Eur. Phys. J. Special Topics **226**, 1921–1937 (2017)

DOI: 10.1140/epjst/e2017-70039-y

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Competing influence of common noise and desynchronizing coupling on synchronization in the Kuramoto-Sakaguchi ensemble*

Denis S. Goldobin[1,2,a], Anastasiya V. Pimenova[1,b], Michael Rosenblum[3,c], and Arkady Pikovsky[3,4,d]

[1] Institute of Continuous Media Mechanics, UB RAS, 614013 Perm, Russia
[2] Department of Theoretical Physics, Perm State University, 614990 Perm, Russia
[3] Institute for Physics and Astronomy, University of Potsdam, 14476 Potsdam-Golm, Germany
[4] Department of Control Theory, Nizhny Novgorod State University, 606950 Nizhny Novgorod, Russia

Received 4 February 2017 / Received in final form 8 March 2017
Published online 21 June 2017

Abstract. We describe analytically synchronization and desynchronization effects in an ensemble of phase oscillators driven by common noise and by global coupling. Adopting the Ott-Antonsen ansatz, we reduce the dynamics to closed stochastic equations for the order parameters, and study these equations for the cases of populations of identical and nonidentical oscillators. For nonidentical oscillators we demonstrate a counterintuitive effect of divergence of individual frequencies for moderate repulsive coupling, while the order parameter remains large.

1 Introduction

Synchronization in ensembles of oscillators with a weak mutual coupling and, specifically, for the case of global coupling, is a well understood phenomenon [1,2]. This effect is important for various physical applications such as lasers and Josephson junctions, engineered systems (colorfully highlighted by the famous incident with the Millennium bridge in London [3]), neuronal networks (including pathological synchronization of neuronal activity in neurodegenerative diseases) [4], colonies and populations of living organisms, and even for many social systems.

Synchronization of ensembles of oscillators can be also caused by common noise or non-periodic external action [5]; this interesting phenomenon is known in various disciplines under different names: "reliability" for neurons [6], "consistency" for lasers [7], synchronization of non-interacting populations by common varying conditions [8]. Although the mathematical theory of synchronization by common noise has

* Supplementary material in the form of one pdf file available from the Journal web page at: https://doi.org/10.1140/epjst/e2017-70039-y

[a] e-mail: Denis.Goldobin@gmail.com
[b] e-mail: Anastasiya.Pimenova@gmail.com
[c] e-mail: mros@uni-potsdam.de
[d] e-mail: pikovsky@uni-potsdam.de

been significantly advanced [9–11], its physical mechanism is not as evident as the mechanism of synchronization by mutual coupling (see, e.g., [12,13,17]). An important difference is that the synchronization by common noise is not accompanied by the frequency locking or by the frequency entrainment. Another peculiar property is that a weak Gaussian intrinsic noise in two otherwise identical oscillators, synchronized by the common noise, produces state deviations which are non-Gaussian, but Lorentzian ones.

Remarkably, while mutual coupling can both synchronize and desynchronize oscillators in a population (i.e., it can be either attractive or repulsive), noise can only synchronize them. The interplay of de-/synchronization by global coupling and synchronization by common noise is especially interesting due to the difference in their basic mechanisms. Desynchronization by repulsive coupling counteracts synchronization by common noise in a non-trivial way; they cannot simply cancel each other. These subtle effects are important if one tries, say, to counteract noise-induced synchrony by a repulsive coupling, or vice versa, what may be relevant for many technical and biological systems.

Recently, we have reported on nontrivial effects due to competition of the common noise and desynchronizing global coupling in an ensemble of globally coupled Kuramoto oscillators [18]; these results extend previous studies [19,20]. For a moderate desynchronizing coupling, the synchronizing effect of common noise prevails and a synchronous regime establishes. However, for the case of slightly nonidentical oscillators, the average frequencies of individual oscillators are repelled, not attracted. For a synchronizing coupling, the perfect frequency locking does not occur, although the average frequencies of individual oscillators are attracted to each other. In the present paper we provide a more detailed study of these phenomena, and extend the description from the case of the purely dissipative coupling as in [18] to a general global coupling of Kuramoto-Sakaguchi type.

Our basic model is an ensemble of infinitely many phase oscillators with the Kuramoto-Sakaguchi coupling and common noise. This model provides an opportunity for a comprehensive analytical treatment via Watanabe-Strogatz and Ott-Antonsen approaches [21–24] and, thus, allows for a deeper insight into the subtle aspects of the interplay between the mechanisms of de/synchronization by coupling and by common noise. The governing equations are:

$$\dot{\varphi} = \Omega + \mathrm{Im}\left(H(t)e^{-i\varphi}\right) \,, \tag{1}$$

where natural frequencies Ω have distribution $g(\Omega)$. In this thermodynamic limit, the probability density $w(\varphi, t, \Omega)$ of oscillators with natural frequency Ω admits, according to [24], a solution parametrized by a single complex quantity $a(\Omega, t)$: $w(\varphi, t, \Omega) = \frac{1}{2\pi}(1 + \sum_{j=1}^{\infty}[(a(\Omega,t)^j e^{ij\varphi} + c.c.])$, where $a(\Omega, t)$ obeys

$$\dot{a}(\Omega, t) = -i\Omega a + \frac{H^*(t)}{2} - \frac{H(t)}{2}a^2 \,. \tag{2}$$

For the ensemble of globally coupled oscillators driven by common noise and subject to the Kuramoto-Sakaguchi coupling, we set

$$H(t) = \mu R e^{-i\beta} e^{i\Phi} - \sigma\xi(t) \,,$$

where the complex mean field is defined as the average over the whole population

$$Re^{i\Phi} = \int_{-\infty}^{+\infty} \mathrm{d}\Omega\, g(\Omega) \int_{0}^{2\pi} \mathrm{d}\varphi\, e^{i\varphi} w(\varphi, t, \Omega) = \int_{-\infty}^{+\infty} \mathrm{d}\Omega\, g(\Omega)\, a^*(\Omega, t) \,.$$

The equations for the oscillators then take the form

$$\dot{\varphi} = \Omega + \sigma\xi(t)\sin\varphi + \mu R\sin(\Phi - \varphi - \beta) \ . \tag{3}$$

Here σ is the common noise strength, $\xi(t)$ is the normalized Gaussian white noise signal, μ is the coupling strength, β characterizes the phase shift in the coupling, or, in other words, the relative contributions of the "active" ($\sim \cos\beta$) and the "reactive" ($\sim \sin\beta$) components of the coupling terms. The noise term in equation (3) is multiplied by $\sin\varphi$ because any external forcing on an oscillator appears in the phase approximation as this forcing multiplied by the phase sensitivity function, that is the sine function in our case.

The main goal of this paper is to extend the results of reference [18], where the case $\beta = 0$ has been considered, to a general situation with $\beta \neq 0$. We will see that the main parameters appearing in the theory are the effective coupling $\mu_\beta = \mu\cos\beta$ and the effective frequency $\Omega_\beta = \Omega - \mu\sin\beta$; many results will be identical to those for $\beta = 0$ with the corresponding effective parameters. However, the behaviour of the average of the oscillators' frequencies will be more sophisticated; in particular, in the case of $\beta \neq 0$, we will observe a shift of the reference frequency, while it disappears for $\beta = 0$.

The paper is organized as follows. In Section 2, we consider the dynamics of the ensemble of identical oscillators, where perfect synchrony is possible. For identical oscillators the stability properties of the synchronous state and the time-average dynamics of the order parameter are derived for a general case. However, a more subtle characterization, namely, the probability density distribution for the order parameter and calculation of the rate of transition to synchrony, can be made analytically only with additional assumption of a high basic frequency of oscillations. In Section 3, we consider a realistic situation of the ensemble of oscillators with nonidentical frequencies. For nonidentical oscillators, the perfect synchrony becomes impossible, but one can derive the time-average value of the order parameter for a highly synchronous state and characterize the dynamics of the order parameter close to the state of asynchrony $J = 0$. Further, for the case of high basic frequency of oscillators, a comprehensive description of the dynamics of the order parameter is possible, including calculation of the probability density distribution. In Section 4, for the case of nonidentical oscillators, the average oscillators' frequencies are calculated. One observes no frequency locking, as in the noise-free case. However, the attraction of average frequencies (for the synchronizing coupling) and a nontrivial phenomenon of their repulsion (for desynchronizing coupling) accompany synchronization.

