Maximizing Coherence of Oscillations by External Locking

Arkady Pikovsky
Institute for Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Straße 24/25, 14476 Potsdam-Golm, Germany and Department of Control Theory, Nizhni Novgorod State University, Gagarin Avenue 23, 606950 Nizhni Novgorod, Russia
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We study how coherence of noisy oscillations can be optimally enhanced by external locking. Based on the condition of minimizing the phase diffusion constant, we find the optimal forcing explicitly in the limits of small and large noise, in dependence of the phase sensitivity of the oscillator. We show analytically that the form of the optimal force bifurcates with the noise intensity; this is confirmed by the analysis of an optimal locking forcing for an experimentally obtained phase sensitivity of a neural cell. In the limit of small noise, the results are compared with purely deterministic conditions of optimal locking.

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Autonomous self-sustained oscillations may be extremely regular (like, e.g., lasers) or rather incoherent (like many biological oscillators, e.g., ones responsible for cardiac or circadian rhythms). A usual way to improve the quality of oscillations is to lock (synchronize) them by an external pacing [1,2]. This is used in radio-controlled clocks and in cardiac pacemakers; also, circadian rhythms are nearly perfectly locked by the 24-h day-night force.

In this Letter we address a novel question: Which periodic force ensures, via locking, the maximal coherence of a noisy self-sustained oscillator? Of course, one has to fix the amplitude of the force, so the nontrivial problem is in finding the optimal force profile. We will treat this problem in the phase approximation [1], which is valid for general oscillators, provided the noise and the forcing terms are small. In this approximation, the dynamics of the phase reduces to a noisy Adler equation [2,3], and the maximal coherence is achieved if the diffusion constant of the phase is minimal. It should be noted that an optimal locking problem has been recently discussed for purely deterministic oscillations. There, the optimal condition was formulated as the maximal width of the Arnold's tongue (the synchronization region) or as the maximal stability of the locked state [4–7].

In our case there is an additional parameter, the noise intensity, and we will show that the optimal force profile, in contradistinction to the purely deterministic case, depends not only on the phase sensitivity properties, but also on the noise amplitude. Moreover, the optimal forcing form demonstrates bifurcations in dependence on the noise strength; this we show both for the solvable case of biharmonic phase sensitivity, and for a realistic case of experimentally obtained sensitivity of a neural cell. Furthermore, below we will also compare the limit of small noise with purely deterministic setups.

Let us consider a self-sustained oscillator with frequency ω; its phase in the presence of a small Gaussian white noise obeys the Langevin equation

$$\frac{d\varphi}{dt} = \omega + \beta^{-1/2}\xi(t), \qquad \langle \xi(t)\xi(t')\rangle = 2\delta(t-t'), \quad (1)$$

where β^{-1} is the noise intensity. A small periodic forcing with frequency Ω leads, in the first order in the force amplitude, to the following phase dynamics [1,3]:

$$\frac{d\varphi}{dt} = \omega + s(\varphi)f(\Omega t) + \beta^{-1/2}\xi(t). \quad (2)$$

Here $s(\varphi)$ is the phase response curve (PRC) (also know as the phase sensitivity function), and $f(\Omega t)$ is the phase-projected force term. Our goal will be to find such a forcing $f(\cdot)$ that maximizes the coherence, i.e., minimizes the diffusion constant of the phase φ. This optimal force will depend on the phase sensitivity function $s(\cdot)$ and on the noise intensity β.

As the first step, we introduce the slow phase $\phi = \varphi - \Omega t$ (the diffusion constant of this slow phase is the same as the original phase φ, because they differ by a constant deterministic rotation) and perform the standard averaging over the period $2\pi\Omega^{-1}$ [1,2]; this yields

$$\frac{d\phi}{dt} = \omega - \Omega + g(\phi) + \beta^{-1/2}\xi(t) = -\frac{dv(\phi)}{d\phi} + \beta^{-1/2}\xi(t), \quad (3)$$

where

$$g(\phi) = \frac{1}{2\pi}\int_0^{2\pi} dy s(\phi + y)f(y), \quad (4)$$

and we introduced the "potential"

$$v(\phi) = (\Omega - \omega)\phi - \int^{\phi} g(y)dy. \quad (5)$$

Let us consider a situation where the mean frequency of oscillations is exactly that of the forcing; this means that the slow phase ϕ performs a random walk without a bias.

