of the complex state (or, more generally, as an angle of rotation with respect to some point on the embedding plane). This approach gives a good approximation, but not the exact phase demodulation. The quality of the analytic signal based phase demodulation is better for slow modulation, although also in this situation, as will be illustrated below, it is not exact.

One can attribute non-exactness of the HT embedding to the fact, that the HT generally mixes phase and amplitude modulations: if one starts with a purely phase modulated signal, then the HT embedded signal will have amplitude modulation, and if an original signal is purely amplitude modulated, after HT embedding a phase modulation appears (we illustrate this in Appendix A). In signal processing, workarounds exist [14, 18] that introduce filtering before application of the HT and/or decomposition of the signal into several functions to avoid the mixing.

In this paper we propose a phase demodulation method based on iterative HT embeddings. The first step is the same as in the traditional approach, it yields an approximation, but the signal is not fully demodulated. We use the phase obtained in the first step as a new time variable to perform the next embedding, with better demodulation. The repetition of this procedure provides better and better approximations. We will demonstrate that the iterations result in a very accurate demodulation with errors related to numerical errors in calculating the Hilbert transform integral. Noteworthy, our method is not restricted to simple waveforms like cosine function; we will present its implementation for arbitrary (but continuous) waveforms. Thus, no filtering is involved in the approach we describe.

For completeness, we mention another class of approaches to phase demodulation, where not the phase itself is determined, but the instantaneous frequency is evaluated via a time-frequency decomposition (see, e.g., [19–21]). Recently, a variant of this approach – wavelet-based synchrosqueezing – have been suggested [15, 22, 23]. Comparison with these methods is a subject of future research.

The paper is organized as follows. First, in section 2, we elaborate on our definition of the waveform and the phase for a signal, and discuss connections to dynamical systems analysis. Then we present our method in sections 3,4. Analytic results on the convergence of the method, based on a perturbation analysis, are described in section 5. Numerical results, confirming accuracy of the technique for a range of strongly modulated signals and for complex waveforms, are presented in section 6. In section 7, a transformation from a phase variable, obtained in the demodulation procedure, to another phase is discussed. We end with conclusions in section 8.

2. Phase modulated signals

2.1. Definition of a phase modulated signal

The subject of our study are phase-modulated signals. First we define a waveform $S(\varphi) = S(\varphi + 2\pi)$ as a 2π -periodic function of its argument. The argument, interpreted as a *phase*, is a monotonous function of time $d\varphi(t)/dt > 0$. If $d\varphi(t)/dt \neq \text{const}$, one speaks of a *phase modulated signal*

$$x(t) = S(\varphi(t)) . \tag{1}$$

Our task is, given the signal x(t), to reconstruct the phase $\varphi(t)$ and the waveform $S(\varphi)$.

Our method below is not restricted to simple waveforms: $S(\varphi)$ can be a rather complex function with several maxima and minima. Below in the numerical examples and in theoretical considerations we will use

$$S_1(\varphi) = \cos(\varphi) \tag{2}$$

as a simple waveform, and

$$S_2(\varphi) = \cos(\varphi) - 0.7\sin(2\varphi) + \cos(3\varphi) \tag{3}$$

as a "complex" waveform. In fact, the latter waveform is not extremely complex, it is used mainly to introduce novel methods of phase definition, compared to the simple waveform. To demonstrate robustnes of the approach, we explore highly complex waveforms in Sect. 6.4 below.

2.2. Phase modulation in driven oscillators

The motivation for this setup comes from the inverse problems for nonlinear dynamical systems. Phase modulated signals naturally appear if a dynamical system with stable self-sustained oscillations (i.e. with a limit cycle) is driven by a weak external force (or interacts with another dynamical system) [16, 24–26], as we briefly outline below.

Suppose we have an autonomous dynamical system

$$\frac{d}{dt}\vec{y} = \vec{F}(\vec{y})$$

with a stable limit cycle $\vec{y}_0(t)$. This means that the system generates periodic oscillations and a generic scalar observable $x(\vec{y})$ is a periodic function of time $x(\vec{y}_0(t))$. On the limit cycle and in its basin of attraction one can introduce the phase variable $\varphi(t) = \Phi(\vec{y}(t))$, such that the limit cycle is parametrized by this phase $\vec{y}_0(\varphi + 2\pi) = \vec{y}_0(\varphi)$, and the phase grows linearly in time with the natural frequency of the limit cycle $d\varphi(t)/dt = \omega$.

If there is a weak external force acting on the system

$$\frac{d}{dt}\vec{y} = \vec{F}(\vec{y}) + \varepsilon \vec{P}(\vec{y}, t),$$

then, in the first approximation in ε , a phase-modulated dynamics on the limit cycle represents the full oscillation of $\vec{y}(t)$ close to $\vec{y}_0(\varphi(t))$, where the phase obeys the equation

$$\frac{d}{dt}\varphi(t) = \omega + \varepsilon Q(\varphi(t), t) \tag{4}$$

and $Q = \operatorname{grad} \Phi|_{\vec{y}_0(\varphi(t))} \cdot \vec{P}(\vec{y}_0(\varphi(t)), t)$ is determined by the phase response curve of the limit cycle and the force $\vec{P}(t)$ [27]. Accordingly, the scalar observable x(t) is the phase-modulated signal as described above:

$$x(t) = x(\vec{y}_0(\varphi(t))) \; .$$

Here the waveform is determined by the form of the limit cycle and of the scalar observable $S(\varphi) = x(\vec{y}_0(\varphi))$. The inverse problem in the context of the theory above is formulated as follows: From scalar observations x(t) of a driven dynamical system, one seeks to reconstruct the dynamics of the phase $\varphi(t)$. The first step here is the reconstruction of the phase $\varphi(t)$, what is the topic of the present study.

2.3. Phase vs protophase

In this section we discuss whether the decomposition of a signal (1) into the phase and the waveform is unique. The answer is obvious: No. Indeed, one can perform a transformation

$$\psi = \Psi(\varphi), \qquad \Psi(s + 2\pi) = \Psi(s) + 2\pi, \qquad \Psi' > 0, \tag{5}$$

to a new phase variable ψ , in terms of which the signal is represented as

$$x(t) = \tilde{S}(\psi(t)), \qquad \tilde{S} = S(\Psi^{-1}(\psi(t)))$$

with the new 2π -periodic waveform $\tilde{S}(\psi) = \tilde{S}(\psi + 2\pi)$. Generally, there is no particular reason to prefer the representation in terms of the waveform S and the phase $\varphi(t)$ to the representation in terms of the waveform \tilde{S} and of the phase $\psi(t)$.

