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NONLINEAR PHENOMENA

Breathing Chimera in a System of Phase Oscillators

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Chimera states consisting of synchronous and asynchronous domains in a medium of nonlinearly coupled phase oscillators have been considered. Stationary inhomogeneous solutions of the Ott–Antonsen equation for a complex order parameter that correspond to fundamental chimeras have been constructed. The direct numerical simulation has shown that these structures under certain conditions are transformed to oscillatory (breathing) chimera regimes because of the development of instability.

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1. INTRODUCTION

A number of key fundamental phenomena inherent in nonlinear oscillatory media of various natures can be considered in the phase approximation [1, 2]. In particular, synchronization and its various manifestations belong to these phenomena [2, 3]. The transition from particular models to a universal description by means of dynamic equations for phase variables makes it possible to reveal common characteristics of the behavior of physical, chemical, biological, and social systems [2].

The formation of chimera states, which are characterized by the coexistence of synchronous and asynchronous groups of oscillators, in ensembles of identical elements is one of the most attractive and intriguing effects. The possibility of formation of such long-lived nontrivial states was revealed for the first time in 2002 by Y. Kuramoto and D. Battogtokh [4]. In the past 15 years, chimera regimes were found numerically in a wide set of spatially distributed models [5]. Furthermore, the existence of these regimes was confirmed experimentally [5–8]. Chimeras are currently under active theoretical and laboratory studies. However, many problems remain unsolved. In particular, necessary and sufficient conditions for the appearance of chimera structures are unknown, these structures are classified incompletely, and it is not reliably determined when these structures are stable and transient [5].

In this work, we study a ring system of nonlinearly coupled identical phase oscillators. To analyze chimeras in this system, we use the Ott–Antonsen approach [9, 10], which allows obtaining closed equations for a complex order parameter. Using this approach, we first construct "stationary" chimeras and analyze their linear stability. Then, performing numerical simulations within the initial model, we show that such structures at certain parameters can be transformed to oscillatory (breathing) long-lived chimera states or to more complex regimes with an irregular behavior of the local order parameter.

2. MODEL

We consider a system of nonlocally coupled identical phase oscillators, which have the natural frequency ω and are continuously distributed in the interval $[0, L)$ with periodic boundary conditions at its boundaries. This configuration is equivalent to the situation where elements of a one-dimensional oscillatory medium are located on a ring with the length L. We describe this system by a dynamic variable $\phi(x,t)$, which is specified at each point $x \in [0, L)$ and satisfies the integrodifferential equation [5, 11]

$$\partial_t \phi(x,t) = \omega + [\hat{H}(x,t)\, e^{-i\phi(x,t) - i\alpha(\hat{H}(x,t))}], \qquad (1)$$

where the field acting on oscillators is defined in terms of the convolution operator:

$$\hat{H}(x,t) = \int_0^L G(x-\tilde{x}) \exp(i\phi(\tilde{x},t)) d\tilde{x}. \qquad (2)$$

Here, the kernel $G(y)$ characterizes the interaction in the considered medium and is normalized to unity.

We take the kernel $G(y)$ in the form

$$G(y) = \kappa \cosh(\kappa(|y| - L/2))/2\sinh(\kappa L/2), \qquad (3)$$

which approximates well the case of a weakly nonlinear coupling [11]. We emphasize that Eq. (3) at $\kappa L \to \infty$ is transformed to the kernel $G_{\mathrm{KB}}(y) = \frac{\kappa}{2}\exp(-\kappa|y|)$ used in [4]. The quantity $\alpha(\hat{H})$ in Eq. (1) determines the phase shift of the force acting on oscillators and is most often taken as constant α_0 [5]. However, according to [12–14], nonlinear effects of the phase shift significantly affect the dynamics of the system. To take into account these effects, we set $\alpha(\hat{H}) = \alpha_0 + \alpha_1|\hat{H}|^2$ as in [14].

