Comment on "Asymptotic Phase for Stochastic Oscillators"

The definition of the phase of oscillations is straightforward for deterministic periodic processes but nontrivial for stochastic ones. Recently, Thomas and Lindner suggested using the argument of the complex eigenfunction of the backward density evolution operator with the smallest real part of the eigenvalue, as an asymptotic phase of stochastic oscillations [1]. Here, I show that this definition does not generally provide a correct asymptotic phase.

The notion of the phase of periodic oscillators lies at the heart of characterization of oscillatory processes. In deterministic systems, the phase close to a limit cycle is defined via construction of isochrons (isophases)—the surfaces of Poincaré sections of the flow with the return time being exactly the period of oscillations. Recently, the notion of isochrons has been extended to stochastic systems, as the surfaces of constant mean first return time [2]. A different definition of the asymptotic phase of stochastic oscillations has been suggested in [1], based on properties of the Fokker-Planck operator describing the evolution of the probability density. If the nontrivial eigenvalue of the backward Fokker-Planck (BFP) operator with the least negative real part is complex, the systems has been called robustly oscillatory. For such systems, the asymptotic phase was defined in [1] as the argument of the least stable complex eigenfunction.

While the phase is well defined for periodic self-sustained oscillations, already for chaotic ones, it is in many cases ill defined. It exists when one can find a good foliation of the attractor by Poincaré surfaces of sections (isochrons in the periodic case). Then, all trajectories cross them in one direction, and a return to the particular surface after one loop defines one oscillation cycle. When applying this approach to noisy oscillators, one has to skip the condition of crossing surfaces of the section in one direction, but if, nevertheless, a return to the same surface happens after performing a global loop, the oscillations and the phase are well defined.

It is important that the foliation should be good, i.e., nonsingular. For example, for a damped linear oscillator

$$\dot{x} = -\gamma x + \omega y, \qquad \dot{y} = -\gamma y - \omega x, \tag{1}$$

the lines $\arctan(y/x) = \text{const}$ are crossed by nontrivial trajectories, but this foliation is singular at the origin. In physical terms, existence of a good foliation corresponds to oscillations with nonvanishing amplitude; it is clear that when the amplitude vanishes, the phase is not well defined.

If white Gaussian noise terms are added on the rhs of Eqs. (1), the latter become the Langevin equations. The eigenvalue problem for the BFP operator reduces to the quantum-mechanical problem of a two-dimensional harmonic oscillator, yielding the complex eigenvalues $\lambda_{mn} = -im\omega - \gamma n$, $n = 0, 1, 2, \ldots$, $m = -n, -n+2, \ldots, n-2, n$. The lines of the constant argument of the least stable eigenfunction ($m = \pm 1$, $n = 1$) are the lines $\arctan(y/x) = \text{const}$. However, this dynamics cannot be characterized as limit-cycle-type oscillations, as the amplitude fluctuates around zero, and the corresponding phase does not exist.

The dissipative rotating dynamics described by (1) may be combined with a true limit cycle. For example, in a three-dimensional system, a deterministic limit cycle can exist, with the phase obeying $\dot{\theta} = \Omega$, while locally, in the transverse direction (x, y), the attraction to the cycle is described by system (1). With small noise, such a system will have proper oscillations, with Poincaré surfaces of section $\theta = \text{const}$. Now the BFP equation also includes relaxation along the phase θ. If this relaxation is weak, then the corresponding eigenvalue will be the least stable one, and the eigenfunction, indeed, provides $\theta = \text{const}$ as surfaces of constant argument. However, if the noise along θ is strong, then the least stable will be the eigenfunction describing the transverse dynamics (1); in this case, the argument of the corresponding eigenmode will not provide the correct oscillation phase θ. On the contrary, the proper foliation by the Poincaré surfaces of sections $\theta = \text{const}$ exists in both cases, thus, the approach of [2] provides a proper phase.

In conclusion, while the phase is a rotating variable, in the phase space of a noisy dynamical system, there can be many rotations corresponding to complex eigenvalues of the density evolution operator. Therefore, one generally cannot identify the phase using the least stable complex eigenfunction.

Two questions remain, in my opinion, open. (i) In the case where the method [1] provides a proper phase variable, are the corresponding surfaces of constant argument of the eigenfunction also the surfaces of the constant mean first return time [2]? (ii) Is it possible to distinguish between proper and improper phase variables purely on the basis of the BFP approach (e.g., by inspecting the invariant probability density in the domain where the amplitude of the relevant eigenfunction vanishes)?

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