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First and second sound in disordered strongly nonlinear lattices: numerical study

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Abstract. We study numerically secondary modes on top of a chaotic state in disordered nonlinear lattices. Two basic models are considered, with or without a local on-site potential. By performing periodic spatial modulation of displacement and kinetic energy, and following the temporal evolution of the corresponding spatial profiles, we reveal different modes which can be interpreted as first and second sound.

Keywords: disordered systems (theory), connections between chaos and statistical physics

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1. Introduction

Statistical properties of dynamical regimes in nonlinear lattices have been a focus of research since the discovery of the Fermi–Pasta–Ulam (FPU) paradox [1]. Generally, one expects that such lattices at large energy densities demonstrate chaos. However, statistical properties of the irregular state can be quite peculiar, e.g. demonstrating anomalous statistics of Lyapunov vectors [2] and anomalous thermal conductivity [3]. One of the ways to characterize chaotic dynamics is to look at the excitations on top of it. In different contexts such modes are referred to as hydrodynamic or sound-type modes. Wave-like modes evolving from the perturbations of density and temperature can be roughly classified as the first and the second sound, respectively. The goal of this paper is to study such modes numerically in strongly nonlinear disordered lattices.

Perturbations on top of a disordered background can be considered in two basic setups. In one setup one produces a local in space perturbation and studies its diffusive spreading (see, e.g. [4–7]). For a hard-point chain (elastically colliding particles on a line), in [8] an anomalous diffusion corresponding to a Lévy walk of the energy was found. Interestingly, the same approach can be applied to the spreading of a localized energy hump over a vacuum background (instead of on top of a finite energy density). This problem has been intensively studied for disordered lattices in the context of Anderson localization destruction by nonlinearity, mainly for a nonlinear Schrödinger equation [9–11], but also for disordered strongly nonlinear lattices of the type we study in this paper [12–14]. In the latter case the spreading has been successfully described by a nonlinear diffusion equation.

In another setup, which we also follow below, a spatially periodic perturbation is imposed, and its relaxation to equilibrium is followed. In [15, 16] acoustic modes on top of a chaotic background have been studied in this way. In [17, 18] a spatially periodic temperature profile has been imposed in lattices of the FPU type with and without on-site potential, by coupling to a white noise thermostate, whose intensity was spatially modulated; after release of the thermostate the relaxation to a uniform temperature was followed.

These two types of modes are related to concepts of first and second sound in the quantum theory of solids [19]. While the first sound just corresponds to a classical sound wave, the second sound is typically interpreted in terms of a quantum-mechanical description of oscillations. They are represented as a gas of phonons with different interaction mechanisms interpreted as scattering processes. Local temperature perturbations in a gas of phonons may, at low temperatures, propagate as waves, similar to the second sound in superfluidic systems. To the best of our knowledge, a classical theory of second sound is missing. Nevertheless, in classical systems one can attack the problem numerically, like in [17, 18]. On the other hand, in quantum-mechanical treatment one starts with ideal lattices without defects, and effects of disorder on the second sound are mainly unexplored. In classical setups, it is easy to introduce disorder and nonlinearity to lattice systems and explore the effects of these factors.

In this paper we present numerical studies of sound-like modes in disordered, strongly nonlinear lattices. In these lattices coupling and on-site terms in the potential energy start with powers larger than two, so that in the linear approximation there are no propagating waves (so called sonic vacuum). Correspondingly, there are no linear phonons in the quantum formulation of the problem [16]. The most prominent example is the Hertzian chain of hard balls [20], see also [21]. In such lattices nonlinear waves, so-called compactons [22, 23], can propagate, but in the presence of disorder compactons are not possible. For a finite energy density such lattices demonstrate chaos: contrary to usual FPU-type chains, there is no threshold for chaos due to the absence of linear terms (in some sense, the strongly nonlinear lattices can be considered as the high-energy limit of usual lattices with linear and nonlinear terms; our model A below is thus a high-energy limit of the FPU lattice). Moreover, as we also show below, in homogeneous strongly nonlinear lattices, where all nonlinear terms have the same power, there is no essential dependence on the energy density level, just the time scale depends on it. This universality strongly simplifies the study, as there is no qualitative dependence on the energy density level. Otherwise, we expect that usual nonlinear lattices for high energy densities will demonstrate similar properties.