In contrast to most of the previous studies, we primarily focus on possible states of the considered one-dimensional oscillatory medium depending on its length L. It is also noteworthy that Eqs. (1)–(3) are invariant under the scaling transformation in x; for this reason, $\kappa = 1$ can be taken without loss of generality.

According to numerical calculations, the system specified by Eqs. (1)–(3) exhibits complex regimes of behaviors where groups of elements rotate synchronously, but the motion of a significant part of phase oscillators is asynchronous. Such spatial structures are called chimeras [4–8, 11, 14, 15]. They are characterized by the presence of sections in the interval $[0, L)$ where the dynamic variable $\phi(x,t)$ is not a smooth function of x. However, using the procedure of averaging over a small neighborhood of the point x, one can introduce a local order parameter $Z(x,t) = \langle e^{\mathrm{i}\phi}\rangle_{\mathrm{loc}}$, which is a continuous complex-valued function of the coordinate x and time t and satisfies the inequality $|Z(x,t)| \le 1$. In regions where $|Z(x,t)| = 1$, the behavior of neighboring elements of the medium is synchronous. When $|Z(x,t)| < 1$, phase oscillators rotate asynchronously. The authors of [9, 10] proposed a reduction that makes it possible to establish a relation between $Z(x,t)$ and the probability density $\rho(\phi, x, t)$ of the distribution $\phi(x,t)$. In addition, it was shown in [9, 10] that the evolution of $Z(x,t)$ is described by the Ott–Antonsen equation (see also [11, 14, 15])

$$\partial_t Z = \mathrm{i}\omega Z + \left(e^{-\mathrm{i}\alpha(H)}H - e^{\mathrm{i}\alpha(H)}H^* Z^2\right)/2. \quad (4)$$

Here,

$$H(x,t) = \int_0^L G(x-\tilde{x})\, Z(\tilde{x},t)\, d\tilde{x} \quad (5)$$

is an integral operator similar to Eq. (2) applied to $Z(x,t)$ and $\alpha(H) = \alpha_0 + \alpha_1|H|^2$.

Further, using Eq. (3), it is easy to pass from Eq. (5) to the equivalent differential equation

$$\partial^2_{xx}H - H = -Z \quad (6)$$

with periodic boundary conditions $H(0,t) = H(L,t)$ and $\partial_x H(0,t) = \partial_x H(L,t)$.

3. STATIONARY SOLUTIONS OF THE OTT–ANTONSEN EQUATION

To analyze possible stationary (in absolute value) solutions of the problem specified by Eqs. (4) and (5) with Eq. (3) depending on the parameters α_0, α_1, and L, we represent the complex-valued functions $Z(x,t)$ and $H(x,t)$ in the form

$$Z(x,t) = z(x)e^{\mathrm{i}(\omega+\Omega)t}, \quad H(x,t) = h(x)e^{\mathrm{i}(\omega+\Omega)t}, \quad (7)$$

where Ω is a parameter.

Below, for definiteness, we consider the case where $\alpha_0, \alpha_1 \in [0, \pi/2]$ and $\Omega \le 0$, which corresponds to the choice in [4, 11, 14].

We first consider states homogeneous in x whose existence is independent of the length L of the system. Assuming $z(x) = z_0$ and $h(x) = h_0$, substituting Eq. (7) into Eqs. (4) and (5), and using the normalization of the kernel $G(y)$, we can easily conclude that two solutions (i) $|h_0| = |z_0| = 0$ with an arbitrary Ω value and (ii) $|h_0| = |z_0| = 1$ at $\Omega = \Omega_s = -\sin(\alpha_0 + \alpha_1)$ exist at any α_0 and α_1 values in the range from 0 to $\pi/2$. The former solution corresponds to the completely asynchronous behavior of elements of the medium under consideration. In the latter case, all phase oscillators rotate synchronously with the frequency $\omega + \Omega_s$. At $\alpha_1 > \pi/2 - \alpha_0$, there is an additional stationary homogeneous solution for which $|h_0| = |z_0| = \sqrt{(\pi - 2\alpha_0)/2\alpha_1}$ and $\Omega = \Omega_{ps} = ((2\alpha_0 - \pi)/2\alpha_1 - 1)/2$. This state is called partially synchronous [14] because $0 < |z_0| < 1$ for it.