2 Ensemble of identical oscillators

For the case of identical oscillators $\Omega_k = \Omega$, and we can rewrite (2), taking into account that $a^* = R\exp(i\Phi)$, as

$$\begin{aligned}
\dot{R} &= \frac{\mu}{2}\cos\beta(1 - R^2)R - \frac{\sigma\xi(t)}{2}(1 - R^2)\cos\Phi \,, \\
\dot{\Phi} &= \Omega - \frac{\mu}{2}\sin\beta(1 + R^2) + \frac{\sigma\xi(t)}{2}\left(\frac{1}{R} + R\right)\sin\Phi \,.
\end{aligned} \tag{4}$$

It is convenient to introduce a new order parameter $J = R^2/(1 - R^2)$ (with the inverse relation $R = \sqrt{J/(1 + J)}$); in new variables we get stochastic equations in the Stratonovich sense:

$$\dot{J} = \mu\cos\beta\, J - \sigma\xi(t)\sqrt{J(1 + J)}\cos\Phi \,, \tag{5}$$

$$\dot{\Phi} = \Omega - \mu \sin\beta \frac{J+1/2}{J+1} + \sigma\xi(t) \frac{J+1/2}{\sqrt{J(1+J)}} \sin\Phi \,. \tag{6}$$

In terms of J, synchronization $R \lesssim 1$ corresponds to $J \to \infty$, while asynchrony $R = 0$ corresponds to $J = 0$. The latter equation system can be treated analytically for $J \gg 1$ (approaching the synchronized state) and $J \ll 1$ (evolution of the asynchronous state $J = 0$).

2.1 Stability of the synchronous state: $J \gg 1$

For $J \gg 1$, to the leading order, equations (5), (6) read

$$\dot{J} = \mu_\beta J - \sigma\xi(t) J \cos\Phi \,, \tag{7}$$

$$\dot{\Phi} = \Omega_\mu + \sigma\xi(t) \sin\Phi \,, \tag{8}$$

where $\mu_\beta \equiv \mu\cos\beta$ and $\Omega_\mu \equiv \Omega - \mu\sin\beta$. Equation (7) can be recast as $\frac{\mathrm{d}}{\mathrm{d}t}\ln J = \mu_\beta - \sigma\xi(t)\cos\Phi$, and the average rate of the exponential growth λ, namely the Lyapunov exponent, $\lambda \equiv \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \ln J \right\rangle$ reads

$$\lambda = \mu_\beta - \sigma\langle\xi(t)\cos\Phi\rangle = \mu_\beta + \sigma^2\langle\sin^2\Phi\rangle \,, \tag{9}$$

where the equality $\langle\xi(t)\cos\Phi\rangle = -\sigma\langle\sin^2\Phi\rangle$ follows from the Furutsu-Novikov formula. Positive value of λ means that the synchronous state $J = \infty$ is stable. One can see that λ has a positive contribution from the noise, and a contribution from the coupling which can change sign.

The dynamics of Φ is governed by equation (8) and is independent of J. Equation (8) yields the Fokker-Planck equation for the probability density $W(\Phi, t)$;

$$\frac{\partial}{\partial t}W + \frac{\partial}{\partial\Phi}(\Omega_\mu W) - \sigma^2 \frac{\partial}{\partial\Phi}\left(\sin\Phi \frac{\partial}{\partial\Phi}\big(\sin\Phi\, W\big)\right) = 0. \tag{10}$$

This Fokker-Planck equation possesses a steady-state solution (which is π-periodic)

$$W(\Phi) = \frac{C}{\sin\Phi} \int_\Phi^\pi \frac{\mathrm{d}\Phi_1}{\sin\Phi_1} e^{\frac{\Omega_\mu}{\sigma^2}(\cot\Phi_1 - \cot\Phi)},$$

where constant C is determined by the normalization condition $\int_0^{2\pi} W(\Phi)\,\mathrm{d}\Phi = 1$. Thus,

$$\langle\sin^2\Phi\rangle = \frac{\displaystyle\int_0^\pi \mathrm{d}\Phi\,\sin\Phi \int_\Phi^\pi \frac{\mathrm{d}\Phi_1}{\sin\Phi_1} e^{\frac{\Omega_\mu}{\sigma^2}(\cot\Phi_1 - \cot\Phi)}}{\displaystyle\int_0^\pi \frac{\mathrm{d}\Phi}{\sin\Phi} \int_\Phi^\pi \frac{\mathrm{d}\Phi_1}{\sin\Phi_1} e^{\frac{\Omega_\mu}{\sigma^2}(\cot\Phi_1 - \cot\Phi)}}\,, \tag{11}$$

where $\langle\cdot\rangle$ denotes averaging over different realization of noise.

The dependence of $\langle\sin^2\Phi\rangle$ on Ω_μ/σ^2 is plotted in Figure 1. This dependence reveals that the influence of common noise on the stability of the synchronous state [Eq. (9)] is more efficient for fast oscillations, and its efficiency monotonously

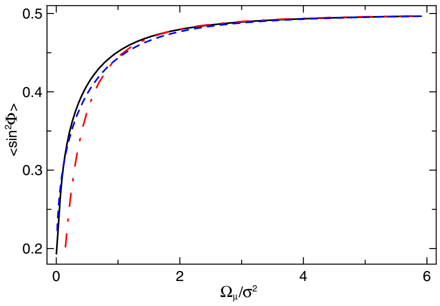

Fig. 1. The dependence of $\langle\sin^2\Phi\rangle$ on Ω_μ/σ^2 determines the average exponential rate of approach to synchronous state λ [see Eq. (9)]. It is plotted with the black solid line for the exact solution (11), with the blue dashed line for the Galerkin approximation (13)–(14), and with the red dash-dotted line for the asymptotic expansion (12). For the exact solution and the Galerkin approximation, $\langle\sin^2\Phi\rangle$ tends to a non-zero finite value (approximately 0.2) as $\Omega_\mu/\sigma^2 \to 0$.

decreases with the decrease of the natural frequency of oscillators. Below we provide two practical ways to calculate the Lyapunov exponent, instead of calculating the integrals in (11) directly.

Small noise limit $\sigma^2 \ll \Omega_\mu$. The integrals in equation (11) cannot be evaluated analytically. However, one can analyse the asymptotic behaviour of $\langle\sin^2\Phi\rangle$ for $\Omega_\mu \gg \sigma^2$. Using σ^2 as a small parameter in equation (10), one can evaluate the series $W(\Phi) = (2\pi)^{-1} + \sigma^2 W^{(1)}(\Phi) + \sigma^4 W^{(2)}(\Phi) + \dots$ and find $\langle\sin^2\Phi\rangle = \frac{1}{2} - \frac{\sigma^4}{8\Omega_\mu^2} + \dots$. Hence,

$$\lambda = \mu_\beta + \sigma^2\left[\frac{1}{2} - \frac{\sigma^4}{8\Omega_\mu^2} + \mathcal{O}\left(\frac{\sigma^6}{\Omega_\mu^3}\right)\right]. \tag{12}$$

Galerkin approximation. Here we employ (see supplementary material for details) a 2-parametric normalized function $W_{m,\Phi_0}(\Phi) = a_m[1 + m\sin^2(\Phi - \Phi_0)]^{-1/2}$, where $a_m = \left(4\mathrm{K}\sqrt{-m}\right)^{-1}$, as a Galerkin approximation function [25]. This allows us to evaluate approximately

$$\frac{\Omega_\mu}{\sigma^2} = \frac{\sqrt{1+m}}{m}\,\frac{\dfrac{2}{2+m}\dfrac{\mathrm{E}(\sqrt{-m})}{\mathrm{K}(\sqrt{-m})} - 1}{1 - \dfrac{1}{\sqrt{1+m}}\left(\dfrac{\pi}{2\mathrm{K}(\sqrt{-m})}\right)^2} \tag{13}$$

and

$$\langle\sin^2\Phi\rangle = \frac{1}{2+m}\,\frac{\mathrm{E}(\sqrt{-m})}{\mathrm{K}(\sqrt{-m})}. \tag{14}$$

With equations (13) and (14), one has a dependence between $\langle\sin^2\Phi\rangle$ and Ω_μ/σ^2, parameterised by m.