The ambiguity above can be reformulated as follows: It is not possible to demodulate components in the phase modulation, corresponding to high harmonics of the basic frequency. Indeed, consider a waveform $S(\varphi) = S(\varphi + 2\pi)$, and a modulation $\varphi(t)$ containing just a k-th harmonic of the basic frequency: $\varphi(t) = t + a \sin(kt) + b \cos(kt)$ (here k is an arbitrary integer). One can simply proof that $S(t) = S(t + 2\pi)$. Thus, one can cannot distinguish such a modulation from a variation of the waveform, and therfore demodulation of high basic harmonics is not possible.

In applications, "preferable" phase variables can exist. For example, for a dynamical system where the equation for the phase in form (4) is used, the "true" phase has the property of a uniform (on average) rotation, so that the distribution density of this phase does not depend on the phase itself. See more discussion in Refs. [6], where general possible phase variables have been named "protophases", and a transformation of form (5) from a protophase to the phase having uniform distribution density has been suggested as a step

in the phase reconstruction. However, in other applications other conditions on the phase may be imposed. On the other hand, there can be a "preferable" waveform. For example, a waveform with one maximum and one minimum over the period, can be transformed to a cosine waveform. The ambiguity of the phase definition results also in the ambiguity of the instantaneous frequency, if the latter is defined as the time derivative of the phase.

Summarizing, we state that the decomposition of a signal (1) is not unique, and in different methods, described below, different protophases and correspondingly different waveforms will be obtained. We will, however, consider demodulation as successful, if at least one particular protophase and the corresponding waveform can be found. The success of demodulation means that a protophase $\psi(t)$ is found, such that values of $|x(\psi) - x(\psi + 2\pi)|$ are very small. Quantitatively, we define the squared integrated error of demodulation in Eq. (9) below.

2.4. Link function approach

There is another formulation of the demodulation problem, based on the *link function* approach [M. Holschneider, private communication]. Given a modulated signal (1), one looks for a function of time L(t), called link function, such that

$$x(t+L(t)) = x(t) \; .$$

If a protophase $\psi(t)$ of the signal is known, then one can define the link function as $\psi(t + L(t)) = \psi(t) + 2\pi$, what can be resolved via the inverse function $t(\psi)$ as $L(t) = t(\psi(t) + 2\pi) - t$. One can easily see that the link function is independent on the choice of the protophase, and in this sense is invariant. For the dynamical systems described in section 2.2, the link function gives the return time for the Poincaré map from the value of the phase to the next value shifted by 2π . The task of finding a protophase if the link function is known, is, to the best of our knowledge, not solved.

3. Proxi-phase extraction

Our main method is based on an iteration procedure. At each iteration stage, an approximation to the phase is extracted from the phase-modulated signal x(t); we call the result of this extraction "proxi-phase" below. In fact, this extraction step is performed only once in most existing approaches to extract the phase dynamics from data (for example, see [6, 10, 28, 29]).

The usual approach is to perform a two-dimensional embedding $x(t) \rightarrow [x(t), y(t)]$, so that a phasemodulated signal is represented by "rotations" or "loops" on the plane [x(t), y(t)]. In the field of dynamical system reconstruction, different choices for y(t) have been discussed, two popular are a delayed signal $y(t) = x(t - \tau)$ and the derivative y(t) = dx(t)/dt [30–32]. However, the mostly stable and reliable results are achieved with the Hilbert transform [3]:

$$y(t) = \hat{H}[x(t)] = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{x(t')}{t - t'} dt' .$$
(6)

The resulting trajectory on the plane [x(t), y(t)] we call HT-embedding.

Below we will consider two basic ways to extract a proxi-phase from the HT embedding. In the simplest case, where the waveform is close to a $\cos(\varphi)$ function, the HT embedding is close to a circle, surrounding the origin (see Fig. 1(a) below), and one can define the proxi-phase $\theta^{(a)}(t)$ as the angle from the origin (or another point close to the origin) to the position of the signal:

$$\theta^{(a)}(t) = \arg[x(t) + iy(t)] . \tag{7}$$

This definition corresponds to the proxi-phase as the argument of the analytic signal z(t) = x(t) + iy(t). Therefore we will call $\theta^{(a)}(t)$ the *analytic proxi-phase*. The proxi-phase $\theta^{(a)}(t)$ should be additionally unwrapped, so that it is a monotonously growing function of time.

This simple definition of the proxi-phase will not work for a complex waveform with several maxima and minima on the period. In this case the embedding on the plane [x(t), y(t)] has many loops (see Fig. 1(e) below), and the evaluation of the proxi-phase according to (7) provides a non-monotonous growth of the analytic proxi-phase, what contradicts to the physical notion of the phase. A complex waveform is a sufficient condition for the analytic proxi-phase to be non-monotonous, but not a necessary one. Even for a simple waveform $S(\varphi) = \cos(\varphi)$, if the phase modulation is strong, nonmonotonicity of the proxi-phase (7) may occur. We, however, do not know a necessary and sufficient condition for this. Altogether, this makes application of the analytic proxi-phase impossible for such cases.

A more universal proxi-phase definition was suggested in Ref. [6] and adopted for the analysis of an electrocardiogram signal in Ref. [9]. This proxi-phase is based on the curve length, to avoid the mentioned problems. The length-based definition of a proxi-phase $\theta^{(b)}(t)$ is

$$\theta^{(b)}(t) = \int_0^t d\tau \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} \,. \tag{8}$$

This expression is written under assumption that the curve is a continuously differentiable function. This is not necessary, in general a Lipschitz-continuity suffice (cf. Lemma 5.4 in [33]), and we will consider a corresponding example in Section 6.4 below. In any case this definition provides a monotonously growing *length proxi-phase*, even if the embedded trajectory has loops.

A small drawback of this definition is that the proxi-phase $\theta^{(b)}(t)$ is not normalized to intervals of 2π at each rotation, contrary to the analytic proxi-phase (7). This does not affect further iteration steps to be described below, because the HT is invariant under rescaling of the function argument. At the very end of the procedure, one can perform the normalization by multiplying $\theta^{(b)}$ with a factor $2\pi \hat{L}^{-1}$, if needed. Here \hat{L} is the average period of the signal in terms of the length (8) after the iteration procedure. To define it, we first define the normalized periodicity error of a function $x(t(\theta))$, with respect to a period guess L, as

$$\tilde{\varepsilon}(L) = \frac{\int_{\theta_{min}}^{\theta_{max}} (x(t(\theta'+L)) - x(t(\theta')))^2 d\theta'}{\int_{\theta_{min}}^{\theta_{max}} x^2(t(\theta')) d\theta'} .$$
(9)

In this expression, to avoid boundary effects, we integrate not over the whole domain corresponding to the time interval 0 < t < T, but drop 10% of it on each side: $\theta_{min} = \theta(0.1T)$, $\theta_{max} = \theta(0.9T)$.