To seek stationary inhomogeneous states, we use Eqs. (4) and (6). The substitution of Eq. (7) into Eq. (4) gives an algebraic relation, which can be treated as a quadratic equation for $z(x)$. For the relation between $z(x)$ and $h(x)$ to be unambiguous, one of two roots of this equation should be chosen. This can be easily done on the basis of the condition $|Z(x,t)| \le 1$. As a result, $z(x)$ is related to $h(x)$ as follows:

$$z(x) = -\mathrm{i}\left(\Omega + \sqrt{\Omega^2 - |h(x)|^2}\right)/h^*(x)e^{\mathrm{i}\alpha(h(x))}. \quad (8)$$

Substituting Eq. (7) into Eq. (6) and using Eq. (8), we obtain the following ordinary differential equation for the complex field $h(x)$:

$$h'' - h = i\left(\Omega + \sqrt{\Omega^2 - |h|^2}\right)/h^* e^{\mathrm{i}\alpha(h)}, \quad (9)$$

where the prime stands for the derivative with respect to the coordinate x.

We recall that the conditions of periodicity $h(0) = h(L)$ and $h'(0) = h'(L)$ should be satisfied at the bounds of the interval $[0, L)$. In view of the relation (8) between $z(x)$ and $h(x)$, it is also easy to conclude that phase oscillators rotate synchronously in regions where $|h(x)| \geq |\Omega|$ because $|z(x)| = 1$ in these regions, whereas the behavior of the elements of the medium is asynchronous in regions where $|h(x)| < |\Omega|$ because $|z(x)| < 1$ in this case.

An efficient procedure for the numerical calculation of stationary (in absolute value) inhomogeneous structures (in particular, chimeras) appearing in the space–time problem specified by Eqs. (4) and (5) with the kernel (3) was proposed in [11] on the basis of Eq. (9). The idea of this approach is in the search for periodic trajectories in the phase space of the system of third-order ordinary differential equations at a fixed Ω value to which Eq. (9) is reduced if the complex field $h(x)$ is represented in the form $h(x) = r(x)\exp(\mathrm{i}\theta(x))$, where $r(x)$ and $\theta(x)$ are real-valued functions of x (details see in [11]). It is noteworthy that the dimension can be reduced because the function $\theta(x)$ is defined up to a constant shift. For the same reason, $\theta(0) = 0$ can be set without loss of generality. As a result, the solution $h(x)$ that is exactly repeated with the period L, which is determined by the parameter Ω, is obtained from Eq. (9) at a given Ω value. The local order parameter $z(x)$ is calculated by Eq. (8) and the distribution of the initial dynamic variable $\phi(x,t)$ is then obtained from $z(x)$ taking into account Eq. (7) according to [9, 10, 15]. Thus, diverse (chimera and nonchimera) families of stationary inhomogeneous structures can be constructed for the model under consideration. Each such family is characterized by an individual dependence $\Omega(L)$ calculated in an implicit form using the procedure described above.