In Figure 1, one can see quite a fair agreement between the Galerkin approximation and the exact solution (11), and assess the applicability domain of the asymptotic formula (12).

2.2 Dynamics of $\langle J\rangle_\xi(t)$ and the evolution of asynchronous state $J = 0$

The equation for the dynamics of the average order parameter $\langle J\rangle_\xi$ (where $\langle\dots\rangle_\xi$ denotes averaging over noise realizations) can be derived from system (5)–(6): $\frac{\mathrm{d}}{\mathrm{d}t}\langle J\rangle_\xi = (\mu_\beta + \sigma^2)\langle J\rangle_\xi + \sigma^2/2$. For the system starting from the asynchronous state, $J(0) = 0$, the solution is $\langle J\rangle_\xi(t) = (e^{(\mu_\beta+\sigma^2)t} - 1)\,\sigma^2/[2(\mu_\beta+\sigma^2)]$. The condition for growth of $\langle J\rangle_\xi$

$$\mu_\beta + \sigma^2 > 0 \tag{15}$$

differs from the condition of positivity of the Lyapunov exponent

$$\mu_\beta + \sigma^2\langle\sin^2\Phi\rangle > 0\,, \tag{16}$$

because for the (15) the contribution of large values of J is weighted heavier. While equation (16) determines whether the system asymptotically (and irreversibly) tends to the perfect synchrony $J=\infty$, equation (15) determines whether the system tends to avoid the asynchronous state, but does not forbid excursions back to low-synchrony states from large values of J. Thus, the condition (15) is weaker than (16).

2.3 Transition from asynchrony ($J = 0$) to the synchronized state

There is a remarkable difference between the statistical properties of the evolution toward the states of perfect synchrony, $J=\infty$, and of asynchrony, $J=0$. Where the synchronous state is attracting, $\lambda = \mu_\beta + \sigma^2\langle\sin^2\Phi\rangle > 0$, the transition to it is irreversible, i.e., this state is an absorbing one. In contradistinction, the drift-attraction $\mu_\beta + \sigma^2 < 0$ for the asynchronous state does not mean that trajectories converge to this state. Indeed, since the noise term in (4) does not vanish at $R=0$, the noise "kicks-out" the system from this state. Thus, where the synchronous state is attracting, the transition to synchrony is unidirectional, and the transition rate can be of interest.

This transition rate can be generally found from the Fokker–Planck equation for the probability density $W(J,\Phi,t)$. For stochastic system (5)–(6), it reads

$$\frac{\partial}{\partial t}W + \frac{\partial}{\partial J}(\mu_\beta J\,W) + \frac{\partial}{\partial\Phi}\left(\left[\Omega - \mu\sin\beta\frac{J+1/2}{J+1}\right]W\right) - \sigma^2\hat{Q}^2W = 0\,, \tag{17}$$

where operator $\hat{Q}$ is defined as $\hat{Q} \equiv \frac{\partial}{\partial J}\left(-\sqrt{J(1+J)}\cos\Phi\right) + \frac{\partial}{\partial\Phi}\left(\frac{J+1/2}{\sqrt{J(1+J)}}\sin\Phi\right)$. This equation can be treated analytically for a realistic case of large frequency $\Omega \gg \mu \sim \sigma^2$.

2.3.1 Averaging over high frequency oscillations

For vanishing μ and σ, the probability density distribution is $W(J,\Phi,t) = (2\pi)^{-1}w(J)$, where $\int_0^\infty w(J)\,\mathrm{d}J = 1$. Hence, for $\mu\sim\sigma^2\ll\Omega$, one can assume $\mu = \sigma^2\mu_1$, $\Omega=\Omega_0$ and employ the standard method of multiple scales; $t_n = \sigma^{2n}t$ and $W = W^{(0)}(J,t_1,t_2,\dots) + \sigma^2W^{(1)}(J,\Phi,t_0,t_1,t_2\dots) + \dots$. To the leading order, equation (17) yields $W^{(0)} = (2\pi)^{-1}w(J,t_1,t_2,\dots)$. In the order σ^2, equation (17) yields

$$\frac{\partial W^{(1)}}{\partial t_0} + \Omega_0\frac{\partial W^{(1)}}{\partial\Phi} + \frac{\partial W^{(0)}}{\partial t_1} + \frac{\partial}{\partial J}\left(\mu_1\cos\beta\,J\,W^{(0)}\right) + \frac{\partial}{\partial\Phi}\left(-\mu_1\sin\beta\frac{J+1/2}{J+1}W^{(0)}\right) - \hat{Q}^2W^{(0)} = 0\,.$$

Integrating the latter equation over Φ from 0 to 2π, one finds $\frac{\partial}{\partial t_0}\int\limits_0^{2\pi} W^{(1)}\mathrm{d}\Phi +$ $\frac{\partial w(J,t_1)}{\partial t_1} + \frac{\partial}{\partial J}\left(\mu_1 \cos\beta\, J\, w(J,t_1)\right) - \frac{1}{2\pi}\int\limits_0^{2\pi} \hat{Q}^2 w(J,t_1)\,\mathrm{d}\Phi = 0$. To avoid linear growth of $W^{(1)}$ with t_0, one has to set the first term to zero. Calculation of the last integral yields $\frac{1}{2\pi}\int_0^{2\pi} \hat{Q}^2 w(J,t_1)\,\mathrm{d}\Phi = \frac{1}{2}\frac{\partial}{\partial J}(\sqrt{J(1+J)}[\frac{\partial}{\partial J}(\sqrt{J(1+J)}\,w) - \frac{J+1/2}{\sqrt{J(1+J)}}w])$. Thus, the probability density $w(J,t)$ is governed by the equation

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial J}\left(\left[\mu_\beta J + \frac{\sigma^2}{2}\left(J + \frac{1}{2}\right)\right] w\right) - \sigma^2 \hat{Q}_J^2 w = 0\,, \tag{18}$$

where $\hat{Q}_J(\cdot) \equiv \frac{1}{\sqrt{2}}\frac{\partial}{\partial J}\left(\sqrt{J(1+J)}\,(\cdot)\right)$. Equation (18) can be treated as the Fokker-Planck equation for the stochastic equation

$$\dot{J} = \mu_\beta J + \frac{\sigma^2}{2}(J + 1/2) + \sigma\sqrt{\frac{J(1+J)}{2}}\,\zeta(t)\,, \tag{19}$$

where effective noise ζ is Gaussian and delta-correlated, $\langle\zeta(t)\zeta(t+t')\rangle = 2\delta(t')$.

Strong desynchronising coupling ($\mu_\beta < -\sigma^2/2$). Here equation (18) admits a steady state probability density without probability flux:

$$w_0(J) = \left(\frac{2(-\mu_\beta)}{\sigma^2} - 1\right)(1+J)^{-\frac{2(-\mu_\beta)}{\sigma^2}}\,, \tag{20}$$

which means that there are no adsorbing states in the system. Recall, the synchronous state is not asymptotically attracting in this case ($\lambda < 0$). Probability density (20) yields the average value and the variance of the original mean field

$$\langle R\rangle = \left\langle\sqrt{\frac{J}{1+J}}\right\rangle = \frac{\sqrt{\pi}}{2}\frac{\Gamma\big(2(-\mu_\beta)/\sigma^2\big)}{\Gamma\big(2(-\mu_\beta)/\sigma^2 + 1/2\big)}, \quad \langle R^2\rangle = \left\langle\frac{J}{1+J}\right\rangle = \frac{\sigma^2}{2(-\mu_\beta)}\,.$$

Asymptotically attracting synchrony ($\mu_\beta > -\sigma^2/2$). Here only a solution with finite probability flux j can be formally written:

$$w_j(J) = \frac{2j}{\sigma^2}(1+J)^{2\mu_\beta/\sigma^2}\int_J^\infty \frac{\mathrm{d}J_1}{J_1}(1+J)^{-1-2\mu_\beta/\sigma^2}. \tag{21}$$

The probability "density" $w_j(J)|_{J\gg 1} \propto 1/J_1$ possesses a heavy tail, the integral of which diverges. Hence, after normalization, one finds $w(J<\infty) = 0$, while $\int_J^\infty w_1(J_1)\,\mathrm{d}J_1 = 1$, which corresponds to $j \to 0$ and also means that all states are accumulated at $J = \infty$.

