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Cite as: Chaos **29**, 011105 (2019); https://doi.org/10.1063/1.5079617
Submitted: 01 November 2018 . Accepted: 05 December 2018 . Published Online: 18 January 2019

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Chaos BRIEF COMMUNICATION scitation.org/journal/cha

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Submitted: 1 November 2018 · Accepted: 5 December 2018 ·
Published Online: 18 January 2019

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ABSTRACT

We develop a numerical approach to reconstruct the phase dynamics of driven or coupled self-sustained oscillators. Employing a simple algorithm for computation of the phase of a perturbed system, we construct numerically the equation for the evolution of the phase. Our simulations demonstrate that the description of the dynamics solely by phase variables can be valid for rather strong coupling strengths and large deviations from the limit cycle. Coupling functions depend crucially on the coupling and are generally non-decomposable in phase response and forcing terms. We also discuss the limitations of the approach.

It is widely accepted that the dynamics of weakly interacting limit cycle oscillators can be fully described by their phases, while the amplitudes can be considered as enslaved and thus irrelevant. This idea is commonly used in both theoretical and experimental studies. While in the latter case, the phase dynamics is inferred from data, in the former one, the corresponding equation shall be derived by a perturbation technique. However, the existing theory provides only a recipe for computation of the first order approximation. In spite of numerous recent efforts, phase dynamics description beyond the limit case of weak coupling is still lacking. Here, we provide a numerical approach for the phase dynamics reduction beyond the first order approximation.

I. INTRODUCTION

Phase dynamics approximation plays an important role in the analysis of coupled self-sustained oscillators.[1–6] Reduction to the phase description essentially decreases the dimension of the problem and in some cases provides an analytical solution. Well-known examples of the solvable phase dynamics are the Adler equation[7] for a periodically driven oscillator and the Kuramoto model of (infinitely) many globally coupled systems.[2] However, a practical application of the theory and a derivation of the phase dynamics equation from equations formulated in state variables for an arbitrary driven or coupled oscillator remains an unsolved problem: this derivation requires an analytical expression for isochrons[8] or for the phase response curves, and therefore can be accomplished analytically only in exceptional cases, e.g., for the Stuart-Landau system. Moreover, this equation can be obtained only in the first approximation, as the first-order term of the perturbative expansion with respect to a small coupling parameter.[2] In spite of recent attempts (see a discussion in Refs. 6 and 9), there exist no practical algorithms for inferring phase description in a general case when a perturbed trajectory is not very close to the unperturbed limit cycle.

In this communication, we propose an efficient numerical approach which provides the desired phase dynamics equation. The evolution of the phase is described by a coupling function that is either defined on a fine grid or represented by a finite Fourier series. Next, we use this approach to study the phase dynamics for strongly driven systems, where we cannot expect validity of the first order approximation. For two different examples, we reconstruct the second- and the third-order terms of the coupling functions and verify that this description works with a good precision.

We start by introducing our notations and recalling the main ideas of the phase reduction theory. Let the autonomous system be described by $\dot{\mathbf{X}} = \mathbf{F(X)}$, where $\mathbf{X}$ is the state vector in n-dimensional phase space, $n \geq 2$. This equation defines the flow $(\mathbf{X}, t) \to S^t\mathbf{X}$. Our goal is to find such a function of the phase space $\Phi(\mathbf{X})$, mapping the state of the system onto a unit

circle, that allows to attribute to every trajectory $\mathbf{X}(t)$ the corresponding phase evolution $\varphi(t) = \Phi[\mathbf{X}(t)]$. Depending on the context, we will either treat φ as a wrapped phase (points on a unit circle) or as an unwrapped phase with $-\infty < \varphi < \infty$.

If the system possesses a stable limit cycle $\overline{\mathbf{X}}$ with period T, then the phase definition $\Phi(\mathbf{X})$ can be constructed for all states $\mathbf{X}$ in the basin of attraction of $\overline{\mathbf{X}}$. First, the phase is easily introduced on $\overline{\mathbf{X}}$. One assigns zero value of φ to an arbitrary point $\mathbf{X_0}$ on the limit cycle. Then, for $0 \le t < T$, the phase on the limit cycle is defined as $\Phi(S^t\mathbf{X_0}) = \omega t$, with $\omega = 2\pi/T$.