This happens for a purely periodic, noninclined potential. This condition, as it follows from Eq. (5), defines the optimal frequency of the forcing

$$\bar{\Omega} = \omega + \langle s \rangle \langle f \rangle, \tag{6}$$

where we denote $\langle f(\phi) \rangle = (2\pi)^{-1} \int_0^{2\pi} f(\phi) d\phi$. In other words, the homogeneous term $\langle s \rangle$ in the phase sensitivity function (which is, e.g., significant for type-1 neurons) affects sensitivity of the frequency shift to the form of the forcing, but does not affect coherence properties. Thus, without loss of generality we can assume that $\langle s \rangle = \langle f \rangle = 0$ and $\Omega = \omega$.

The problem of finding the diffusion constant D of a particle in a periodic potential v, driven by a white Gaussian noise, has been solved in Ref. [8] (and generalized to the case of an inclined potential in Ref. [9]):

$$D = \frac{D_0}{\langle \exp(\beta v) \rangle \langle \exp(-\beta v) \rangle}, \tag{7}$$

where D_0 is the bare diffusion constant without potential. Thus, the problem of maximizing the coherence reduces to maximizing the expression

$$C = \langle \exp(\beta v) \rangle \langle \exp(-\beta v) \rangle. \tag{8}$$

As an additional condition we have to fix the intensity of the force:

$$\langle f^2 \rangle = \text{const.} \tag{9}$$

The formulated optimization problem is quite complex to be solved in general. Therefore, below we consider some simplifying cases, and will perform a rather full analysis for a simple biharmonic phase sensitivity function. The main novel feature we will focus on are bifurcations in dependence on the form of this function and on the noise intensity; we will see that different forcing waveforms provide optimal coherence in different domains of the parameter space.

For the analytical consideration below it is convenient to use Fourier transforms, which we will denote by capitals:

$$s(x) = \sum_k S_k \exp[ikx],$$
$$S_k = \frac{1}{2\pi} \int_0^{2\pi} s(x) \exp[-ikx] dx, \tag{10}$$

and the same for functions f, g, v, Fourier harmonics of which we denote as F_k, G_k, V_k, respectively. Because $g(\phi)$ is, according to Eq. (4), a convolution of f and s, and v is the integral of g, we have

$$G_k = S_k F_{-k}, \qquad V_k = ik^{-1} S_k F_{-k}. \tag{11}$$

The condition on the norm of the force, Eq. (9), now reads

$$\sum_k |F_k|^2 = \text{const.} \tag{12}$$

We start with the case of strong noise (small β). Expanding Eq. (8), we obtain a simple expression for the quantity to be maximized:

$$C \approx 1 + \beta^2 \langle v^2 \rangle = 1 + \beta^2 \sum_k k^{-2} |S_k|^2 |F_k|^2. \tag{13}$$

Together with Eq. (12), the maximum can be found by virtue of Lagrange multipliers:

$$|F_k| \sim \delta_{k,K}, \quad \text{where } K = \arg\max(k^{-2}|S_k|^2). \tag{14}$$

Thus, for large noise, the optimal forcing is a purely harmonic one, $f(x) \sim \cos(Kx)$, where K is determined from Eq. (14). Intuitively, this result can be understood as follows. For strong noise, the forcing delivers a small perturbation to an intensive Brownian motion of the phase. Its effect on the random walk is a weak scattering, proportional to the variance of the overall coupling term [see Eq. (13)]. The contributions of forcing harmonics to this variance are just summed, indicating that they can be considered as independent scattering channels. For a fixed overall forcing intensity, for any PRC, it is optimal to concentrate the forcing in the mostly amplified channel, i.e., to apply force at the harmonics with maximal value of $k^{-2}|S_k|^2$. For smaller noise, a multiple scattering becomes significant and the channels are no more independent, so that a more complex force maximizes the coherence.