2.3.2 Transition to the synchronous state: first passage time

In the case where the synchronous state is absorbing, an interesting statistical quantity is the characteristic time to reach it, starting from asynchrony. As the system can approach the perfect synchrony state only asymptotically, the transition time is infinite. However, one can consider how the system approaches the perfect synchrony state and pose the problem of the first passage time for some large value $\bar{J}$. For equation (19) [or Eq. (18)], the first passage time $T(J_0,\bar{J})$ from J_0 to $\bar{J}$ obeys

$$B(J_0)\frac{\partial^2 T(J_0,\bar{J})}{\partial J_0^2} + A(J_0)\frac{\partial T(J_0,\bar{J})}{\partial J_0} = -1\,, \tag{22}$$

with $T(\bar{J},\bar{J})=0$, $\big(\partial T(J_0,\bar{J})/\partial J_0\big)|_{J_0=\bar{J}}=0$, and $A(J_0)=\mu_\beta J_0+\frac{\sigma^2}{2}(J_0+1/2)$, $B(J_0)=\frac{\sigma^2}{2}J_0(1+J_0)$. Notice, in our problem $J_0=0$ is the boundary of the domain of possible states of the system. The solution to equation (22) is

$$T(J_0,\bar{J})=\int_{J_0}^{\bar{J}}\mathrm{d}J_1\int_{J_0}^{J_1}\frac{\mathrm{d}J_2}{B(J_2)}e^{-\int\limits_{J_2}^{J_1}\frac{A(J_3)}{B(J_3)}\mathrm{d}J_3}=\frac{2}{\sigma^2}\int_{J_0}^{\bar{J}}\frac{\mathrm{d}J_1}{J_1}\int_{J_0}^{J_1}\mathrm{d}J_2\frac{(1+J_2)^{2\mu_\beta\sigma^{-2}+2}}{(1+J_1)^{2\mu_\beta\sigma^{-2}+3}}.$$

Integrating over J_2 and assuming $J_0\to 0$, one obtains

$$\begin{aligned}T(0,\bar{J})&=\frac{\sigma^{-2}}{\mu_\beta\sigma^{-2}+3/2}\int_0^{\bar{J}}\frac{\mathrm{d}z}{z}\left(1-\frac{1}{(1+z)^{2\mu_\beta\sigma^{-2}+3}}\right)\\&=\frac{1}{\sigma^2}\left[\frac{\ln(1+\bar{J})}{\mu_\beta\sigma^{-2}+3/2}+\frac{\bar{J}^{-(2\mu_\beta\sigma^{-2}+3)}}{2(\mu_\beta\sigma^{-2}+3/2)^2}+\tau\left(\frac{2\mu_\beta}{\sigma^2},\bar{J}\right)\right],\end{aligned}\tag{23}$$

where $\tau(q,\bar{J})$ is small compared to the sum of the first and second terms in the brackets for $\bar{J}\gg 1$. For $2\mu_\beta\sigma^{-2}+3>0$, the first passage time is logarithmically large $\propto\ln\bar{J}$, meaning that the synchronous state attracts the system trajectories on average. For $2\mu_\beta\sigma^{-2}+3<0$, the first passage time diverges as a power law of $\bar{J}$, meaning that the synchronous state is strongly repelling and the passages of the system trajectories near it are rare events.

Physical picture for identical oscillators. We summarize the results of this section as the following qualitative picture. In the competition between coupling and noise there is a critical value of the coupling parameter $\mu_\beta^{\mathrm{crit}}=-\sigma^2/2$. For $\mu_\beta>\mu_\beta^{\mathrm{crit}}$ noise wins: eventually the ensemble gets fully synchronized. For $\mu_\beta<\mu_\beta^{\mathrm{crit}}$, repulsive coupling prevents full synchronization. However, the order parameter never vanishes, thus one observes partial synchrony with a fluctuating order parameter.

3 Nonidentical oscillators: transition to synchrony

Let us now consider the case of nonidentical natural frequencies Ω and assume the Lorentzian distribution of them with width γ: $g(\Omega)=\frac{\gamma}{\pi[\gamma^2+(\Omega-\Omega_0)^2]}$. According to reference [24], $a(\Omega)$ can be considered as analytic function on the lower half-plane, and one can evaluate $R\exp[-i\Phi]=\int_{-\infty}^{+\infty}\mathrm{d}\Omega\, g(\Omega)\,a(\Omega)=a(\Omega_0-i\gamma)$. Hence, equation (2) written for $a(\Omega_0-i\gamma)$ gives a closed equation for the order parameter; in terms of J and Φ, we obtain

$$\dot{J}=\mu_\beta J-2\gamma J(1+J)-\sigma\xi(t)\sqrt{J(1+J)}\cos\Phi,\tag{24}$$

$$\dot{\Phi}=\Omega_0-\mu\sin\beta\frac{J+1/2}{J+1}+\sigma\xi(t)\frac{J+1/2}{\sqrt{J(1+J)}}\sin\Phi.\tag{25}$$

3.1 Dynamics close to a highly synchronous state

For $\boldsymbol{J}\gg\mathbf{1}$, equations (24)–(25) read

$$\dot{J}=\mu_\beta J-2\gamma J^2-\sigma\xi(t)J\cos\Phi\,,\tag{26}$$

$$\dot{\Phi} = \Omega_{\mu,0} + \sigma\xi(t)\sin\Phi\,. \tag{27}$$

Similarly to equations (7) and (8), one can find $\langle \frac{\mathrm{d}}{\mathrm{d}t}\ln J\rangle = \mu_\beta - 2\gamma\langle J\rangle - \sigma\langle \xi(t)\cos\Phi\rangle = \mu_\beta + \sigma^2\langle \sin^2\Phi\rangle - 2\gamma\langle J\rangle$, where $\langle\sin^2\Phi\rangle$ is determined by equation (11). For nonidentical oscillators, $\gamma \neq 0$ and the system does not attain perfect synchrony; at a steady state the average of the time-derivative of $\ln J$ is zero. Hence,

$$\langle J\rangle \approx \frac{\lambda}{2\gamma}\,. \tag{28}$$

This equation is valid for $\langle J\rangle \gg 1$, which is observed for small diversity $\gamma \ll |\lambda|$.

3.2 Evolution of asynchronous state $J = 0$

Averaging the equation for J, similarly to the case of identical oscillators, one finds $\frac{\mathrm{d}}{\mathrm{d}t}\langle J\rangle_\xi = (\mu_\beta + \sigma^2 - 2\gamma)\langle J\rangle_\xi - 2\gamma\langle J^2\rangle_\xi + \sigma^2/2$. For the system starting from the absolutely asynchronous state, $J(0) = 0$, as long as $J \ll 1$, this yields $\langle J\rangle_\xi(t) \approx (e^{(\mu_\beta+\sigma^2-2\gamma)t} - 1)\,\sigma^2/[2(\mu_\beta + \sigma^2 - 2\gamma)]$. Similarly to the case of identical oscillators, further treatment of the intermediate behaviour of the system is not possible for the general case and requires certain approximations. The detailed analytical consideration is possible for a realistic case of high oscillation frequency $\Omega \gg \mu \sim \sigma^2$, which is considered below.