Extension of this definition to the whole basin of attraction of the cycle $\overline{\mathbf{X}}$ is based on the isochrons.[8] For a point $\mathbf{X}$ in the basin of attraction of $\overline{\mathbf{X}}$, the phase is defined as $\Phi(\mathbf{X}) = \Phi(\mathbf{X}_*)$, where $\mathbf{X}_* = \lim_{m\to\infty} S^{mT}\mathbf{X}$ $(m = 0, 1, 2\ldots)$; obviously, this limiting point belongs to the limit cycle, $\mathbf{X}_* \in \overline{\mathbf{X}}$. In other words, isochrons, as the sets of equal phases, are stable manifolds of the corresponding points on the limit cycle under the action of the stroboscopic map $\mathbf{X} \to S^T\mathbf{X}$.

Knowledge of isochrons, i.e., of the relation $\varphi = \Phi(\mathbf{X})$ in a vicinity of the limit cycle, allows one to obtain phase equations for a *perturbed* system

$$\dot{\mathbf{X}} = \mathbf{F(X)} + \varepsilon\mathbf{p}(\mathbf{X}, t), \tag{1}$$

where $\mathbf{p}$ describes external forcing or coupling with the strength ε. Then, for small ε, exploiting the perturbation approach, *using the unperturbed definition of the phase*, and neglecting deviations from the limit cycle (see Refs. 2 and 4 for details), one obtains

$$\dot{\varphi} = \omega + \varepsilon \sum_k \frac{\partial\Phi[\overline{\mathbf{X}}(\varphi)]}{\partial X_k} p_k[\overline{\mathbf{X}}(\varphi), t] = \omega + \varepsilon Q_1(\varphi, t), \tag{2}$$

where $\varepsilon Q_1(\varphi, t)$ is called the coupling function and the subscript emphasizes that the latter is obtained in the first order approximation in ε. In typical cases, e.g., for periodic external forcing or coupling to another limit cycle oscillator, $\mathbf{p}$ can be parameterized by its phase, ψ [it means that we can write $\mathbf{p}(\mathbf{X}, \psi)$], and then Eq. (2) can be written as

$$\dot{\varphi} = \omega + \varepsilon Q_1(\varphi, \psi). \tag{3}$$

Generally, the perturbation approach implies that the dynamics can be represented by a power series in ε

$$\dot{\varphi} = \omega + \varepsilon Q_1(\varphi, \psi) + \varepsilon^2 Q_2(\varphi, \psi) + \cdots = \omega + Q(\varphi, \psi, \varepsilon). \tag{4}$$

One can expect that this description is valid not only for small coupling, but as long as the dynamics of the driven system occurs on a smooth attractive torus. Although the methods for computation of the high-order terms are not known, the function Q, and the corresponding contributions Q_k, $k = 1, 2, \ldots$, can be obtained numerically, as is demonstrated below.

To complete the introduction, we mention that if only one component of the force in (1) is present, and this force does not depend on the state of the system, i.e., $\mathbf{p}(\mathbf{X}, t) = p(t)$, then, according to (2), the coupling function in the first approximation can be represented as a product

$$Q_1(\varphi, t) = Z(\varphi)p(t), \tag{5}$$

where $Z(\varphi)$ is the phase sensitivity function, also called the phase response curve (PRC). This representation is often called the Winfree form.

II. NUMERICAL COMPUTATION OF COUPLING FUNCTIONS

In this section, we describe the numerical aspects of the technique and present our algorithm step-by-step.

A. Inference of the coupling function

1. The first, preparatory step is to compute period T of the autonomous oscillation and to choose a zero-phase point $\mathbf{X_0} \in \overline{\mathbf{X}}$ on the limit cycle. (We use the known Henon-trick[16] for precise determination of the times when the trajectory returns to $\mathbf{X_0}$.)
2. The next step is to obtain the time series $\mathbf{X}(t_k)$. This is accomplished by virtue of numerical integration of ODEs (1), e.g., by a Runge-Kutta method.
3. For each point $\mathbf{X}(t_k)$, the phase $\varphi(t_k) = \Phi[\mathbf{X}(t_k)]$ is calculated. The key issue here is a numerical evaluation of the function $\Phi(\tilde{\mathbf{X}})$, which should provide the phase for any state $\tilde{\mathbf{X}}$. According to the theory outlined above, this function is determined solely by the autonomous dynamics. Then, for each $\mathbf{X}(t_k)$, we proceed as follows.
   (a) We integrate the autonomous equations $\dot{\mathbf{Y}} = \mathbf{F(Y)}$, taking $\mathbf{Y}(0) = \mathbf{X}(t_k)$ for the initial condition. The integration time is chosen to be NT, where an integer N shall be taken sufficiently large so that we can assume that the trajectory is already attracted to the limit cycle. (In fact, N is the only parameter of our procedure, it crucially depends on the second largest multiplier of the limit cycle.)
   (b) Next, we compute the phase of the point $\mathbf{Y}(NT)$. For this goal, we determine the time τ required for the trajectory started from $\mathbf{Y}(NT) \in \overline{\mathbf{X}}$ to achieve $\mathbf{X_0}$. This provides $\Phi[\mathbf{X}(t_k)] = \Phi[\mathbf{Y}(NT)] = 2\pi(T - \tau)/T$.