The actual value of $\tilde{\varepsilon}(L)$ only weakly depends on the proto-phase definition. If it tends to zero for one protophase, then it tends to zero for other protophases as well.

In case of the analytic proxi-phase, we can set $L = 2\pi$, then expression (9) provides the quality of the phase reconstruction, without further adjustments. In contradistinction, for the length proxi-phase, the value of L is not known a priori. Here, we obtain an "optimal" period by minimizing the error (9) with respect to L:

$$\hat{L} = \operatorname{argmin} \tilde{\varepsilon}(L) , \qquad \varepsilon = \min_{L} \tilde{\varepsilon}(L) = \tilde{\varepsilon}(\hat{L}) .$$

This error will be used in figures 5 and 6 below to quantify the quality of demodulation.

Note that the periodicity error is a measure of cumulative energy. If the integration domain contains parts of large deviation from the actual phase, the approximate waveform is also not 2π -periodic such that the integral is not vanishing. However, if it converges to zero, the energy of all harmonics in the artificial modulation of proxi-phases has vanished as well.

The proxi-phase extraction can be viewed as a (nonlinear) operator \hat{P} , providing for a given signal x(t) a proxi-phase

$$\theta(t) = \hat{P}[x(t)]$$

with one of the methods above.

4. Iterated Hilbert transform embeddings

Application of the proxi-phase extraction by one of the methods described in section 3 above shows that this proxi-phase is not the phase, because in the function $x(t(\theta))$ some phase modulation is still present such that it is not exactly periodic.

We suggest to proceed in an iterative manner: We consider the obtained proxi-phase (which we supply with an index that counts the steps in the iteration process) $\theta_1(\theta_0(t))$ as a new time variable, and apply the proxi-phase extraction to the signal $x(t(\theta_1))$. Namely, we calculate $\theta_2(\theta_1) = \hat{P}[x(\theta_1)]$, and so on. In other words, we perform iterations

$$\theta_{n+1}(\theta_n) = \hat{P}[x(t(\theta_n))] \tag{10}$$

The original time can be treated as the zero-index proxi-phase $\theta_0 = t$. To illustrate, we write explicitly the steps for the analytic signal proxi-phase $\theta^{(a)}$:

$$y(t(\theta_n)) = \hat{H}[x(\theta_n)] = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{x(t(\theta'_n))}{\theta_n - \theta'_n} d\theta'_n,$$

$$\theta_{n+1} = \arg \left[x(\theta_n) + iy(\theta_n) \right].$$
(11)



Figure 1: Iterated HT embeddings for the phase-modulated signals $x_{1,2}(t) = S_{1,2}(\varphi(t))$ (see expressions (2,3)) with modulation $\varphi(t) = t + 1.2(\sin(0.25\sqrt{2}t) + \cos(0.25\sqrt{3}t))$. Panels (a)-(d) show the simple waveform case, here the analytic proxi-phase was adopted. Panels (e)-(h) show the complex waveform, here the length proxi-phase was used. The iteration steps: (a),(e): n = 0; (b),(f): n = 1; (c),(g): n = 2, and (d),(h): n = 10.

An important remark regarding the expressions (11) is as follows. Because the proxi-phases θ_n at different iterations are different, the functions $t(\theta_n)$ are different two. To avoid unnecessary complex notations, we stress this fact by writing explicitly the index n at the argument of $t(\theta_n)$. Thus, also $x(t(\theta_n))$ are different functions of θ_n for different n. This means that we cannot drop index n in the integration variable θ'_n in (11).

In numerical implementation of the iterations, one has to perform the HT for a function defined on a non-uniform grid; see Appendix B for the implementation we used.

Our numerical simulations show that this iterative procedure effectively reduces the errors (9), and, with high accuracy, $\theta_n(t)$ converges to a correct protophase for large n. Just visually one can check this by comparison of embedded plots $[x(\theta_n), y(\theta_n)]$ for different iteration steps n, see Fig. 1. If the signal $x(\theta_n)$ is a periodic function of θ_n , then $y(\theta_n)$ is also a periodic function, and on the embedding plane one observes just a closed curve. Contrary to this, if θ_n is just an approximation and $x(\theta_n)$ is not periodic but contains rest modulation, then one observes a non-closed trajectory forming a sequence of non-equal loops. Another illustration of the iterations is presented in Appendix C, where we demonstrate the original modulated signal x(t) together with extracted phases θ_n for n = 1, 2, 3, 10.

In Fig. 1 one can see that just one step in the phase extraction does not provide a good phase – the embeddings in panels (b),(f) look like quite wide bands. In course of iterations, these bands become narrower, and already at the 10-th step one observes very narrow lines (panels (d),(h)).

As has been mentioned above, a reconstruction of a phase means also a reconstruction of the waveform.



Figure 2: Depicted are the reconstructed waveforms (black dots: $x(\theta_n^{(b)}) = x(t(\theta_n^{(b)}))$) and the actual wave form $S_2(\varphi)$ (red lines) for the same modulation as in Fig. 1(e-h). We show here iterations: (a): n = 1, (b): n = 2, (c): n = 3, (d): n = 100. Note that $\theta^{(b)}$ is normalized by the mean periodicity L_n .

Having found $\varphi(t)$, we obtain the waveform $S(\varphi)$ by plotting $x(t) = S(\varphi(t))$ vs. $\varphi(t)$. We illustrate this in Fig. 2. One can see that for small *n* the points of the plot *x* vs θ_n are scattered, what means that the proper waveform is not reconstructed. For large *n*, when the procedure converges, the points lie on a curve which reproduces the waveform (up to proto-phase ambiguity described in Section 2.3).

Below we first present a theory, showing that the proposed iteration method indeed converges for a cosine waveform and a weak modulation. Then, in section 6, we will first check predictions of the theory; and then explore convergence of the method for strong modulations, by virtue of the error analysis according to expression (9).