3.3 Statistics at high oscillation frequency

The averaging over high-frequency oscillations can be performed exactly like in the case of identical oscillators. Equation (19) takes now the form

$$\dot{J} = \mu_\beta J - 2\gamma J(1+J) + \frac{\sigma^2}{2}(J + 1/2) + \sigma\sqrt{\frac{J(1+J)}{2}}\,\zeta(t)\,. \tag{29}$$

For nonzero γ the state of perfect synchrony is not possible, thus the order parameter fluctuates in the range $0 \leq J < \infty$. For this statistically stationary regime the corresponding Fokker-Planck equation for $w(J)$ can be integrated yielding a steady state probability density

$$w(J) = \frac{(1+J)^{2\mu_\beta/\sigma^2}\exp\left[-\dfrac{4\gamma}{\sigma^2}(1+J)\right]}{\left(\dfrac{\sigma^2}{4\gamma}\right)^{2\mu_\beta/\sigma^2+1}\Gamma\left(\dfrac{2\mu_\beta}{\sigma^2}+1,\dfrac{4\gamma}{\sigma^2}\right)}\,, \tag{30}$$

where $\Gamma(m,x)$ is the upper incomplete Gamma function. This solution allows to express the moments of order parameters as

$$\langle R^2\rangle = 1 - \frac{4\gamma}{\sigma^2}\frac{\Gamma\left(\dfrac{2\mu_\beta}{\sigma^2},\dfrac{4\gamma}{\sigma^2}\right)}{\Gamma\left(\dfrac{2\mu_\beta}{\sigma^2}+1,\dfrac{4\gamma}{\sigma^2}\right)}\,,\quad \langle J\rangle = \frac{\sigma^2}{4\Gamma}\frac{\Gamma\left(\dfrac{2\mu_\beta}{\sigma^2}+2,\dfrac{4\Gamma}{\sigma^2}\right)}{\Gamma\left(\dfrac{2\mu_\beta}{\sigma^2}+1,\dfrac{4\Gamma}{\sigma^2}\right)} - 1\,. \tag{31}$$

In Figure 2 the average value of the order parameter $\langle J\rangle$ is plotted vs. coupling strength μ_β for different values of γ; one can assess the impact of the natural frequency

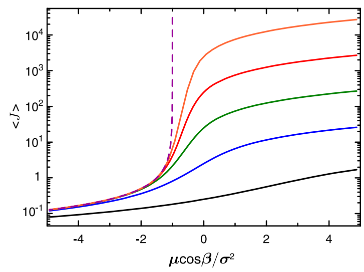

Fig. 2. The order parameter $\langle J \rangle$ is plotted vs. $\mu\cos\beta/\sigma^2$ for different values of γ; the solid curves from bottom to top correspond to $\gamma = 1$, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4} [Eq. (31)], and the dashed curve, which runs to infinity at $\mu\cos\beta/\sigma^2 = -1$, corresponds to identical oscillators, $\gamma = 0$.

dispersion on the system synchrony. Notice that for identical oscillators, the perfect synchrony state is attracting for $\mu\cos\beta/\sigma^2 > -1/2$.

Physical picture for the collective dynamics of nonidentical oscillators. The ensemble of nonidentical oscillators under common noise neither synchronizes nor desynchronizes. For all values of coupling and noise, the order parameter fluctuates in the range $0 \le R < 1$.

4 Nonidentical oscillators: frequency entrainment and repulsion

Let us now consider the dynamics of individual oscillators in the population. It is convenient to study the phase relative to that of the synchronous cluster – specifically, the dynamics of the phase difference $\varphi_\Omega - \Phi$. We have to complement equations (24) and (25) with the equation for the phase deviation of a particular oscillator $\theta_\omega = \varphi_\Omega - \Phi$ [see Eq. (3)];

$$\dot{J} = \mu_\beta J - 2\gamma J(1+J) - \sigma\xi(t)\sqrt{J(1+J)}\cos\Phi, \tag{32}$$

$$\dot{\Phi} = \Omega_0 - \mu\sin\beta\frac{J+1/2}{J+1} + \sigma\xi(t)\frac{J+1/2}{\sqrt{J(1+J)}}\sin\Phi, \tag{33}$$

$$\begin{aligned}\dot{\theta}_\omega = \omega - \mu\left(\sqrt{\frac{J}{1+J}}\sin(\theta_\omega+\beta) - \frac{J+1/2}{J+1}\sin\beta\right)\\ +\sigma\xi(t)\left[\sin(\Phi+\theta_\omega) - \frac{J+1/2}{\sqrt{J(1+J)}}\sin\Phi\right],\end{aligned} \tag{34}$$

where $\omega = \Omega - \Omega_0$. For simplicity of notations we omit hereafter the subscript ω.

The Fokker-Planck equation for the time-dependent probability density $W(J, \Phi, \theta, t)$ of the states of stochastic system (32)–(34) reads

$$\begin{aligned}\frac{\partial}{\partial t}W + \frac{\partial}{\partial J}\Big[(\mu_\beta J - 2\gamma J(1+J))\,W\Big] + \frac{\partial}{\partial\Phi}\left[\left(\Omega_0 - \mu\sin\beta\frac{J+1/2}{J+1}\right)W\right]\\ +\frac{\partial}{\partial\theta}\left[\left(\omega - \mu\sqrt{\frac{J}{1+J}}\sin(\theta+\beta) + \mu\frac{J+1/2}{J+1}\sin\beta\right)W\right] - \sigma^2\hat{Q}^2W = 0\,,\end{aligned} \tag{35}$$

where the operator $\hat{Q}$ is

$$\hat{Q} \equiv \frac{\partial}{\partial J}\Big(-\sqrt{J(1+J)}\cos\Phi\Big) + \frac{\partial}{\partial \Phi}\Big(\frac{J+1/2}{\sqrt{J(1+J)}}\sin\Phi\Big) + \frac{\partial}{\partial \theta}\Big(\Big(\sin(\Phi+\theta) - \frac{J+1/2}{\sqrt{J(1+J)}}\sin\Phi\Big)\Big). \tag{36}$$

4.1 Distribution of the relative phase for high frequency of oscillations

For vanishing μ, σ, and γ, the probability density is $W(J,\Phi,\theta,t) = (2\pi)^{-1}w(J,\theta)$, where $\int_0^{+\infty}\mathrm{d}J\int_0^{2\pi}\mathrm{d}\theta\, w(J,\theta) = 1$. Hence, for $\mu \sim \sigma^2 \sim \gamma \ll \Omega_0$, one can assume $\mu = \sigma^2\mu_1$, $\gamma = \sigma^2\Gamma_1$, $\omega = \sigma^2\omega_1$ and employ the standard method of multiple scales; $t_n = \sigma^{2n}t$ and $W = W^{(0)}(J,\theta,t_1,t_2,...) + \sigma^2 W^{(1)}(J,\Phi,\theta,t_0,t_1,t_2...) + \dots$. To the leading order, equation (35) yields $W^{(0)} = (2\pi)^{-1}w(J,\theta,t_1,t_2,...)$. In the order σ^2, equation (35) yields

$$\frac{\partial W^{(1)}}{\partial t_0} + \Omega_0\frac{\partial W^{(1)}}{\partial \Phi} + \frac{\partial W^{(0)}}{\partial t_1} + \frac{\partial}{\partial J}\Big[\big(\mu_{\beta,1}J - 2\gamma_1 J(1+J)\big)W^{(0)}\Big] + \frac{\partial}{\partial \Phi}\Big[\Big(-\mu_1\sin\beta\frac{J+1/2}{J+1}\Big)W^{(0)}\Big] + \frac{\partial}{\partial\theta}\Big[\Big(\omega_1 - \mu_1\sqrt{\frac{J}{1+J}}\sin(\theta+\beta) + \mu_1\frac{J+1/2}{J+1}\sin\beta\Big)W^{(0)}\Big] - \hat{Q}^2W^{(0)} = 0\,.$$

Integrating the latter equation over Φ from 0 to 2π, one finds

$$\frac{\partial}{\partial t_0}\int_0^{2\pi}W^{(1)}\mathrm{d}\Phi + \frac{\partial w(J,\theta,t_1)}{\partial t_1} + \frac{\partial}{\partial J}\Big[\big(\mu_{\beta,1}J - 2\gamma_1J(1+J)\big)w(J,\theta,t_1)\Big] + \frac{\partial}{\partial\theta}\Big[\Big(\omega_1 - \mu_1\sqrt{\frac{J}{1+J}}\sin(\theta+\beta) + \mu_1\frac{J+1/2}{J+1}\sin\beta\Big)w(J,\theta,t_1)\Big] - \frac{1}{2\pi}\int_0^{2\pi}\hat{Q}^2w(R,\theta,t_1)\,\mathrm{d}\Phi = 0\,.$$