   We emphasize that steps 3a and b are not required if the function $\Phi(\tilde{\mathbf{X}})$ is known explicitly, like for the Stuart-Landau oscillator.[4] However, even in this exceptional case, further numerical steps for determination of the coupling function beyond the first order approximation appear to be unavoidable.
4. Next, we evaluate numerically $\dot{\varphi}_k = \dot{\varphi}(t_k)$. Practically, we use the Savitzky-Golay smoothing filter for numerical differentiation of φ_k, see Refs. 14 and 19 for details.
5. As a result of the previous steps, we obtain three scalar time series $\{\dot{\varphi}_k, \varphi_k, \psi_k\}$, where $\psi_k = \nu t_k$. Now, we use these series to fit the r.h.s. of Eq. (4) and thus obtain $Q(\varphi, \psi)$ on the torus $0 \le \varphi, \psi < 2\pi$. Practically, we obtain Q on a grid 100×100 using the kernel density function estimation;[17] alternatively, it can be obtained as a finite double Fourier series. Notice that the fit is possible if the systems remain

in a quasiperiodic state so that the trajectory densely fills the torus.

Our final remark is related to software used. For the steps 4 and 5, we used functions co_phidot1 and co_kcplfct1 from our open-source Matlab toolbox DAMOCO, available at www.stat.physik.uni-potsdam.de/~mros/damoco2.html. An example of C code for computation of phases (steps 1–3) is available upon request.

B. Inference of the coupling function in the Winfree form

As follows from Eq. (5), for a known force $p(t)$, the determination of the coupling function reduces to finding PRC. For infinitely weak coupling, the latter can be obtained from equations of the autonomous system by solving an adjoint problem.[6,18] Since our goal here is to explore the strong coupling case and to check whether the representation via a product still holds, we compute PRC for different values of ε from Eq. (5). Since $\{\dot\varphi_k, \varphi_k, \psi_k\}$ are already computed, this computation can be accomplished by expanding PRC in a Fourier series and obtaining the coefficients of this series via linear fit.

C. Decomposition in powers of the coupling strength ε

Here, we shortly discuss how a general coupling function $Q(\varphi,\psi,\varepsilon)$ can be decomposed in a power series of ε, cf. Eq. (4). In our simulations, we restrict the order of the series to three; hence, we are looking for a third-degree polynomial representation of $Q(\varphi,\psi,\varepsilon)$ and determine $Q_{1,2,3}$. For this goal, it suffices to compute $Q(\varphi,\psi,\varepsilon)$ for at least three different values of coupling ε. Practically, in our examples shown below, we use more than 10 values of ε. Then, with account of the condition $Q(\varphi,\psi,0)=0$, the polynomial ε-dependence can be found by least mean squares fit. Notice that this procedure can be performed point-wisely for coupling functions given on a grid (below we use this option), or for each Fourier harmonics for the Fourier representation of Q. As a result, the partial coupling functions $Q_{1,2,3}$ defined in (4) are obtained.

III. RESULTS

In order to illustrate the advantages and limitations of our approach, we analyze the phase dynamics of two simple paradigmatic models.

A. The Rayleigh oscillator

In the first example, we consider the harmonically forced Rayleigh oscillator

$$\ddot x - \mu(1-\dot x^2)\dot x + x = \varepsilon\cos(\nu t). \tag{6}$$

The nonlinearity parameter is $\mu = 4$; for this value, the limit cycle is strongly stable (its transversal Lyapunov exponent is ≈ -5.62, what for the period of the cycle ≈ 10.2 yields a multiplier $\approx 10^{-24}$). Notice that in this particular case, the perturbation is scalar and therefore the first-order coupling function can be written in the Winfree form (5) with $p(t) = \cos\psi = \cos(\nu t)$.