5. Theory

Our strategy here is to show that at each iteration (10) the residue between the true phase and the calculated phase decreases. We represent the modulation of the true phase as

$$\varphi(t) = t + \epsilon q(t), \qquad \epsilon \ll 1 ,$$

and restrict our analysis to terms of first order in ϵ . We have fixed the basic frequency of the signal to one, without loss of generality. Thus, the signal and its HT are represented as

$$x(t) = S(\varphi(t)) \approx S(t) + \epsilon S'(t)q(t) ,$$

$$\hat{H}[x](t) \approx \hat{H}[S(t)] + \epsilon \hat{H}[S'(t)q(t)] ,$$
(12)

where prime denotes derivative with respect to the argument (which in the first iteration step is time, and in other steps is the previous proxi-phase $\theta_n(t)$). We calculate both proxi-phases in the first order in ϵ based on (7) and (8) respectively, and obtain general expressions, valid for arbitrary waveforms S(t) and modulations q(t):

$$\theta^{(a)}(t) = \arg \left(S(t) + i\hat{H}[S(t)]\right) + \epsilon \frac{\hat{H}[S'(t)q(t)]S(t) - S'(t)q(t)\hat{H}[S(t)]}{(S(t))^2 + H[S(t)]^2}$$

$$\theta^{(b)}(t) = \int_0^t \sqrt{(S'(\tau))^2 + \hat{H}[S'(\tau)]^2} d\tau$$

$$+ \epsilon \int_0^t \frac{S'(\tau)(q'(\tau)S'(\tau) + S''(\tau)q(\tau)) + \hat{H}[S'(\tau)](\hat{H}[q'(\tau)S'(\tau)] + \hat{H}[q(\tau)S''(\tau)])}{\sqrt{(S'(\tau))^2 + \hat{H}[S'(\tau)]^2}} d\tau$$
(13)

We are able to give compact tractable expressions for the simplest waveform $S(\varphi) = \cos(\varphi)$ only. This case is treated below.

The difference $\Delta^{(a,b)}(t) = \varphi(t) - \theta^{(a,b)}(t)$ between the true phase and the reconstructed one should be interpreted as the rest modulation after the first iteration. For the analytic proxi-phase (7) and the length based proxi-phase (8), in the case $S(\varphi) = \cos(\varphi)$, the expressions for the rest modulation, obtained through simple calculations from (13), read

$$\Delta^{(a)}(t) = \epsilon \left(q(t) + \cos(t)\hat{H}[q(t)\sin(t)] - q(t)\sin^2(t) \right) , \qquad (14)$$

$$\Delta^{\prime(b)}(t) = \frac{\epsilon}{2}q'(t) + \frac{\epsilon}{2} \left(\cos(2t)q'(t) - \sin(2t)q(t) + 2\cos(t)\hat{H}[q'(t)\sin(t)] + 2\cos(t)\hat{H}[q(t)\cos(t)] \right) . (15)$$

Here, to get rid of the integral in the expression for $\Delta^{(b)}$, we take derivative with respect to the argument.

We now use the Bedrosian's theorem [34], which states that the HT of a product of a low-frequency $Q_L(t)$ and a high-frequency $Q_H(t)$ signals and can be represented as

$$\hat{H}[Q_L(t)Q_H(t)] = Q_L(t)\hat{H}[Q_H(t)] .$$

The condition for the Bedrosian's theorem is that the spectra of Q_L and Q_H do not overlap: the whole spectrum of Q_L lies left to the spectrum of Q_H .

Thus, to evaluate the terms $\hat{H}[q(t)\sin(t)]$, $\hat{H}[q(t)\cos(t)]$, and $\hat{H}[q'(t)\sin(t)]$, we represent q(t) as a sum of two components $q(t) = \ell(t) + h(t)$, where the Fourier spectrum of the slow component $\ell(t)$ contains frequencies $|\omega| < 1$, while the Fourier spectrum of the fast component h(t) contains frequencies that obey $|\omega| > 1$. Then, we obtain

$$\hat{H}[q(t)\sin t] = \hat{H}[(\ell(t) + h(t))\sin(t)] = -\ell(t)\cos(t) + \hat{H}[h(t)]\sin(t) ,$$

and similar for $\hat{H}[q(t)\cos(t)]$ and $\hat{H}[q'(t)\sin(t)]$. Substituting this in (14),(15), we see that the component $\ell(t)$ does not contribute to the rest modulation. Only the contribution of the high-frequency component remains:

$$\Delta^{(a)}(t) = \frac{\epsilon}{2}h(t) + \frac{\epsilon}{2}\left(h(t)\cos(2t) + \sin(2t)\hat{H}[h(t)]\right) , \qquad (16)$$

$$\Delta^{\prime(b)}(t) = \frac{\epsilon}{2}h^{\prime}(t) + \frac{\epsilon}{2}\left(\hat{H}[h(t)] + \cos(2t)h^{\prime}(t) - \sin(2t)h(t) + \sin(2t)\hat{H}[h^{\prime}(t)] + \cos(2t)\hat{H}[h(t)]\right) .$$
(17)

At this stage, it is more convenient to operate with the Fourier spectra $\mathcal{F}_h(\omega), \mathcal{F}_{\Delta}(\omega)$ of functions $\epsilon h(t), \Delta(t)$. The Hibert transform in the Fourier space has the form $\mathcal{F}(\hat{H}[h(t)])(\omega) = -i \operatorname{sgn}(\omega) \mathcal{F}_h(\omega)$. Application of the Fourier transform to expressions (16) and (17) yields:

$$\mathcal{F}_{\Delta^{(\alpha)}}(\omega) = \frac{1}{2} \left[\mathcal{F}_{h}(\omega) + \frac{1}{2} (1 - \operatorname{sgn}(\omega - 2)) \mathcal{F}_{h}(\omega - 2) + \frac{1}{2} (1 + \operatorname{sgn}(\omega + 2)) \mathcal{F}_{h}(\omega + 2) \right] ,$$

$$\mathcal{F}_{\Delta^{(b)}}(\omega) = \frac{1}{2} \left[\frac{\omega - \operatorname{sgn}(\omega)}{\omega} \mathcal{F}_{h}(\omega) + \left(\frac{\omega - 1}{2\omega} (1 - \operatorname{sgn}(\omega - 2)) \mathcal{F}_{h}(\omega - 2) + \frac{\omega + 1}{2\omega} (1 + \operatorname{sgn}(\omega + 2)) \mathcal{F}_{h}(\omega + 2) \right) \right] .$$
(18)

We now use that $\mathcal{F}_h(\omega) = 0$ for $|\omega| < 1$, what allows us to simplify expressions (18) to

$$\mathcal{F}_{\Delta}(\omega) = \begin{cases} A(\omega)\mathcal{F}_{h}(\omega) + B(\omega)\mathcal{F}_{h}(\omega+2) & \text{if } \omega > 1 , \\ B(\omega)\mathcal{F}_{h}(\omega+2) + A(\omega)\mathcal{F}_{h}(\omega-2) & \text{if } -1 < \omega < 1 , \\ B(\omega)\mathcal{F}_{h}(\omega) + A(\omega)\mathcal{F}_{h}(\omega-2) & \text{if } \omega < -1 , \end{cases}$$
(19)

where

$$A(\omega) = B(\omega) = 1/2 \text{ for the analytic proxi-phase and } \Delta^{(a)},$$

$$A(\omega) = \frac{\omega - 1}{2\omega}, \quad B(\omega) = \frac{\omega + 1}{2\omega} \text{ for the length proxi-phase and } \Delta^{(b)}.$$
(20)