In this work, we analyze only solutions for which $|h(x)|$ has only two extrema (maximum and minimum) because these solutions in the case $\alpha_1 = 0$ correspond to fundamental single-cluster chimeras, which have one synchronous and one asynchronous domain and are stable with respect to linear perturbations [11]. Figure 1 shows the functions $\Omega(L)$ for two combinations of the coefficients α_0 and α_1 ($\alpha_1 \neq 0$). This figure also shows the spatial distributions $\phi(x, t = 0)$ reconstructed from local order parameters $z(x)$, which were found with three different Ω values and correspond to different branches of $\Omega(L)$ shown in Figs. 1a and 1b. In particular, the dependence $\Omega(L)$ at chosen α_0 and α_1 values when $\alpha_1 < \pi/2 - \alpha_0$ approaches closely the horizontal straight line $\Omega = \Omega_s$ determined by a synchronous state. In contradistinction, the chimera branch of $\Omega(L)$ at $\alpha_1 > \pi/2 - \alpha_0$ begins near the value $\Omega = \Omega_{ps}$ associated with a partially synchronous homogeneous state. We emphasize that an additional class of nonmonotonic periodic solutions $h(x)$ and $z(x)$ for which $|h(x)| < \Omega$ and $|z(x)| < 1$ at all values $x \in [0, L)$ can be found in the case of $\alpha_1 > \pi/2 - \alpha_0$. For such structures, the behavior of phase oscillators in a certain region becomes more synchronous as compared to other regions of the medium, but the complete synchronization of motion does not occur (see Fig. 1e). This class of solutions appears near considered stationary chimeras and then merges with the homogeneous partially synchronous state (see the inset of Fig. 1b).

4. BREATHING CHIMERAS

We analyze the time stability of structures considered in the preceding section. First, we linearize the integrodifferential equation (4) with the integral operator (5) and the kernel (3) near one of the stationary solutions (7), which are characterized by the parameter Ω and length L. To this end, we represent $Z(x,t)$ in the form

$$Z(x,t) = (z(x) + \tilde{z}(x,t))e^{\mathrm{i}(\omega+\Omega)t}, \tag{10}$$

where $\tilde{z}(x,t)$ describes x-periodic small deviations from the $z(x)$ profile. As a result, we obtain the expression

$$\begin{aligned}\partial_t \tilde{z} = &-\left(\mathrm{i}\Omega + e^{\mathrm{i}\alpha(h)} z h^*\right)\tilde{z} + \left(e^{-\mathrm{i}\alpha(h)}\tilde{h} - e^{\mathrm{i}\alpha(h)} z^2 \tilde{h}^*\right)/2 \\ &- \mathrm{i}\alpha_1\left(e^{-\mathrm{i}\alpha(h)} h + e^{\mathrm{i}\alpha(h)} z^2 h^*\right)\left(h^*\tilde{h} + h\tilde{h}^*\right)/2,\end{aligned} \tag{11}$$

where the relation between $\tilde{h}(x,t)$ and $\tilde{z}(x,t)$ is similar to Eq. (5).

Using Eq. (11), it is easy to determine the type of stability of stationary homogeneous solutions by considering perturbations $\tilde{z}(x,t) \propto \exp(\lambda t + \mathrm{i}k_n x)$, where $k_n = 2\pi n/L$ ($n \in \mathbb{N}$), and taking into account that the action of the convolution operator is reduced to multiplication by $\left(1 + k_n^2\right)^{-1}$. As a result, at $\alpha_1 < \pi/2 - \alpha_0$, the fully asynchronous state is unstable, whereas the synchronous state is stable; in the opposite case, both of these homogeneous solutions are unstable. For the partially synchronous state, which exists only at $\alpha_1 > \pi/2 - \alpha_0$, $\mathrm{Re}\lambda < 0$ holds only if k_n is larger than a certain critical value $k_{\mathrm{cr}}(\alpha_0, \alpha_1)$. Consequently, when $\alpha_1 > \pi/2 - \alpha_0$, stable spatially homogeneous regimes are absent above a certain critical length $L_{\mathrm{cr}}(\alpha_0, \alpha_1) 2\pi/k_{\mathrm{cr}}(\alpha_0, \alpha_1)$ [14].