We fix the frequency of the forcing at $\nu = 0.8$ and increase the amplitude from a small value $\varepsilon = 0.01$ (for which we can expect that the first-order phase approximation is valid) up to $\varepsilon = 0.55$, which is not small at all: this forcing is comparable to the amplitude of $x(t)$. Notice that for larger values of ε, the system gets locked to the force and determination of the coupling function with our method becomes impossible.

First, we demonstrate that the phase description is well-defined for the whole explored range of ε. For this purpose we compute, for all available φ_k, $\dot\varphi_k$, $\psi_k = \nu t_k$, the rest term of our full model,

$$\xi_k = \xi(t_k) = \dot\varphi_k - \omega - Q(\varphi_k,\psi_k,\varepsilon), \tag{7}$$

and quantify the quality of the fit by

$$\sigma = \mathrm{std}[\xi]/\mathrm{std}[\dot\varphi], \tag{8}$$

where $\mathrm{std}[u] = \left[\overline{(u-\bar u)^2}\right]^{1/2}$ and bar denotes the time averaging over the available time series. The dependence $\sigma(\varepsilon)$ is shown in Fig. 1(a), together with a similar measure for the Winfree form (5). The latter increases with coupling strength, what clearly demonstrates that the Winfree form is valid for small

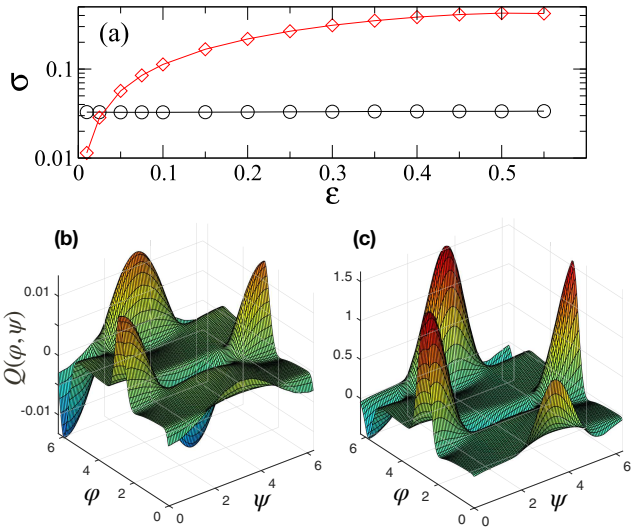

FIG. 1. (a) Quality of phase dynamics description in terms of the coupling function [see Eq. (4); circles] and in terms of PRC [see Eq. (5), diamonds] is quantified by the normalized error of fit σ, as defined in Eq. (8). Notice that in the former case, σ is small and practically independent of ε; here σ is due to inaccuracy of numerical differentiation of φ and of the kernel estimation procedure. In the latter case, the error becomes quite large for increasing ε, what cannot be explained by numerical precision only; large values of σ demonstrate that the coupling function cannot be anymore represented as $Q(\varphi,\psi) = Z(\varphi)p(\psi)$. (b) and (c) Coupling functions of the harmonically driven Rayleigh oscillator for weak, $\varepsilon = 0.01$ [panel (b)], and strong, $\varepsilon = 0.55$ [panel (c)], forcing.

couplings only. In contradistinction, the full coupling function describes the dynamics with accuracy ≈ 3% in the whole range of couplings.

In Figs. 1(b) and 1(c), we also show the inferred coupling functions for weak, $\varepsilon = 0.01$ (here Q nearly coincides with εQ_1), and strong, $\varepsilon = 0.55$, forcing amplitudes. Although qualitatively similar, the coupling functions are clearly different, indicating the importance of higher-order components Q_2 and Q_3.

Next, we obtain functions $Q_{1,2,3}$ as discussed in Sec. II C and verify the validity of the series representation by Eq. (4). To this end, we compute the error of the first-, second-, and third-order approximations

$$\sigma_m(\varepsilon) = \mathrm{std}\left[\mathrm{Q}(\varepsilon) - \sum_{i=1}^{m} \varepsilon^i \mathrm{Q}_i\right] / \mathrm{std}\left[\mathrm{Q}(\varepsilon)\right], \tag{9}$$

where $m = 1, 2, 3$ and $\mathrm{std}[u] = \langle (u - \langle u \rangle)^2 \rangle^{1/2}$. Here, the averaging is performed over the torus on which the coupling function is defined

$$\langle w \rangle = (4\pi^2)^{-1} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \, w(\varphi, \psi).$$

These errors are shown in Fig. 2, while the functions $Q_{1,2,3}$ are given in Fig. 3.