Relation (19) can be considered as a transformation of the rest phase modulation at a step of our iteration procedure. Denoting the spectrum of the high-frequency modulation $\epsilon h(t)$ at the *n*-th step of the iteration procedure as \mathcal{F}_n , we can write a general recursion formula for the evolution of this spectrum under iterations:

$$\mathcal{F}_{n+1}(\omega) = \begin{cases} A(\omega)\mathcal{F}_n(\omega) + B(\omega)\mathcal{F}_n(\omega+2) & \text{if } \omega > 1, \\ B(\omega)\mathcal{F}_n(\omega+2) + A(\omega)\mathcal{F}_n(\omega-2) & \text{if } -1 < \omega < 1, \\ B(\omega)\mathcal{F}_n(\omega) + A(\omega)\mathcal{F}_n(\omega-2) & \text{if } \omega < -1. \end{cases}$$
(21)

This relation is the main result of our weak modulation analysis. We now discuss its meaning for the iteration procedure of demodulation (where we focus, due to the evident symmetry, on positive frequencies only).

1. The first observation is that in both cases of proxi-phase definition, slow modulation (i.e. with frequencies lower than the basic oscillation frequency) is resolved exactly already in the first iteration. Indeed, as it follows from (21), if the spectrum \mathcal{F}_0 lies in the interval $-1 < \omega < 1$, then $\mathcal{F}_1 = 0$. This is a well-known empirical fact that the traditional HT phase extraction (where only the first iteration is used) works well for slow modulation, but is not so good for fast modulation. This efficiency of the HT embedding is valid, according to our linear theory, for both proxy-phase definitions, the one based on the analytic signal, and the one based on the length definition. We stress here, that this result is derived for very weak modulation only. We will consider below cases of relatively strong low-frequency modulation, where one indeed needs several iterations to achieve a good demodulation. Contrary to low-frequency components, the high-frequency components do not disappear immediately: if $\mathcal{F}_0(\omega) \neq 0$ for $|\omega| > 1$, then $\mathcal{F}_1 \neq 0$. According to equation (21), at each iteration step, there are two processes: (i) the harmonics at frequency ω is multiplied by factor $A(\omega)$, and (ii) there is a spectral transfer toward lower frequencies $\omega \to \omega - 2$, with the corresponding factor $B(\omega)$. Important is that both these factors are less than one. Schematically, this is illustrated in Fig. 3. This scheme shows that for all spectra decaying at infinity, i.e. with $\lim_{\omega\to\infty} |\mathcal{F}_0(\omega)| = 0$, the modulation eventually disappears, i.e. $\lim_{n\to\infty} |\mathcal{F}_n(\omega)| = 0$. For some important classes of spectra, one can show that the eigenvalues of the transformation operator (21) are smaller than one (see Appendix D), so that the iteration procedure converges exponentially. In particular, for spectra $\mathcal{F}_h(\omega)$ which decay exponentially $\mathcal{F}_h(\omega) \sim \exp(-\alpha\omega)$ (what corresponds to a smooth modulation h(t)), we have $|\mathcal{F}_n| \sim \left(\frac{1+\exp(-2\alpha)}{2}\right)^n \xrightarrow[n\to\infty]{} 0$.

2. In the analysis above we have not considered components of modulation having frequencies 1, 3, 5, ..., i.e. harmonics of the basic oscillation frequency. Such components cannot be demodulated, because one cannot distinguish them from the original phase – in fact, a modulation with these frequencies is equivalent to changing the shape of the waveform $S(\varphi)$.



Figure 3: Depicted is the damping of Fourier modes in the range $2K < \omega < 2K + 2$ for the simple waveform $S(\varphi) = \cos(\varphi)$. In each iteration step, the method damps $\mathcal{F}_n(\omega)$ (dashed vertical arrows) and additionally generates a new Fourier mode at a smaller frequency (dashed diagonal arrows). At step K, a Fourier mode with frequency less than 1 is generated in the low-frequency region (bold green-white arrow), where it disappears at the next iteration. The damping factors are given by (20).



Figure 4: We show the most essential spectral components of the phase demodulation error $\theta_n^{(a)} - \varphi$ for the first 4 iterations (squares, circles, up and bottom triangles, respectively). The observed spectral components are marked with arrows on top of panels. In all cases the original signal is $x(t) = \cos(\varphi(t))$, where the modulation is given by Eq. (22). The vertical scale is logarithmic, with the ticks marking factor 2, to guide a visual comparison with the theory, where the damping factors are 1/2.

6. Numerical tests

6.1. Testing theoretical relations

Our first numerical example is intended to illustrate the theory of Section 5. Therefore, we use here the simple waveform $S(\varphi) = \cos(\varphi)$, and the modulation is a function of time

$$\varphi(t) = t + a\sin(ft) + b\cos(gt) , \qquad (22)$$

possessing two frequencies f and g. We performed iterative HT embeddings as described above, calculated at each step the analytic proxi-phase, and found the spectral components of the differences $\theta_n^{(a)}(t) - \varphi(t)$. These spectral components are presented in Fig. 4 for four numerical setups. In case (a) the modulation is weak and here one can directly compare the observed spectral amplitudes with theoretical predictions (nonlinear effects are small). In all other cases (b-d) the modulation is relatively strong, so that the nonlinear effects become dominant. In this subsection we focus on the spectral composition of the errors at first several iterations. The cumulative "energy" of all spectral components is measured by the error (9), we discuss this quantity in the next subsection.

Case (a). Panel (a) of Fig. 4 shows the case a = 0, b = 0.2, g = 3.73 of a single-harmonic modulation with high

frequency. Here the modulation is relatively weak, and we expect relations (19) to hold. Let us follow these relations in details. Here only components $\mathcal{F}_0(\pm g)$ are present. Thus, at the first iteration one has two equal components $\mathcal{F}_1(\pm g) = \frac{1}{2}\mathcal{F}_0(\pm g)$ and $\mathcal{F}_1(\pm g \mp 2) = \frac{1}{2}\mathcal{F}_0(\pm g)$. These componenta are depicted with red squares in Fig. 4. At the second iteration, one has components $\mathcal{F}_2(\pm g) = \frac{1}{4}\mathcal{F}_0(\pm g)$; $\mathcal{F}_2(\pm g \mp 2) = \frac{1}{2}\mathcal{F}_0(\pm g)$, and $\mathcal{F}_2(\pm g \mp 4) = \frac{1}{4}\mathcal{F}_0(\pm g)$ (green filled circles). At the third iteration, one has components $\mathcal{F}_3(\pm g) = \frac{1}{8}\mathcal{F}_0(\pm g)$; $\mathcal{F}_3(\pm g \mp 2) = \frac{3}{8}\mathcal{F}_0(\pm g)$, and $\mathcal{F}_3(\pm g \mp 4) = \frac{1}{4}\mathcal{F}_0(\pm g)$ (blue triangles). At the fourth iteration, one has components $\mathcal{F}_4(\pm g) = \frac{1}{16}\mathcal{F}_0(\pm g)$; $\mathcal{F}_4(\pm g \mp 2) = \frac{1}{4}\mathcal{F}_0(\pm g)$, and $\mathcal{F}_4(\pm g \mp 4) = \frac{3}{16}\mathcal{F}_0(\pm g)$ (magenta downside triangles). In addition, one can see weak nonlinear components at frequency 2g - 4 (frequencies 2g and 2g - 2, at which also nonlinear effects are present, are outside the shown range of frequencies).