The analysis of the stability of the stationary inhomogeneous solution of the Ott–Antonsen equation (4) with the integral operator (5) and the kernel

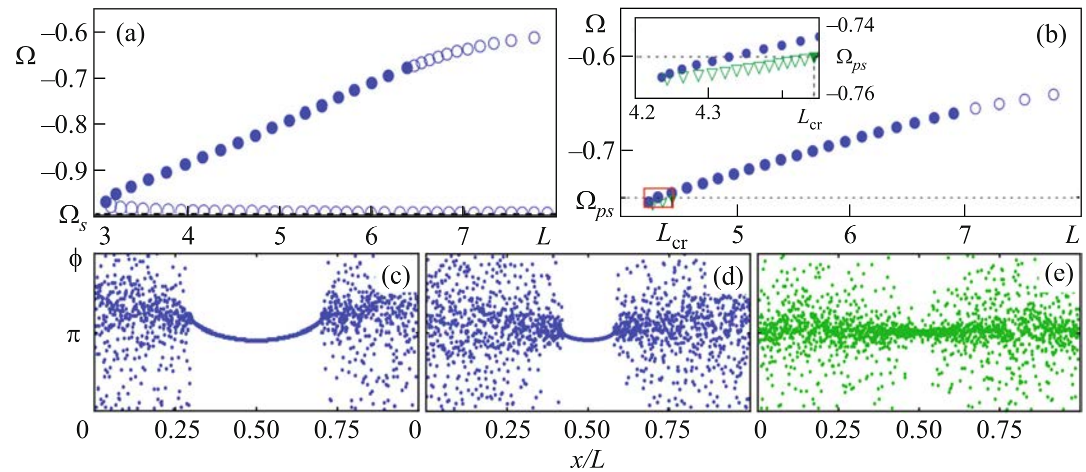

Fig. 1. (Color online) (a, b) Dependences $\Omega(L)$ for stationary inhomogeneous solutions of the Ott–Antonsen equation at (a) $\alpha_0 = 1.1$, $\alpha_1 = 0.4$ and (b) $\alpha_0 = 0.4\pi$, $\alpha_1 = 0.2\pi$. Circles and triangles correspond to single-cluster chimeras and partially synchronous states of the system, respectively. Closed and open points correspond to stable and unstable structures, respectively. (c–e) Spatial distributions of phase oscillators reconstructed from local order parameters $z(x)$ calculated with the parameters (c) $\alpha_0 = 1.1$, $\alpha_1 = 0.4$, and $\Omega = -0.7$; (d) $\alpha_0 = 0.4\pi$, $\alpha_1 = 0.2\pi$, and $\Omega = -0.7$; and (e) $\alpha_0 = 0.4\pi$, $\alpha_1 = 0.2\pi$, and $\Omega = -0.755$. Panels (c) and (d) show stable single-cluster chimeras with the lengths $L \approx 6.147$ and 5.668, respectively. Panel (e) demonstrates a characteristic form of an inhomogeneous partially synchronous state (the length of the medium in this case is $L \approx 4.253$).

(3) is much more complicated primarily because of the specific features of the $z(x)$ profile for fundamental chimeras [11, 16, 17]. Omitting details of the analysis, we briefly present some of its points important for the understanding of the results below. We rewrite Eq. (11) in the form

$$\partial_t \tilde{\mathbf{z}}(x,t) = \left(\hat{\mathbf{M}}(x) + \hat{\mathbf{K}}(x)\right)\tilde{\mathbf{z}}(x,t), \quad (12)$$

where $\tilde{\mathbf{z}}(x,t)$ is the two-component vector consisting of the real $\tilde{z}_{\mathrm{Re}}(x,t)$ and imaginary $\tilde{z}_{\mathrm{Im}}(x,t)$ parts of the complex function $\tilde{z}(x,t)$, $\hat{\mathbf{M}}(x)$ is the multiplicative operator, and $\hat{\mathbf{K}}(x)$ is the integral operator, which is compact for any piecewise-smooth kernel $G(y)$. Lengthy expressions for $\hat{\mathbf{M}}$ and $\hat{\mathbf{K}}$ are not presented.