Figure 2 shows that, while the first order approximation is quite accurate for $\varepsilon \lesssim 0.1$, for larger coupling strengths the higher order terms become relevant. The third-order approximation provides a rather good accuracy (better than 1%) in the whole explored range of coupling strengths. Remarkably, the coupling functions Q_1, Q_2, Q_3 are very different. While the first-order coupling function Q_1 reproduces the known theoretical result (5), the second-order term Q_2 contains strong second harmonics in both phases. Preliminary computations also show that Q_1 is independent of the driving frequency ν, as expected, while the higher-order functions depend on ν; this issue shall be analyzed in detail separately.

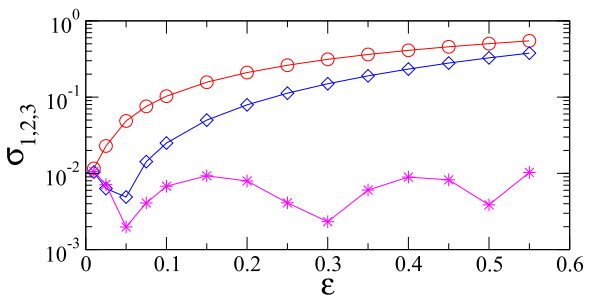

FIG. 2. Error of the 1st-order (circles), 2nd-order (diamonds), and 3rd-order (stars) approximations of the coupling functions for the Rayleigh oscillator. Notice that the precision of the third-order approximation is below 1% for the whole range of the driving strength.

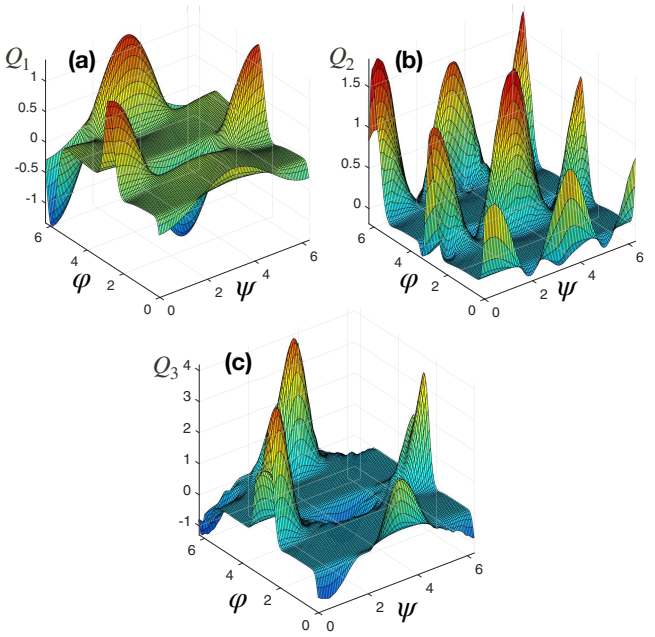

FIG. 3. Coupling function of strongly perturbed Rayleigh oscillator can be well approximated by a power series Eq. (4) with linear, quadratic, and qubic components, shown in (a), (b), and (c) correspondingly.

B. The Rössler oscillator

For the second example, we take a three-dimensional system, namely, the harmonically driven Rössler oscillator

$$\begin{aligned} \dot{x} &= -y - z + \varepsilon \cos(\nu t), \\ \dot{y} &= x + 0.34y, \\ \dot{z} &= 0.8 + z(x - 2). \end{aligned} \tag{10}$$

For the chosen parameter values, the unforced system has a limit cycle with complex multiplicators $(-8.71 \pm$

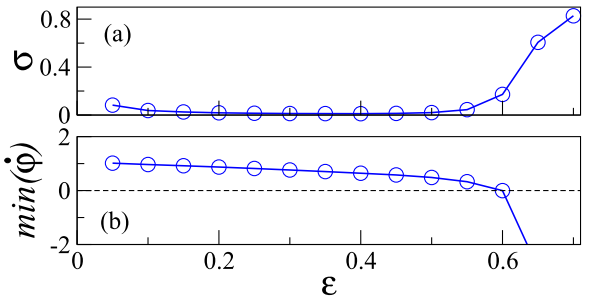

FIG. 4. (a) The error of the phase model description of the Rössler oscillator (10) remains small for the coupling strength $\varepsilon \lesssim 0.55$. For stronger coupling, the description fails. This is related to the fact that the phase becomes not monotonic, as can be seen from the plot of $\min(\dot{\varphi})$ vs. ε; see text and Fig. 7 for a discussion.