To eliminate the linear growth of $W^{(1)}$ with t_0, which would break the hierarchy of smallness of expansion terms, one has to set the first term to zero. This integral can be expressed as

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\Phi\,\hat{Q}^2w(J,t_1) = \frac{\partial}{\partial J}\left(-\frac{J+1/2}{2}w\right) + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{2}\frac{J+1/2}{\sqrt{J(1+J)}}w\right) + \hat{Q}^2_{J,\theta}w + \hat{Q}^2_\theta w\,,$$

where

$$\hat{Q}_{J,\theta} \equiv \frac{\partial}{\partial J}\Big(-\sqrt{\frac{J(1+J)}{2}}\Big) + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\sqrt{2}}\right), \quad \hat{Q}_\theta \equiv \frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sqrt{2}} - \frac{J+1/2}{\sqrt{2J(1+J)}}\right).$$

Thus, the probability density $w(J,\theta,t)$ is governed by the equation

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial J}\left(\left[\mu_\beta J-2\gamma J(1+J)+\frac{\sigma^2}{2}(J+1/2)\right]w\right)$$
$$+\frac{\partial}{\partial\theta}\left[\left(\omega-\mu\sqrt{\frac{J}{1+J}}\sin(\theta+\beta)+\mu\frac{J+1/2}{J+1}\sin\beta-\frac{\sigma^2}{2}\frac{J+1/2}{\sqrt{J(1+J)}}\sin\theta\right)w\right]$$
$$-\sigma^2\hat{Q}^2_{J,\theta}w-\sigma^2\hat{Q}^2_{\theta}w=0\,. \quad (37)$$

Equation (37) can be treated as the Fokker-Planck equation for the stochastic equation system with two independent noises $\zeta_1(t)$ and $\zeta_2(t)$:

$$\dot{J}=\mu_\beta J-2\gamma J(1+J)+\frac{\sigma^2}{2}(J+1/2)-\sigma\sqrt{\frac{J(1+J)}{2}}\zeta_1(t)\,, \quad (38)$$

$$\dot{\theta}=\omega-\mu\sqrt{\frac{J}{1+J}}\sin(\theta+\beta)+\mu\frac{J+1/2}{J+1}\sin\beta-\frac{\sigma^2}{2}\frac{J+1/2}{\sqrt{J(1+J)}}\sin\theta+\sigma\frac{\sin\theta}{\sqrt{2}}\zeta_1(t)$$
$$+\sigma\left(\frac{\cos\theta}{\sqrt{2}}-\frac{J+1/2}{\sqrt{2J(1+J)}}\right)\zeta_2(t). \quad (39)$$

The original noise $\xi(t)$ generates two independent effective noises $\zeta_1(t)$ and $\zeta_2(t)$, which are Gaussian and delta-correlated, $\langle\zeta_n(t)\zeta_l(t+t')\rangle=2\delta_{n,l}\delta(t')$, as the signals $\xi(t)\cos\Omega_0 t$ and $\xi(t)\sin\Omega_0 t$ are uncorrelated on time scales large compared to $2\pi/\Omega_0$.

4.2 Highly synchronous dynamics for small disorder

Let us consider the case of small disorder $\gamma\ll\sigma^2\sim|\mu|$ and $2\mu/\sigma^2>-1$. (Notice that compared to Ω_0, γ is considered to be of the same order of magnitude as σ^2 and μ, i.e., $\sigma^3\ll\gamma\ll\sigma^2$.) For this case, the system states are accumulated at $J\gg 1$, which can be used for integration of equation (37) over J. For $\overline{w}\equiv\int_0^{+\infty}w\,\mathrm{d}J$, with $\overline{f(J)w}\approx\overline{w}\lim_{J\to+\infty}f(J)$, one finds from equation (37):

$$\frac{\partial\overline{w}}{\partial t}+\frac{\partial}{\partial\theta}\left(\left[\omega-\mu\big(\sin(\theta+\beta)-\sin\beta\big)-\frac{\sigma^2}{2}\sin\theta\right]\overline{w}\right)$$
$$-\frac{\sigma^2}{2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\,\overline{w}\right)\right)-\frac{\sigma^2}{2}\frac{\partial}{\partial\theta}\left((\cos\theta-1)\frac{\partial}{\partial\theta}\left((\cos\theta-1)\,\overline{w}\right)\right)=0\,,$$

which can be simplified to

$$\frac{\partial\overline{w}}{\partial t}+\frac{\partial}{\partial\theta}\left(\big[\omega-\mu\big(\sin(\theta+\beta)-\sin\beta\big)\big]\,\overline{w}\right)-\sigma^2\frac{\partial^2}{\partial\theta^2}\left((1-\cos\theta)\overline{w}\right)=0\,. \quad (40)$$

Let us consider the stochastically steady state, which corresponds to a constant in θ probability flux j in (40): $\left(\omega-\mu\big(\sin(\theta+\beta)-\sin\beta\big)\right)\overline{w}-\sigma^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left((1-\cos\theta)\overline{w}\right)=j$. In terms of the latter equation, the frequency locking, $\langle\dot{\theta}\rangle=0$, corresponds to $j=0$. For $\sigma\neq 0$, it can be shown that the solution with $j=0$ exists for one special value of ω, meaning the complete frequency locking is not possible even for a small deviation of the natural frequency from the ensemble mean frequency.

For $j \neq 0$, one obtains

$$\overline{w}(\theta) = \frac{j}{\sigma^2} \int_{\theta}^{2\pi} \mathrm{d}\psi \frac{(1-\cos\psi)^{\mu_\beta/\sigma^2}}{(1-\cos\theta)^{1+\mu_\beta/\sigma^2}} \exp\left[-\frac{\omega}{\sigma^2}\left(\cot\frac{\theta}{2} - \cot\frac{\psi}{2}\right) - \frac{\mu \sin\beta}{\sigma^2}(\psi - \theta)\right]. \tag{41}$$

This expression possesses good convergency properties at $\theta = 0$ and 2π: $\overline{w}(0) = \overline{w}(2\pi) = j/\omega$; at nonzero ω it converges for any μ. The normalization condition yields the state flux j as a function of μ, β, σ^2, and ω: $j = \sigma^2 f\left(\frac{\mu}{\sigma^2}, \frac{\omega}{\sigma^2}, \beta\right)$.

4.3 Average oscillator frequency for imperfect synchrony

Let us now calculate the average frequency of oscillations for the case of an imperfect synchrony. Technically, our task is to derive the average frequency for equation (39) with finite but still large J. For an approximate calculation, let us recast equation (39) as

$$\dot{\theta} = \omega_\beta - \mu b \sin(\theta + \beta) - \frac{\sigma^2}{2} c \sin\theta - \frac{\sigma}{\sqrt{2}} \sin\theta\, \zeta_1(t) + \frac{\sigma}{\sqrt{2}}(\cos\theta - c)\, \zeta_2(t)\,, \tag{42}$$

and consider it for constant coefficients ω_β, b and c calculated as average values:

$$\omega_\beta \equiv \omega + \mu \left\langle \frac{J+1/2}{J+1} \right\rangle \sin\beta = \omega + \mu \left(1 - \frac{1}{2}\left\langle \frac{1}{J+1} \right\rangle\right) \sin\beta\,,$$

$$b \equiv \left\langle \sqrt{\frac{J}{1+J}} \right\rangle = 1 - \frac{1}{2}\left\langle \frac{1}{1+J} \right\rangle - \frac{1}{8}\left\langle \frac{1}{(1+J)^2} \right\rangle + \dots\,,$$

$$c \equiv \left\langle \frac{J+1/2}{\sqrt{J(1+J)}} \right\rangle = 1 + \frac{1}{8}\left\langle \frac{1}{(1+J)^2} \right\rangle + \dots\,.$$