The case of small noise is the limit, $\beta \to \infty$. In this case, the integrals in Eq. (8) can be asymptotically estimated as Laplace integrals:

$$\langle \exp(\beta v) \rangle \approx (2\pi)^{-1} \exp(\beta v_{\max}),$$
$$\langle \exp(-\beta v) \rangle \approx (2\pi)^{-1} \exp(-\beta v_{\min}), \tag{15}$$

which gives

$$C \sim \exp[\beta(v_{\max} - v_{\min})]. \tag{16}$$

Suppose now that $v_{\min} = v(x_2)$ and $v_{\max} = v(x_1)$. Then

$$\ln C \sim v_{\max} - v_{\min} = \int_{x_1}^{x_2} g(\phi) d\phi = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx \int_{x+x_1}^{x+x_2} s(y) dy. \tag{17}$$

Using additionally Eq. (9) with a Lagrange multiplier, we obtain

$$f(x) = \text{const} \int_{x+x_1}^{x+x_2} s(y)dy. \quad (18)$$

Substituting this into conditions $g(x_1) = g(x_2) = 0$, we get an equation for $\Delta = x_2 - x_1$ (only this difference is important, but not the values of x_1, x_2):

$$p(\Delta) = \int_0^{2\pi} s(z) \int_0^{\Delta} s(z+y)dzdy = 2\pi \sum_k k^{-1}|S_k|^2 \sin k\Delta = 0. \quad (19)$$

This equation has always a solution $\Delta = \pi$, but depending on the form of the phase sensitivity function s there can be other solutions, corresponding to local maxima of C; one has to compare different possible values of Δ to find the global maximum. Once Δ is found, the corresponding force can be expressed as

$$f(\phi) \sim \int_{\phi}^{\phi+\Delta} s(y)dy, \qquad F_k \sim S_k \frac{\exp[ik\Delta] - 1}{ik}. \quad (20)$$

Here below we present the simplest nontrivial example, where it is possible, in addition to the asymptotic cases of small and large noise considered above, to perform the analysis for intermediate noise levels. We consider the biharmonic phase sensitivity function

$$s(x) = 2\sqrt{q}\cos x + 2\sqrt{1-q}\cos 2x, \quad (21)$$

where parameter q describes the relative weight of the harmonics [as has been mentioned above, a constant term in $s(x)$ affects only a possible frequency shift, but not the coherence, and therefore is omitted here].

The limit of strong noise, Eq. (14), with $|S_1|^2 = q$, $|S_2|^2 = 1 - q$, yields

$$f(x) \sim \begin{cases} \cos 2x & \text{if } 0 \le q < 1/5 \\ \cos x & \text{if } 1/5 < q \le 1. \end{cases} \quad (22)$$

The limit of weak noise leads to the following expression for Eq. (19):

$$p(\Delta) = q \sin\Delta + \frac{1-q}{2}\sin 2\Delta. \quad (23)$$

For $q > 1/2$, the only root in Eq. (23) is $\Delta = \pi$, while for $q<1/2$ there is an additional root $\Delta_1 = \arccos[-q/(1-q)]$. Substituting this into Eq. (20), we obtain for small noise

$$|F_1|^2 = 1 - |F_2|^2 = \begin{cases} 2q & \text{if } 0 \le q < 1/2 \\ 1 & \text{if } 1/2 < q \le 1. \end{cases} \quad (24)$$

Let us now consider general noise intensities. The forcing in this case should be also generally biharmonic [higher harmonics disappear according to Eq. (11)]:

$$f(x) = a\cos x + b\cos 2x + c\sin 2x, \quad (25)$$

with unknown constants a, b, c satisfying $a^2 + b^2 + c^2 = 1$. In this representation the potential $v(x)$ reads

$$v(x) = -\sqrt{q}a\sin x - \frac{\sqrt{1-q}b}{2}\sin 2x - \frac{\sqrt{1-q}c}{2}\cos 2x. \quad (26)$$