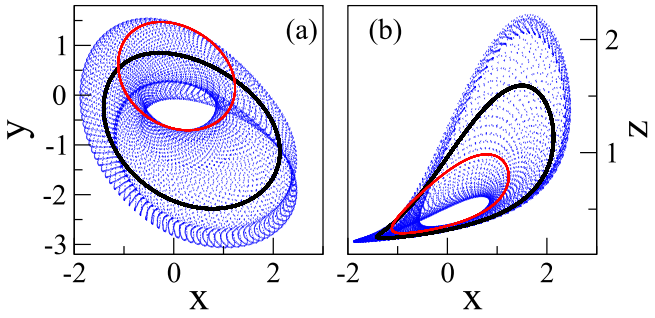

FIG. 5. Two projections of the phase space of the Rössler oscillator (10). In both panels, black bold line shows the unperturbed limit cycle, while the blue points show the trajectory of the driven system with $\varepsilon = 0.55$. The red line shows the Poincaré section of the attractor obtained by fixing the phase of the external force; this section confirms that the trajectory of the strongly driven system lies on the torus.

$12.4i) \cdot 10^{-3}$. Thus, the cycle is weakly stable; moreover, complex-valued multipliers mean that the dynamics cannot be reduced to a two-dimensional one. However, our approach successfully constructs the phase dynamics model. To demonstrate this, we infer the coupling functions for $\nu = 0.5$ and forcing strengths varied from $\varepsilon = 0.05$ to 0.7. The quality of the model for different ε is illustrated in Fig. 4, where we show the normalized standard deviation $\sigma(\varepsilon)$, defined according to (8). One can see that the model remains valid for the coupling values as large as $\varepsilon = 0.55$. We emphasize that for such a driving the deviation of the trajectory from the unperturbed limit cycle is not small at all, see Fig. 5, where the attracting torus at this regime is depicted. For the coupling functions for weak and strong coupling, see Fig. 6. Qualitatively, the results are similar to that for the Rayleigh oscillator: The coupling function is strongly nonlinear, as one can conclude from the comparison of two panels of Fig. 6.

We now discuss, why the technique fails for larger coupling, $\varepsilon \gtrsim 0.6$. In the case of the Rayleigh oscillator, the value of ε was limited by the effect of locking to the drive, what makes the inference of the coupling function impossible. In the case

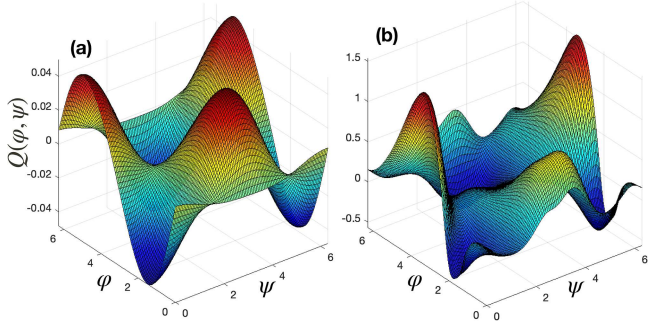

FIG. 6. Coupling functions of the harmonically driven Rayleigh oscillator for weak, $\varepsilon = 0.05$ [panel (a)], and strong, $\varepsilon = 0.5$ [panel (b)], forcing.