With distribution (30), one can evaluate

$$\left\langle \frac{1}{(1+J)^n} \right\rangle = \left(\frac{4\gamma}{\sigma^2}\right)^n \frac{\Gamma\left(\frac{2\mu_\beta}{\sigma^2} + 1 - n, \frac{4\gamma}{\sigma^2}\right)}{\Gamma\left(\frac{2\mu_\beta}{\sigma^2} + 1, \frac{4\gamma}{\sigma^2}\right)}\,. \tag{43}$$

Stochastic equation (42) with constant coefficients yields the Fokker-Planck equation

$$\frac{\partial w(\theta,t)}{\partial t} + \frac{\partial}{\partial\theta}\left[\Big(\omega_\beta - \mu b \sin(\theta+\beta)\Big) w(\theta,t)\right] - \frac{\sigma^2}{2}\frac{\partial^2}{\partial\theta^2}\left[\Big(1 + c^2 - 2c\cos\theta\Big)\, w(\theta,t)\right] = 0\,. \tag{44}$$

For a time-independent distribution $w(\theta)$, one can integrate the latter equation with respect to θ and obtain

$$(\omega_\beta - \mu b \sin(\theta+\beta)) w - \frac{\sigma^2}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[(1 + c^2 - 2c\cos\theta) w\right] = j\,, \tag{45}$$

where $j = \mathit{const}$ is the integration constant, which is the probability flux in the system. The flux j counts the average number of crossings of certain state θ_0 per time or, equivalently, the number of phase turnovers per time. Hence, $\langle\dot{\theta}\rangle = 2\pi j$.

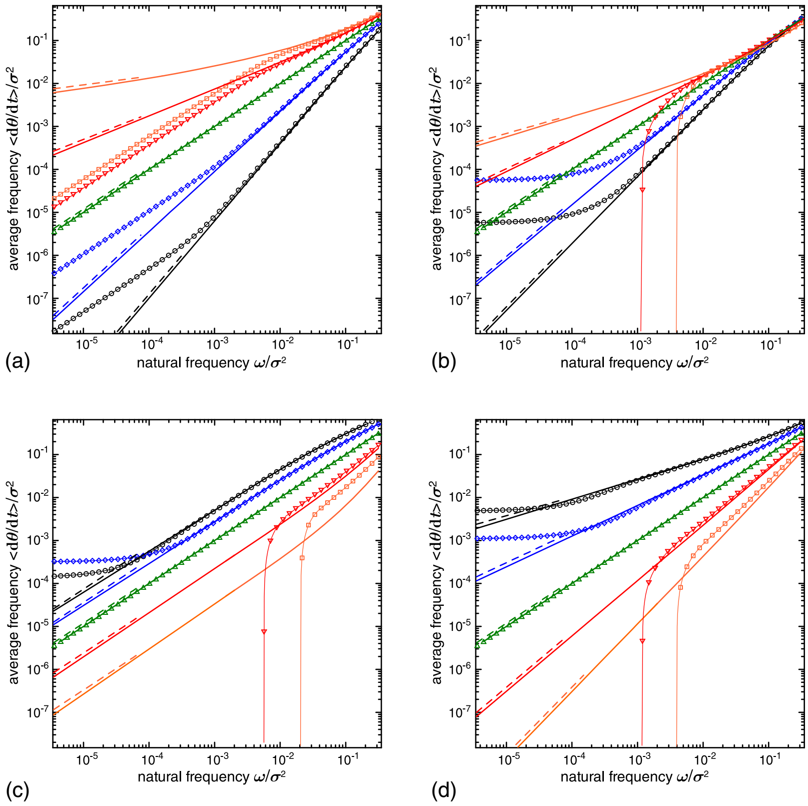

Fig. 3. The dependencies of the average frequency $\langle\dot{\theta}\rangle$ on the frequency mismatch ω are presented for $\beta = 0$ (a), $\pi/4$ (b), $\pi/2$ (c), and $3\pi/4$ (d). The results of numerical calculations for $\gamma = 0.01$ with continuous fractions are plotted in (a), (b), (d), where black circles correspond to $\mu/\sigma^2 = 0.4$, blue diamonds to $\mu/\sigma^2 = 0.2$, green up-pointed triangles to $\mu/\sigma^2 = 0$, red down-pointed triangles to $\mu/\sigma^2 = -0.2$, and orange squares to $\mu/\sigma^2 = -0.4$. The results in (c) correspond to a reactive coupling, $\mu/\sigma^2 = 0.8, 0.4, 0, -0.4$, -0.8, respectively (doubled compared to (a), (b), (d)). Solid lines present the results for specified μ/σ^2 and $\Gamma = 0$; dashed lines show the slope $1 + 2\mu\cos\beta/\sigma^2$, which means asymptotic behaviour $\langle\dot{\theta}\rangle \propto \omega^{1+2\mu\cos\beta/\sigma^2}$.

Unfortunately, we cannot give a practical closed expression for the density w. In the supplementary material we describe a way to calculate it numerically using a continuous fraction expansion. The results of calculation of $\langle\dot{\theta}\rangle$ are plotted in Figures 3 and 4. The case of $\beta \in [\pi, 2\pi)$ does not require additional consideration because of the symmetry $(\mu, \beta) \leftrightarrow (-\mu, \beta + \pi)$.

For small ω, one can evaluate the frequency using an approximation method, described in the supplementary material. In the approximation the frequency ω_0 corresponding to $j = 0$ is

$$\omega_0 = -\frac{\mu}{4}\left\langle(1 + J)^{-2}\right\rangle \sin\beta\,. \tag{46}$$

Notice that for imperfect synchrony, where J does not tend to infinity, the natural frequency of an oscillator locked to the order parameter is nonzero for $\sin\beta \neq 0$. Its deviation from zero is stronger for a weaker synchrony. The average oscillator

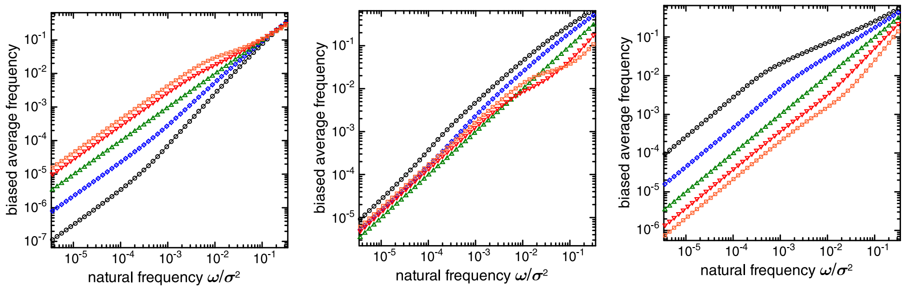

Fig. 4. The dependence of the biased average frequency $\langle\dot{\theta}\rangle - \langle\dot{\theta}\rangle_{\omega=0}$ on the frequency mismatch ω for $\gamma = 0.01$ and $\beta = \pi/4$ (a), $\pi/2$ (b), $3\pi/4$ (c); marks are the same as in Figure 3.

frequency in this approximation is

$$\langle\dot{\theta}\rangle = \left[\frac{\pi\,\Gamma\!\left(\frac{\mu e^{i\beta}}{\sigma^2}+1\right)}{\Gamma\!\left(\frac{2\mu\cos\beta}{\sigma^2}+1\right)\,\Gamma\!\left(-\frac{\mu e^{-i\beta}}{\sigma^2}\right)\,\sin\!\left(\frac{\mu e^{-i\beta}}{\sigma^2}\pi\right)}\right]^2 \left(\frac{\langle(1+J)^{-2}\rangle}{4}\right)^{\frac{2\mu\cos\beta}{\sigma^2}} (\omega-\omega_0), \tag{47}$$

where the average value $\langle(1+J)^{-2}\rangle$ is determined by equation (43). Notice, that the magnitude of the first multiplier in equation (47) is of the order of 1, while $\langle(1+J)^{-2}\rangle$ is small for $\mu\cos\beta/\sigma^2 > -1/2$ and, hence, the proportionality coefficient between $\langle\dot{\theta}\rangle$ and $(\omega-\omega_0)$ is small for a synchronizing coupling ($\mu\cos\beta > 0$) and large for a desynchronizing coupling ($\mu\cos\beta < 0$). For the limit $\gamma \to 0$, where $\langle(1+J)^{-2}\rangle \to 0$, the coefficient tends to zero and infinity, for a synchronizing and a desynchronizing couplings, respectively.