According to Eq. (12), to determine the character of the behavior of small perturbations $\tilde{z}(x,t)$, it is sufficient to find the eigenvalues λ of the time-independent composite operator $\hat{\mathbf{M}} + \hat{\mathbf{K}}$. Since $\hat{\mathbf{K}}$ is compact, the significant spectrum λ_e of the sum $\hat{\mathbf{M}} + \hat{\mathbf{K}}$ coincides with the corresponding spectrum of the first term $\hat{\mathbf{M}}$. Using this property and relation (8) between $z(x)$ and $h(x)$, one can show that λ_e can be either an imaginary or negative real number. In particular, the ordered set λ_e for chimera states has a T shape [11, 16, 17]. Therefore, the stability of the studied regimes of motion of the system of phase oscillators is determined only by the point spectrum λ_p of the operator $\hat{\mathbf{M}} + \hat{\mathbf{K}}$. However, the eigenvalues λ_p cannot be reliably calculated by the direct method of discretization and replacement of integral operators by a large-dimension matrix. Such procedures hardly affect the eigenvalues λ_p, but they violate the compactness of the operator $\hat{\mathbf{K}}$ and noticeably distort the form of the significant spectrum λ_e, which complicates the separation of true λ_p values [11]. For this reason, in addition to the standard approach, it is necessary to use a number of modifications whose details are described in [11].

The results of our numerical analysis are shown in Fig. 1 by different points. Closed and open points correspond to stable and unstable stationary inhomogeneous solutions, respectively. In particular, it is seen in Fig. 1a that two branches $\Omega(L)$ of single-cluster chimeras appear when the size L of the oscillatory medium becomes equal to $L_{1a} \approx 3.127$. One of them is close to the straight line $\Omega = \Omega_s$ up to $L_{2a} \approx 10.418$. All solutions belonging to this branch are unstable. The second of the fundamental chimera structures marked in Fig. 1a (see Fig. 1c) is stable in the range from L_{1a} to $L_{3a} \approx 6.403$. A bifurcation similar to a saddle–node bifurcation occurs in fact at L_{1a}. A similar picture is observed in the situation shown in Fig. 1b. However, an unstable nonchimera state (see Fig. 1e) and a stable chimera (see Fig. 1d) appear in this case near $L_{1b} \approx 4.241$. As was mentioned above, the first class of solutions merges with the homogeneous par-