- Case (b). Panel (b) illustrates nonlinear effects at a strong low-frequency single-harmonic modulation. Here a = 1, f = 0.41, and b = 0. One can see that although the frequency of modulation is less than one, this component at f does not disappear completely in $\theta_1^{(a)} \varphi$ as predicted by the linear theory, but is finite (already rather small). In the first iteration, higher harmonics nf and the combinational frequencies nf 2 appear, and they are relatively large. During next iterations the amplitudes of these components decrease with a factor $\approx 1/2$. This example shows efficiency of phase demodulation based on the analytic proxi-phase for slow modulation. It is illustrated here in terms of the amplitudes of Fourier modes of the phase. One can see that the first iterations are dominated by nonlinear effects, and at this stage the spectrum broadens. Only subsequent iterations lead to decrease of the Fourier components. For cumulative error see Fig. 5 below.
- Case (c). Panel (c) shows the case of relatively strong two-frequency modulation, where both frequencies are low: a = b = 0.3, f = 0.41, g = 0.73. Due to nonlinearity, many combinational frequencies kf + lg, kl + fg - 2 with integers k and l are exited. It appears that all the new frequencies, like in case (b), are created in the first iteration, further iterations follow roughly the law (21).
- Case (d). Panel (d) shows, like case (c), a strong two-frequency modulation, but with high basic frequencies a = b = 0.1, f = 2.41, g = 3.73. One can see that at some combinational frequencies, the level even initially increases due to the cascade process of amplitude shifting (see, e.g., components with f + g 4), and one needs more iterations to reduce error at these components.

In conclusion, the presented numerical tests show that the theory based on the linear approximation works well for weak modulation, but for strong modulation, there are essential nonlinear effects at the first few iterations. Because the effective modulations become weaker in the course of iterations, the theory better describes higher iterations.

6.2. Accuracy of the iterative procedure

In the following, we present numerical results for the signals $x(t) = S_{1,2}(\varphi(t))$, where the waveforms are given by expressions (2),(3), and the phase modulation is a quasiperiodic function of time

$$\varphi(t) = t + \frac{a}{R}\sin(R\omega_1 t) + \frac{b}{R}\cos(R\omega_2 t) .$$
(23)

We have chosen $\omega_1 = \sqrt{2}$, $\omega_2 = \sqrt{3}$, a = b = 0.3. Parameter *R* allows us to vary the basic frequencies, keeping the range of the phase modulation constant. This range is quite large, $\min(d\varphi/dt) = 1 - a(\omega_1 + \omega_2) \approx$ 0.056, so the first-order theoretical analysis of section 5 is not applicable. In all runs we used discretisation step $\Delta t = 0.002$, the length of the time series was about 95 basic periods. Convergence was characterized by evaluating error (9), where, to discard boundary effects, only the central 80% of the time series where used.

Our goal is to quantify performance of the iterative procedure described, in dependence on the characteristic time scale defined by parameter R, on the properties of the waveform (simple vs complex), and on the choice of proxi-phase in the iterative process (analytic vs length-based).

6.2.1. Simple waveform: comparison of analytic and length-based proxi-phases

For the simple waveform (2), one can use both proxi-phases - the analytic one and the length-based one. The results are shown in Fig. 5. One can see that both proxi-phases ensure demodulation with a very small rest error, if the modulation is slow ($R \leq 1$). For the lowest-frequency case, the length method performs even significantly better at high iterations.

For larger values of $R \ge 2$, the length method appears to be the only applicable one. Here, the analytic proxi-phase fails because at one of the first iterations the analytic proxi-phase becomes a non-monotonous function of time, and iterations break. In contradistinction, the length-based phase always provides a monotonous estimate of the phase. One can see that for the fastest modulation explored (R = 8), the error decreases not monotonously with n (it increases at $n \ge 10$), but even in this case, the error of the iteration procedure eventually becomes rather small.

6.2.2. Complex waveform

For the complex waveform (3), only the length-based phase can be used. In Fig. 6 we show the errors vs the iteration number n, for the same values of the parameter R as in Fig. 5. In all cases the final error is very small, close to $10^{-7} - 10^{-8}$. Similarly to the case of the simple waveform, here at the fastest example (R = 8), the demodulation is not monotonous.

The presented results confirm that the described method provides an effective demodulation of the complex signal. We attribute the rest error to numerical inaccuracies in calculation of the HT, of the length of the trajectory on the plane [x(t), y(t)], and in the evaluation of the error itself. This is supported by the observation that the asymptotic errors grow significantly if a larger discretization step is used.



Figure 5: Errors of demodulation for simple waveform S_1 and analytic proxi-phase (a) and length based proxi-phase (b) in dependence on the iteration index n and R. Note that R = 0.25, 0.5, 1 are depicted for both proxi-phases in the same color, but with open and filled markers, respectively. For R > 1 the analytic proxi-phase is not suitable for reconstruction, as at high iterations it becomes non-monotonous (cf. discussion in Section 3 above). One can also see that the performance of the iterations is the best for the lowest considered frequency of modulation (case R = 0.25).



Figure 6: Errors of demodulation in dependence on the iteration index n and R for the complex waveform S_2 . Similar to Fig. 5 above, here the best performance is for the slowest modulation frequency (R = 0.25).



Figure 7: Demodulation of a signal having complex waveform S_2 and the instantaneous frequency (24) (panel (b)). Panel (a) shows embeddings for the original signal $x(\theta_0)$ (blue line) and for the iteration $x(\theta_{64})$ (red line). Panel (c): Errors in dependence on the iteration number.

It is important to mention, that contrary to the analytic proxi-phase (7) which is 2π -periodic, the lengthbased proxi-phase (8) does not have a "natural" period: strictly speaking, a proper normalization to period 2π is only possible at the end stage of the iteration procedure, where the periodicity of $x(\theta^{(b)})$ is well-defined.