Physical picture for the oscillators' frequencies in a non-identical population. All individual frequencies of the oscillators are different. For attractive coupling, the frequencies become close to each other (are attracted to each other). For repulsive coupling, the frequencies become more different (are repulsed). Qualitatively, one can understand this effect as follows. In the absence of coupling, the frequencies remain the same, although the phases typically form a cluster; in the cluster the instantaneous frequencies are nearly the same. This cluster is however temporary: the phases stay together for some period of time, then the phase of a fast oscillator makes an extra rotation with respect to a slow oscillator; this rotation exactly compensates the closeness of the instantaneous frequencies during the stage when the phases stay together. In the case of attractive coupling, the rotations are more seldom, so they only partially compensate the closeness of the instantaneous frequencies during the cluster stage. In the case of repulsive coupling, the rotations are more frequent, thus they "overcompensate" the closeness of the instantaneous frequencies during the cluster stage. This is illustrated in Figure 5.

5 Conclusion

In this paper we have presented a theory of synchronization of a population of phase oscillators subject to global coupling and common noise. The coupling of the Kuramoto-Sakaguchi type and the multiplicative noise can be both incorporated in the Ott-Antonsen framework, allowing for a set of closed stochastic equations for the order parameters. In our analysis of these equations we tried, whenever possible, to

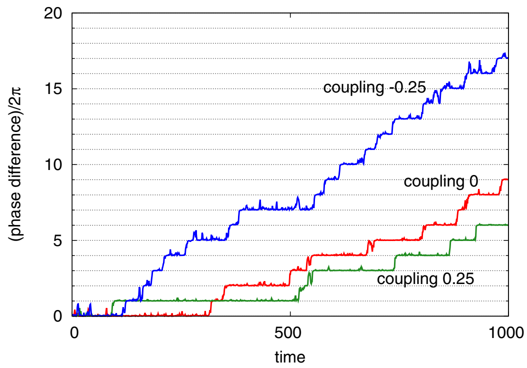

Fig. 5. Dynamics of the phase difference of two oscillators in an ensemble, for different coupling strengths.

present analytical results for the statistical characteristics of the order parameters. We studied also in detail an important situation of fast natural oscillations, where averaging over the fast rotating phase allows one to significantly simplify the stochastic equations.

The most interesting effects appear when the noise and the coupling compete, namely when the coupling is repulsive, tending to desynchronize the ensemble (noise is always synchronizing for phase oscillators). For identical oscillators, synchrony wins up to moderate repulsive couplings, because the fully synchronized state is absorbing one, contrary to the desynchronized state which is only weakly stable. For nonidentical oscillators, full synchrony is impossible, so for all relations between the noise and the coupling one observes finite fluctuations of the order parameter.

For nonidentical oscillators, the most surprising is the dynamics of averaged oscillator frequencies. Here the difference in the synchronization by common noise and by coupling manifests itself in a highly nontrivial way. Coupling without noise pulls frequencies together when it is attractive, but do not influence frequencies if it is repulsive. Noise without coupling does not influence frequencies at all. In the presence of noise and repulsive coupling one observes, quite counterintuitevly, divergence of frequencies, while the order parameter can be relatively large.

Studies presented in Sections 2 and 4 of this work, conducted by Ar.P., were supported by the Russian Science Foundation (grant No. 17-12-01534). D.S.G. and A.V.P. acknowledge support by G-RISC (grant No. M-2016b-8). A.V.P. also acknowledges financial support by a Grant of the President of the Russian Federation (MK-1447.2017.5).

References

1. A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001)
2. J.D. Crawford, Amplitude expansions for instabilities in populations of globally-coupled oscillators, J. Stat. Phys. **74**, 1047 (1994)
3. S.H. Strogatz, D.M. Abrams, A. McRobie, B. Eckhardt, E. Ott, Theoretical mechanics: Crowd synchrony on the Millennium Bridge, Nature **438**, 43 (2005)
4. D. Golomb, D. Hansel, G. Mato, Mechanisms of synchrony of neural activity in large networks in *Handbook of Biological Physics, Volume 4: Neuroinformatics and Neural Modelling*, edited by F. Moss, S. Gielen (Elsevier, Amsterdam, 2001), pp. 887–968

5. A.S. Pikovsky, Synchronization and stochastization of the ensemble of autogenerators by external noise, Radiophys. Quantum Electron. **27**, 576 (1984)
6. Z.F. Mainen, T.J. Sejnowski, Reliability of spike timing in neocortical neurons, Science **268**, 1503 (1995)
7. A. Uchida, R. McAllister, R. Roy, Consistency of nonlinear system response to complex drive signals, Phys. Rev. Lett. **93**, 244102 (2004)
8. B.T. Grenfell, K. Wilson, B.F. Finkenstädt, T.N. Coulson, S. Murray, S.D. Albon, J.M. Pemberton, T.H. Clutton-Brock, M.J. Crawley, Noise and determinism in synchronized sheep dynamics, Nature **394**, 674 (1998)
9. J. Ritt, Evaluation of entrainment of a nonlinear neural oscillator to white noise, Phys. Rev. E **68**, 041915 (2003)
10. J.N. Teramae, D. Tanaka, Robustness of the noise-induced phase synchronization in a general class of limit cycle oscillators, Phys. Rev. Lett. **93**, 204103 (2004)
11. D.S. Goldobin, A.S. Pikovsky, Synchronization of self-sustained oscillators by common white noise, Physica A **351**, 126 (2005)
12. D.S. Goldobin, A. Pikovsky, Synchronization and desinchronization of self-sustained oscillators by common noise, Phys. Rev. E **71**, 045201(R) (2005)
13. D.S. Goldobin, A. Pikovsky, Antireliability of noise-driven neurons, Phys. Rev. E **73**, 061906 (2006)
14. S. Wieczorek, Stochastic bifurcation in noise-driven lasers and Hopf oscillators, Phys. Rev. E **79**, 036209 (2009)
15. D.S. Goldobin, J.-N. Teramae, H. Nakao, G.-B. Ermentrout, Dynamics of Limit-Cycle Oscillators Subject to General Noise, Phys. Rev. Lett. **105**, 154101 (2010)
16. D.S. Goldobin, Uncertainty principle for control of ensembles of oscillators driven by common noise, Eur. Phys. J. Special Topics **223**, 677 (2014)
17. W. Braun, A. Pikovsky, M.A. Matias, P. Colet, Global dynamics of oscillator populations under common noise, EPL **99**, 20006 (2012)
18. A.V. Pimenova, D.S. Goldobin, M. Rosenblum, A. Pikovsky, Interplay of coupling and common noise at the transition to synchrony in oscillator populations, Sci. Rep. **6**, 38518 (2016)
19. D. García-Álvarez, A. Bahraminasab, A. Stefanovska, P.V.E. McClintock, Competition between noise and coupling in the induction of synchronisation, EPL **88**, 30005 (2009)
20. K.H. Nagai, H. Kori, Noise-induced synchronization of a large population of globally coupled nonidentical oscillators, Phys. Rev. E **81**, 065202 (2010)
21. S. Watanabe, S.H. Strogatz, Constant of motion for superconducting josephson arrays, Physica D **74**, 197 (1994)
22. A. Pikovsky, M. Rosenblum, Partially integrable dynamics of hierarchical populations of coupled oscillators, Phys. Rev. Lett. **101**, 2264103 (2008)
23. S.A. Marvel, R.E. Mirollo, S.H. Strogatz, Identical phase oscillators with global sinusoidal coupling evolve by Möbius group action, Chaos **19**, 043104 (2009)
24. E. Ott, T.M. Antonsen, Low dimensional behavior of large systems of globally coupled oscillators, Chaos **18**, 037113 (2008)
25. C.A.J. Fletcher, *Computational Galerkin Methods* (Springer, 1984)