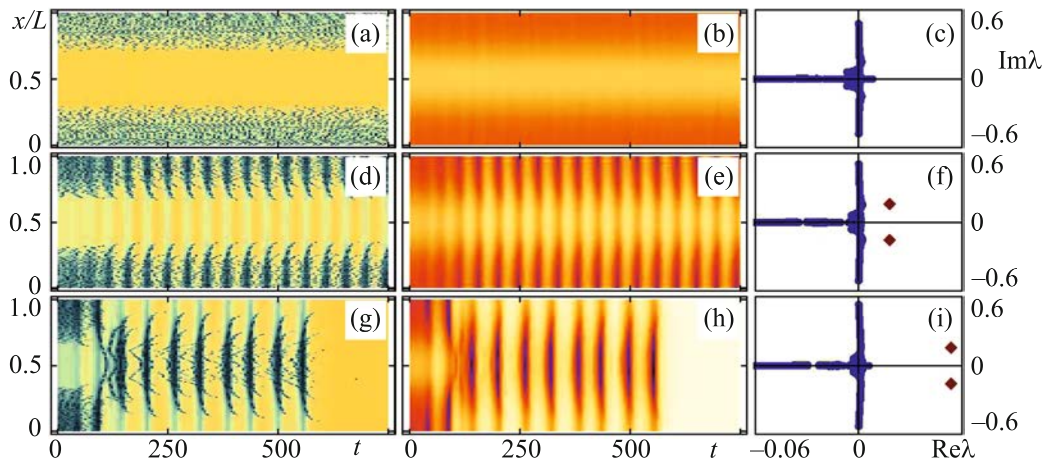

Fig. 2. (Color online) (a, d, g) Dynamics of the spatial distribution of individual frequencies $\partial_t\phi(x,t)$ of phase oscillators. Regions with synchronous and asynchronous motions of neighboring elements of the medium are clearly seen. (b, e, h) Dynamics of the absolute value of the complex field $\hat{H}(x,t)$. The results were obtained by the direct numerical simulation within a discrete analog of the initial model given by Eqs. (1)–(3) with the parameters $\omega = 1$, $\alpha_0 = 1.1$, $\alpha_1 = 0.4$ and the length of the system $L \approx$ (a, b) 6.147, (d, e) 6.668, and (g, h) 7.231 and with the initial state close to a single-cluster chimera at a given length L. (c, f, i) Spectrum λ of linear perturbations for the corresponding stationary inhomogeneous solutions of the Ott–Antonsen equation (4) with the integral operator (5) with the kernel given by Eq. (3). Circles are the eigenvalues corresponding to the significant part of the spectrum λ_e. In view of the problems with discretization, this set consists of not only real negative and imaginary numbers, although it has the characteristic T shape. Diamonds are eigenvalues belonging to the point spectrum λ_p responsible for instability; these eigenvalues are separated from the total set by means of a procedure described in [11].

tially synchronous state when the length L_{2b} coincides with $L_{\rm cr} \approx 4.443$. Stationary chimeras lose their stability at $L_{3b} \approx 6.902$.

Our direct numerical simulation (with 1000 points per unit length along the x axis) within a discrete analog of the model specified by Eqs. (1)–(3) confirms that, as the length L of the oscillatory medium exceeds the critical value L_* depending on the coefficients α_0 and α_1, chimera distributions found by means of the reduction of the Ott–Antonsen equation really lose their stability. In particular, L_* at $\alpha_0 = 1.1$ and $\alpha_1 = 0.4$ coincides with L_{3a}, whereas L_* at $\alpha_0 = 0.4\pi$ and $\alpha_1 = 0.2\pi$ is equal to L_{3b}. Figures 2 and 3 demonstrate the process of destruction of stationary chimeras at $L > L_*(\alpha_0,\alpha_1)$ for each of these sets of α_0 and α_1. According to detailed calculations, this process in the range $L_*(\alpha_0,\alpha_1) < L < L_{**}(\alpha_0,\alpha_1)$ occurs according to the same scenario for various combinations of the parameters α_0 and α_1. As a result of the development of instability, the system of phase oscillators passes to a regime characterized by the oscillatory variation of the spatial structure (in particular, of the integral quantity $\hat{H}(x,t)$) and by the simultaneous existence of domains with the synchronous and asynchronous behaviors of elements of the medium (see Figs. 2d, 2e and Figs. 3d, 3e). The revealed oscillating (breathing) chimera states exist for a long time interval (4000 units) and do not decay at the addition of small perturbations. The scheme of appearance of such a dynamics is similar to the Andronov–Hopf bifurcation in lumped element models when a stable limit cycle appears from the stable equilibrium state at the variation of control parameters and the stable equilibrium state becomes unstable. However, in our case, a typical continuous transition of a pair of eigenvalues from the left complex half-plane to the right half-plane does not occur, but points of the discrete spectrum λ_p of linear perturbations of stationary chimeras responsible for instability evolve from the imaginary axis (see Figs. 2c, 2f, 2i and Figs. 3c, 3f, 3i).

As the length L increases from $L_*(\alpha_0,\alpha_1)$ to $L_{**}(\alpha_0,\alpha_1)$, the period of the resulting oscillatory chimera regime increases. Beginning with $L_{**}(\alpha_0,\alpha_1)$ ($L_{**} = L_{4a} \approx 7.012$ at $\alpha_0 = 1.1$, $\alpha_1 = 0.4$ and $L_{**} = L_{4b} \approx 7.767$ at $\alpha_0 = 0.4\pi$, $\alpha_1 = 0.2\pi$), stationary chimeras are no longer transformed to breathing ones. At $\alpha_1 < \pi/2 - \alpha_0$ and $L > L_{**}(\alpha_0,\alpha_1)$, in the process of destruction of fundamental chimeras, structural oscillations are first observed and all elements of the medium are synchronized (see Figs. 2g and 2h), which is expectable because the homogeneous synchronous distribution is stable in this case. When $\alpha_1 > \pi/2 - \alpha_0$ and $L > L_{**}(\alpha_0,\alpha_1)$, all states considered above are unstable. The direct numerical