6.3. Arbitrary phase modulation

Above we considered phase modulations as quasiperiodic deviations from the linear phase growth. The method is, however, not restricted to such cases. As an example, we present here results for the complex waveform S_2 (Eq. (3)) with phase modulation defined as $d\varphi/dt = \Omega(t)$, with a rather arbitrarily chosen instantaneous frequency

$$\Omega(t) = \frac{-t}{100\pi} + 0.5 \exp\left(\frac{t}{100\pi}\right) + \exp\left(-\frac{t^2}{400\pi^2}\right) + 0.2 \cos\left(3\sqrt{2}\frac{t}{\pi}\right) , \quad -200\pi \le t < 200\pi , \qquad (24)$$

see Fig. 7(b). For the sampling with $\Delta t = 0.005$, the demodulation error $\approx 3 \cdot 10^{-8}$ was achieved after 20 iterations (see Fig. 7(c)). In Fig. 7(a) the quality of demodulation is illustrated visually, like in Fig. 1.

6.4. More complex waveforms and modulations

Above we demostrated the efficiency of the method for relatively simple waveforms and modulations. To show that the method robustly extracts the phase for complex waveforms and modulations, we adopt in this section the waveforms shown in Fig. 8(left panel). The waveform $S_3(\varphi) = 1 - \frac{2}{\pi}|\varphi - \pi|$ is non-smooth. The waveform $S_4(\varphi) = \sum_{k=1}^{10} (\sin(2k\varphi(t)) + \cos(k\varphi(t)))k^{-1}$ is smooth, but contains 20 harmonics



Figure 8: Left panel: Waveforms $S_3(\varphi)$ (red dotted line) and $S_4(\varphi)$ (blue line). Right panel: Errors of phase demodulation in dependence on the iteration index. Case S_3 : red filled circles; case S_4 : blue squares.

and possesses many small-scale oscillations. For both waveforms, we constructed signals with a non-smooth phase modulation according to

$$\varphi(t) = t + a|2 - |1 - \cos(\sqrt{2}t) - \sin(\sqrt{3}t)|| - a, \qquad a = 0.3.$$

The results for the phase modulation reconstruction error are presented in Fig. 8(right panel). They show that the method works here nearly as well as for smooth examples.

7. Transformation of a protophase to true phase

As discussed in section 2.3 above, there is an intrinsic ambiguity in the phase demodulation problem: one can perform a transformation of a waveform together with a transformation of an obtained protophase, such that the signal remains the same. Thus, one needs additional conditions to specify a particular phase and a particular waveform uniquely. In the context of the dynamical system applications (section 2.2), a natural condition is that the phase should grow on average uniformly in time. This condition was adopted in [6], where a procedure for the transformation $\theta \to C(\theta)$ from a protophase to the uniformly on average growing phase was described. Roughly speaking, for a wrapped protophase one determines the probability distribution density on the interval $[0, 2\pi)$, and performs a transformation such that the new phase has the uniform distribution density. To illustrate this approach, we take an example from section 6.2.2 with $x(t) = S_2(\varphi(t))$, where S_2 is given by Eq. (3) and $\varphi(t)$ is given by Eq. (22) with R = 8. In Fig. 9 we show the protophase obtained via the length proxi-phase method, and its transformation according to method [6]. The transformed phase is close to the original one, up to an unavoidable, but unimportant, shift. We stress here, that the accuracy of the transformation $\theta \to C(\theta)$ is rather weak, because it relies on the empirical probability density, which one obtains based on finite-size data.



Figure 9: Red points: values of the reconstructed protophase $\theta_{100}^{(b)}$ at 100-th iteration vs. the true phase φ . Blue bold points: the transformed protophase $C(\theta_{100})$. One can see that the relation between $C(\theta_{100})$ and φ is nearly linear.

8. Conclusion

In summary, we have proposed a method of phase demodulation based on iterative HT embeddings. In all tested cases it leads to an effective accurate reconstruction of the waveform and of the phase of the signal, up to unavoidable ambiguity due to protophase variety. Our method can be applied to smooth simple and complex waveforms as well as to cases where phases and waveforms that are non-smooth. In all cases we have found that the proxi-phase extraction based on the length of the embedded trajectory is superior to a traditional definition based on the analytic signal.

We have performed theoretical analysis valid for a weak modulation of the simplest cosine waveform. This analysis shows that while slow modulation is detected exactly already within the first iteration, for fast components the iterations are indeed needed. Furthermore, this analysis yields exponential (in number of iterations) convergence rate for modulation signals with spectra that decay exponentially at high frequencies. We have shown also that nonlinear effects in the case of strong modulation lead to appearance of combinational frequencies in the modulation spectrum, what nevertheless does not prevent eventual convergence of the procedure. The overall performance of the method is the best for slow modulation.

In this study, we considered only regular (not noisy) signals and used a quite high sampling rate. The later is mainly to achieve the highest possible performance, because larger discretization steps lead to errors in calculations of the Hilbert transforms. The analysis of different factors affecting the accuracy will be subject of further research. For noisy signals, a strategy could be first to smoothen them, for example by virtue of Savtzky-Golay filters, and then to apply the method described. How this influences the accuracy, is now under investigation, and will be reported elsewhere. Also applications of signals from forced dynamical systems are now under consideration. There, the effect of amplitude variations is unavoidable, and we expect the method to work well for weak amplitude modulations only.

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Appendix A. Mixing of amplitude and phase modulation at HT embedding

The goal of this appendix is to illustrate how the HT embedding mixes amplitude and phase modulations. First, we consider a signal with purely amplitude modulation $x(t) = (1 + 0.1 \cos(\sqrt{5}t)) \cos(t)$. We perform the Hilbert transform according to (6), and calculate the phase of the embedded signal according to Eq. (8). The result, shown in fig. A.10(a), illustrates that the extracted phase deviates from the linear growth in time, what means phase modulation. Second, we take a purely phase modulated signal $x(t) = \cos(t + 1.2(\sin(0.25\sqrt{2}t) + \cos(0.25\sqrt{3}t)))$. After performing the HT of this signal according to Eq. (6), we calculate the amplitude $A(t) = (x^2(t) + y^2(t))^{1/2}$, it is depicted in fig. A.10(b). One can see that the embedded signal is amplitude modulated, because $A \neq const$.



Figure A.10: (a): Deviation of the phase, obtained from the HT embedding of a purely amplitude modulated signal. (b): The amplitude A(t) obtained via the HT embedding of a purely phase modulated signal.