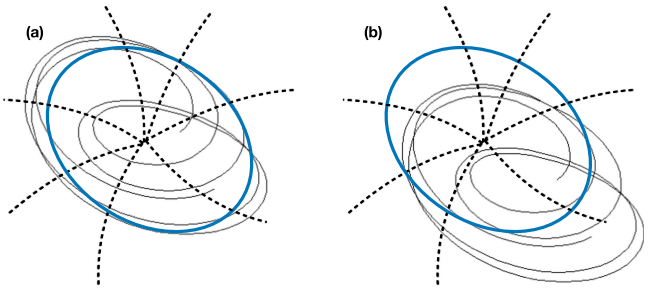

FIG. 7. Qualitative illustration of a limitation of the phase reduction technique. Here, the bold blue line presents the unperturbed limit cycle, while the black solid curve shows a piece of the trajectory in the phase space. In (a), the loops of the trajectory revolve around the origin. The point within one revolution goes sequentially through all the isochrons (dashed lines) and the phase, therefore, grows monotonically. In (b), several loops do not revolve around the origin, what means that two points within each such loop have same phase (this also means that the phase is not monotonically growing). Thus, the definition of a cycle becomes ambiguous and the phase reduction fails.

of the Rössler system, the reason for failure is different. Here, the torus of the driven system in the phase space becomes strongly shifted with respect to the original cycle, so that the trajectory starts to cross "wrong" isochrons, see Fig. 7. In other words, the isochrons do not anymore represent a proper phase coordinate on the attracting torus. This results in a non-monotonic growth of the isochron-based phase, manifested by the violation of the "good-phase-condition" $\dot{\varphi} > 0$, cf. Fig. 4(b).

IV. DISCUSSION

In summary, we have developed a simple and efficient technique for numerical determination of the phase of a perturbed self-sustained oscillator from the equations formulated in state space variables. In its turn, the computation of the phase provides an easy way to reconstruct Eq. (4) for the phase dynamics, where the coupling function is given on an equidistant grid within $0 \le \varphi < 2\pi, 0 \le \psi < 2\pi$, or/and can be given by a double Fourier series in phases φ, ψ. For transparency of the presentation, we have considered driven systems, but an extension to the case of two coupled oscillators is straightforward. The phases can be obtained in the same way for a network of more than two oscillators as well, but the reconstruction of high-dimensional coupling functions requires extremely long data sets and becomes unfeasible, see a discussion in Ref. 10.

We stress here that our approach heavily relies on the availability of full dynamical equations as they are needed to determine the true phase based on the isochrons. The existing approaches for phase reconstruction from a scalar time series[10,12–14] do not provide the exact isochron-based phase, what appears to be crucial for the proper reconstruction of the coupling function. Development of advanced phase reconstruction methods is a subject of ongoing research and our approach provides an easy way to test their efficiency.

We have demonstrated that reduction to the phase description is valid even for quite strong driving/coupling and have confirmed numerically that the coupling function can be represented as a power series of the driving strength. The suggested algorithm for correct computation of the phase offers also an easy way to obtain the PRC of the system. However, our simulations show that description in terms of an (amplitude-dependent) PRC fails for strong coupling: one cannot generalize the Winfree-type coupling (5) to the full coupling function as $Q(\varphi,\psi,\varepsilon) = Z(\varphi,\varepsilon)p(\psi)$. This fact has strong implications for the time-series analysis of experimental data: if only data for a particular forcing amplitude are available, then, to be safe, one should assume a nonlinear regime and reconstruct the coupling function in its general form, $Q(\varphi,\psi)$, but not in the Winfree representation. Thus, one does not have to rely on the assumption of week coupling that can hardly be checked if only passive observation of oscillatory system is possible. Furthermore, if data for several amplitudes of forcing are available, the nonlinearity level could be potentially revealed from such an experiment.

We have illustrated two effects that limit the applicability of the approach. The first one is synchronization of the system to the external perturbation, as in the case of the Rayleigh oscillator. In the example of the Rössler system, the technique does not work if a strong perturbation makes the mapping $\Phi(\mathbf{X})$ not unique within one cycle. Finally, we expect that the approach can also fail if a very strong coupling destroys the smooth torus in the phase space, see Ref. 11 for a rigorous treatment and Ref. 4 for a qualitative discussion of this mechanism.

Our results open a way for further studies on phase reduction, e.g., they provide a numerical tool for an analysis of the dependence of high-order terms of the coupling functions on parameters of the driving, with the utmost goal to extend the analytical techniques. Finally, our approach can be used as a benchmark for the testing technique for coupling functions reconstruction from data,[12–15] where only one observable of the system is available.

ACKNOWLEDGMENTS

The work was supported by ITN COSMOS (funded by the European Union Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant Agreement No. 642563). The numerical part of this work was supported by the Russian Science Foundation (Rayleigh model: Grant No. 17-12-01534; Rössler model: Grant No. 14-12-00811).

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