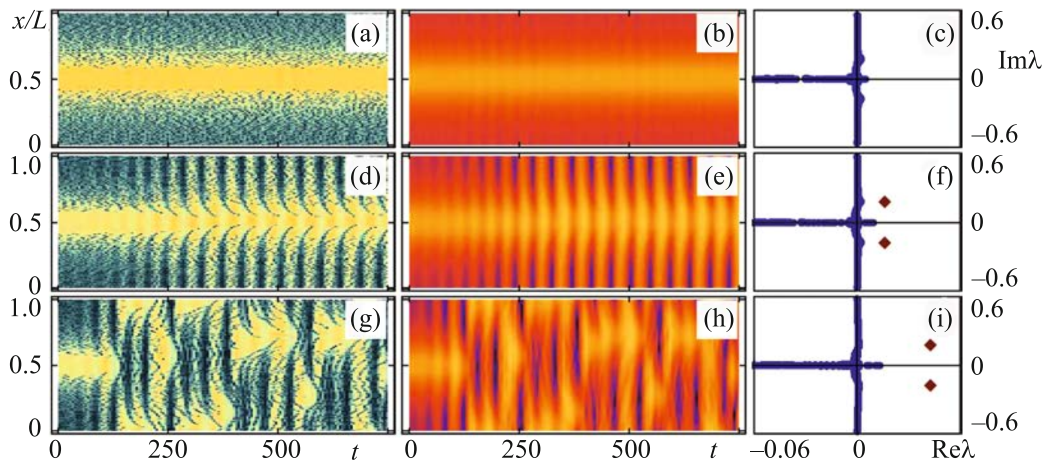

Fig. 3. (Color online) Same as in Fig. 2, but at $\omega = 1$, $\alpha_0 = 0.4\pi$, $\alpha_1 = 0.2\pi$, and the length of the medium $L \approx$ (a–c) 6.899, (d–f) 7.2905, and (g–i) 7.767.

simulation shows that the system in this case exhibits complex space–time regimes with irregular time evolution of synchronous and asynchronous domains (see Figs. 3g and 3h).

5. CONCLUSIONS

To summarize, oscillating (breathing) chimeras have been found for the first time in a system of nonlinearly coupled identical phase oscillators with nonlinear delay continuously distributed on a ring.

The authors of [18, 19] mentioned the possibility of existence of modulated chimera regimes but in discrete populations rather than in a continuous medium. A configuration of two interacting groups of globally coupled phase oscillators was studied in [18]. The generalization of this configuration to the case of three such populations was discussed in [19]. Using the Ott–Antonsen approach, the authors of [18, 19] passed from the initial problem to the study of the time behavior of two or three order parameters whose dynamics is described by low-order ordinary differential equations. The analysis has shown that equilibrium states for these equations corresponding to chimeras can undergo the Andronov–Hopf bifurcation and can be transformed to stable limit cycles, which were attributed to specific regimes with a periodically varying degree of synchronization of oscillators in one of the groups.

Breathing chimeras found in this work provide a positive answer to the question formulated in [18] on the existence of such chimeras in a system of identical phase oscillators distributed on a ring. We emphasize that a similar effect was detected in [15]. However, for the formation of oscillations of chimera states, Laing [15] artificially introduced the inhomogeneous delay $\alpha = \alpha(x)$, used a different kernel $G(y)$, and assumed that the natural frequencies of elements of the medium are randomly dispersed. In [15], the dependence $\alpha = \alpha(x)$ was not constant and satisfied periodic boundary conditions at the bounds of the considered spatial interval. Our nonlinear function $\alpha(h)$ also has this property. This circumstance is apparently decisive for the formation of breathing chimeras.

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