Appendix B. Numerical implementation of the Hilbert transform on a non-uniform grid

The HT of a function $u(\phi)$ is defined as

$$H(\psi) = \frac{1}{\pi} \mathrm{PV} \int_{-\infty}^{\infty} \frac{u(\phi)}{\psi - \phi} d\phi \; .$$

Suppose the function u is given on a discrete set of points $\phi_1, \phi_2, \ldots, \phi_N$ as values u_1, u_2, \ldots . We now want to find H_i in the same set of points $\psi_1 = \phi_1, \ldots$

We apply the trapezoidal rule, similar to the approach of [35] for a uniform grid, and write the integral as a sum of elementary integrals over domains (ϕ_k, ϕ_{k+1}) :

$$H(\phi_i) = \frac{1}{\pi} \sum_{k=1}^{N-1} \int_{\phi_k}^{\phi_{k+1}} \frac{u(x)}{\phi_i - x} dx = \frac{1}{\pi} \sum_{k=1}^{N-1} I_k .$$

We approximate u(x) on the interval (ϕ_k, ϕ_{k+1}) by a linear function:

$$u(x) = u_k + \frac{(u_{k+1} - u_k)}{\phi_{k+1} - \phi_k} (x - \phi_k) = a_k + b_k x ,$$

where

$$a_k = \frac{u_k \phi_{k+1} - u_{k+1} \phi_k}{\phi_{k+1} - \phi_k}$$
, $b_k = \frac{u_{k+1} - u_k}{\phi_{k+1} - \phi_k}$

Thus, the elementary integrals are

$$I_k = \int_{\phi_k}^{\phi_{k+1}} \frac{a_k + b_k x}{\phi_i - x} dx = u_k - u_{k+1} - \frac{u_k \phi_{k+1} - u_{k+1} \phi_k + \phi_i (u_{k+1} - u_k)}{\phi_{k+1} - \phi_k} \ln \left| \frac{\phi_{k+1} - \phi_i}{\phi_k - \phi_i} \right|$$

This expression is valid for $k \neq i, k \neq i - 1$.

To calculate the integrals around ϕ_i , we represent $u = u_i + d_+(x - \phi_i)$ on $\phi_i < x < \phi_{i+1}$ and $u = u_i + d_-(x - \phi_i)$ on $\phi_{i-1} < x < \phi_i$, where

$$d_{+} = \frac{u_{i+1} - u_i}{\phi_{i+1} - \phi_i}, \qquad d_{-} = \frac{u_i - u_{i-1}}{\phi_i - \phi_{i-1}}$$

Then

$$I_{i-1} + I_i = \int_{\phi_{i-1}}^{\phi_{i+1}} \frac{u_i}{\phi_i - x} dx - d_+(\phi_{i+1} - \phi_i) - d_-(\phi_i - \phi_{i-1}) =$$
$$= u_i \ln \frac{\phi_i - \phi_{i-1}}{\phi_{i+1} - \phi_i} - d_+(\phi_{i+1} - \phi_i) - d_-(\phi_i - \phi_{i-1}) = u_i \ln \frac{\phi_i - \phi_{i-1}}{\phi_{i+1} - \phi_i} - u_{i+1} + u_{i-1}$$

Appendix C. Phase Modulations

This appendix illustrates in Fig. C.11 the time series of the phase modulated signal (upper panel) and the extracted analytic proxi-phases for iterations 1,2,3 and 10. We compare these extracted phases with the rue modulation phase, to illustrate the convergence of the method.



Figure C.11: Depicted are the signal $x(t) = S_1(\varphi(t)) = \cos \varphi(t)$ with modulation $\varphi(t) = t + 0.3 \sin(\sqrt{2}t) + 0.3 \cos(\sqrt{3}t))$ (top panel), and the reconstructed in different iterations phase modulation $\theta_n^{(a)} - t$ (black), overlapped on the actual phase modulation $\varphi(t) - t$ (bold line, light blue). This case is similar to that shown in Fig. 1, but here, for clarity of the figures, a higher frequency of modulation is used (for the low modulation frequency already the first iteration gives a good approximation and further improvements are hardly seen).

Appendix D. Eigenvalues and eigenfunctions of the modulation iteration operator

Here we explore properties of the transformation operator, acting on the Fourier spectrum of the modulation according to (21)

$$\hat{T}F(\omega) = A(\omega)F(\omega) + B(\omega)F(\omega+2), \quad \omega > 1.$$

Here we restrict our attention to the case $\omega > 1$ only, because negative frequencies are symmetric to positive ones. The real factors A, B > 0 are given by (20).

First, we employ the triangle inequality

$$||A(\omega)F(\omega) + B(\omega)F(\omega+2)|| \le A(\omega)||F(\omega)|| + B(\omega)||F(\omega+2)||,$$

and restrict our attention to real positive functions $F(\omega)$ only. We look for eigenfunctions of the operator \hat{T} in the class of exponentially decreasing functions of ω at large frequencies: $F(\omega) \sim \exp(-\alpha\omega)$. This means that we consider smooth modulations only. With an ansatz

$$F(\omega) = f(\omega) \exp(-\alpha \omega) ,$$

where $f(\omega)$ has finite variation at $\omega \to \infty$, the problem reduces to finding eigenfunctions and eigenvalues of the operator

$$\hat{T}_{\alpha}f(\omega) = A(\omega)f(\omega) + B(\omega)e^{-2\alpha}f(\omega+2).$$
(D.1)

In the simplest case of the analytic proxi-phase, where $A = B = \frac{1}{2}$, one can see that any periodic function $f(\omega) = f(\omega + 2)$ is an eigenfunction of (D.1) with eigenvalue

$$\lambda(\alpha) = (1 + \exp(-2\alpha))/2 < 1$$
. (D.2)

This shows that under action of the operator \hat{T} , the spectral function tends to zero exponentially, the exponent depends on the asymptotics of the spectrum at large frequencies.

In the case of the length proxi-phase, where $A(\omega) = \frac{\omega-1}{2\omega}$ and $B(\omega) = \frac{\omega+1}{2\omega}$, we cannot find the eigenfunctions, but can show that the eigenvalues are the same as for the case A = B = 1/2. Indeed, the eigenvalue problem $\hat{T}_{\alpha}f = \lambda_{\alpha}f$ reduces, after substitution to (D.1), to a recursion

$$f(\omega + 2) = \frac{\lambda_{\alpha} - A(\omega)}{B(\omega)e^{-2\alpha}}f(\omega)$$
.

Because asymptotically $f(\omega)$ should neither grow nor decay at $\omega \to \infty$, we have

$$\lim_{\omega \to \infty} \frac{\lambda_{\alpha} - A(\omega)}{B(\omega)e^{-2\alpha}} = \frac{\lambda_{\alpha} - \frac{1}{2}}{\frac{1}{2}e^{-2\alpha}} = 1.$$

This yields the same eigenvalue (D.2).