Unfortunately, after substitution of this potential in Eq. (8) for the factor C, we obtain integrals which cannot be expressed in a closed analytic form. However, for a purely first-harmonic forcing ($b = c = 0$) and a purely second-harmonic forcing ($a = 0$), the factor C as well as its derivatives can be expressed via first-order Bessel functions. Thus, it is possible to find the domains of stability of these pure forcing terms analytically, for arbitrary values of noise intensity β. These lengthy but straightforward calculations give the stability boundaries in a parametric form: The first-harmonic force loses stability at the curve on the (β, q) plane, parametrically represented as

$$q = \frac{z[-I_4(z) + I_0(z)]}{8I_1(z) + z[-I_4(z) + I_0(z)]}, \qquad \beta = \frac{z}{\sqrt{q}}. \quad (27)$$

The stability boundary of the second-harmonic solution is

$$q = \frac{I_1(z)}{I_1(z) + 2zI_0(z)}, \qquad \beta = \frac{2z}{\sqrt{1-q}}. \quad (28)$$

We illustrate these domains in Fig. 1. Here we also show numerically obtained dependencies of $|F_1|^2$ (the intensity of the second harmonics is $|F_2|^2 = 1 - |F_1|^2$) on parameters q and β, demonstrating bifurcations on the form of the forcing.

Next, we discuss a relation between different criteria used for the "optimal locking." While here we optimize the coherence in the presence of noise, in Refs. [4,5] purely deterministic criteria have been suggested. It is instructive to compare them with our approach in the limit of small noise. Suppose that the coupling function $g(\phi)$ has zeros at $\phi_{1,2}$ (where ϕ_1 is the stable one) and extrema at $\phi_{3,4}$. In the approach of [5], the linear stability at the stable equilibrium $|g'(x_1)|$ is maximized. In the approach of [4], the width of the synchronization region $\sim|g(\phi_3) - g(\phi_4)|$ is maximized. In our maximization of the coherence, the potential barrier for a noise-induced phase slip $\sim|\int_{\phi_1}^{\phi_2} g(x)dx|$ should be maximal. For the discussed above example of a biharmonic phase sensitivity function, Eq. (21), all the optimal forcings can be found analytically; they are generally also biharmonic. The approach of [5] yields in this case

$$|F_1|^2 = 1 - |F_2|^2 = q/(4 - 3q), \quad (29)$$

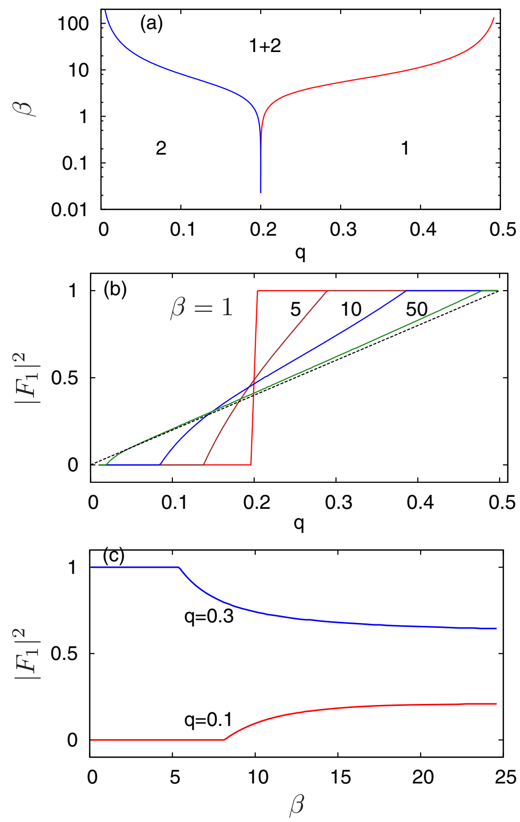

FIG. 1 (color online). (a) Domains on the plane of parameters (q,β) where the optimal force has one harmonics (2: the second one, 1: the first one), and two harmonics $(1+2)$, according to expressions, Eqs. (27) and (28). (b) The intensity of the first harmonics $|F_1|^2$ as a function of q for different noise intensities β. The thin dashed black line shows the limit of small noise, Eq. (24). (c) Two dependencies of $|F_1|^2$ on the noise intensity β showing bifurcations from one-mode to two-mode solutions at critical values of β.

while the approach of [4] gives

$$|F_1|^2 = 1 - |F_2|^2 = \begin{cases} \frac{2q}{4-3q} & \text{if } 0 \le q < 4/5 \\ 1 & \text{if } 4/5 < q \le 1. \end{cases} \tag{30}$$

We compare the results in Fig. 2. One can see that for the minimal coherence, the presence of a strong first-harmonics component in the forcing is more important than for other criteria.

Finally, we present a practical example of application of our approach. We study here the optimal forcing which maximizes the coherence, for an experimentally found PRC of a mitral cell of the mouse olfactory bulb [10]. The PRC presented in Fig. 2(b) of Ref. [10] has three essentially nonzero components; thus, we have to find optimal values of F_k for $k = 1, 2, 3$ using S_k from the experiment. The result, obtained by virtue of the numerical maximization of Eq. (8), is shown in Fig. 3. One can see a transition from the pure first-harmonic forcing at small β to a nontrivial forcing

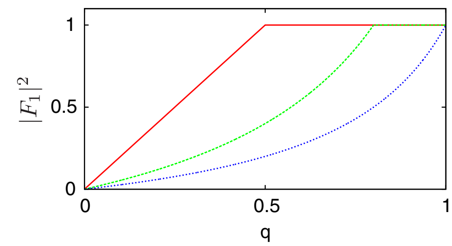

FIG. 2 (color online). The intensity of the first-harmonic component in the optimal force as a function of parameter q, for three optimization criteria. Top solid red line: Maximal coherence in the weak noise limit, Eq. (24). Middle dashed green line: Maximal width of the synchronization region, Eq. (30). Bottom dotted blue curve: Maximal linear stability of the locked state, Eq. (29).

profile that includes all three harmonics in the small noise limit; the second harmonics appears as a supercritical bifurcation at $\beta \approx 3$.

In conclusion, we have studied the problem of maximizing coherence of oscillations by external locking, in the phase approximation. The optimal phase forcing function depends not only on the phase response curve of the system, but also on the noise intensity. For large noise, a purely harmonic forcing is optimal, the number of the harmonic depends on the phase sensitivity. For smaller noise, a bifurcation to a more general, multiharmonic forcing may occur. We have also demonstrated that different optimality conditions in the purely deterministic case lead to different optimal forcing functions, which also differ from the limit of small noise when optimization of the coherence is performed.

The approach of this Letter can be potentially generalized to a broader class of situations. Above we considered an oscillatory system subject to two driving forces: One is white noise that brings incoherence, and another is a purely periodic forcing that brings coherence. More generally, for nonautonomous systems [11] both forcing terms can be nonideal: Noise can be correlated, and the applied force can be not exactly periodic. Moreover, parameters of the oscillator can vary in time. While coherence properties

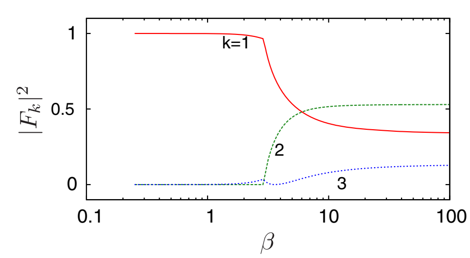

FIG. 3 (color online). Optimal forcing in dependence on the noise intensity for the experimental phase sensitivity of a mitral cell [10]. While for large noise the solution is the pure first-harmonic one, according to the theory, for small noise a forcing with dominating second harmonics is optimal.

in such cases can be studied by direct numerical simulations, an analytic treatment (in contradistinction to the ideal case considered above) remains a challenging task for further studies, as already the phase reduction and the averaging over fast oscillations are not straightforward.

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