2. Lattice models

In this paper we consider two types of strongly nonlinear lattices: without local potential, and with it. We will call them model A and model B, respectively. The Hamilton function for model A reads

$$H = \sum_l \frac{p_l^2}{2} + \kappa_l \frac{(q_{l+1} - q_l)^4}{4}, \qquad (1)$$

and model B is described by Hamiltonian

$$H=\sum_{l}\frac{p_l^2}{2}+\beta_l\frac{q_l^4}{4}+\kappa_l\frac{(q_{l+1}-q_l)^4}{4}. \tag{2}$$

In the case of regular lattices, parameters κ_l, β_l are constants. For disordered lattices they are quenched random variables. For model A we considered uniformly distributed independent values of κ_l: $0.5\leqslant\kappa_l\leqslant 1.5$. For model B parameter κ was set to 1, and the parameter β_l was chosen from a uniform distribution $0.5\leqslant\beta_l\leqslant 1.5$. In fact, most of the calculations have been performed for both cases, regular and disordered lattices, with almost coinciding results. Thus we present here numerics for disordered lattices only.

Regimes in nonlinear lattices generally depend on the energy density. For example, in the FPU lattice, nearly periodic, recurrent regimes are observed for small energy densities, while chaos predominates at large energies. In models A and B, the potential terms are powers of the coordinates, moreover the local and the coupling potentials in model B have the same power. Therefore regimes in these models are independent of the energy density: the latter can be rescaled together with the time so that the Hamiltonian remains invariant: $E\rightarrow\alpha E'$, $t\rightarrow\alpha^{-1/2}t'$. This means that the basic regime does not depend on the energy density, just its characteristic time scale is dependent on it. Thus below we set energy density to 1.

In both lattices a chaotic state is observed, independently of the presence of disorder. We illustrate this in figure 1 by showing time dependencies of the coordinates and the momenta in chains of 32 oscillators.

3. First sound modes

In this section we study the properties of the 'first sound mode' on top of a chaotic state described above. Our procedure is as follows (see [15, 16]). First, equations of the lattice of length L have been integrated from random initial conditions until a statistical equilibrium is achieved. Next, a small harmonic in space perturbation with the wave number $2\pi k$, where $0<k\leqslant\frac{1}{2}$, has been added to all the coordinates: $q_l\rightarrow q_l+\varepsilon\cos 2\pi kl$. From the subsequent integration of the lattice, the average response amplitude of the mode at this wave number has been calculated as a function of time: $Q(k,t)=\langle\sum_l q_l\cos 2\pi kl\rangle$. Practically, we have not used the ensemble average, but instead simulated a very long lattice so that an additional average was not needed. Indeed, the lattice contains kL periods of the perturbation, which due to irregularity of the underlying dynamics can be considered as independent, so that just the summation over the lattice index l ensures averaging over kL effectively independent samples. The only relevant parameter determining the spacial scale is thus the wave number of the perturbation k. The overall length of the lattice L just measures the number of independent samples and the quality of statistical averaging. The smooth curves $Q(t)$, $E(t)$ in figures 2, 4 and 6 below show that the statistics are good indeed. The obtained response amplitudes are illustrated in figure 2.

We have performed a fit of the found dependencies as $Q(t)=C\exp[-\gamma t]\cos\Omega t$, and found dispersion relations $\Omega(k),\gamma(k)$ presented in figure 3. One can see that the two models show different behaviors at small wave numbers: while in model B both the frequency Ω and the damping constant γ tend to finite values (i.e. the phonons are

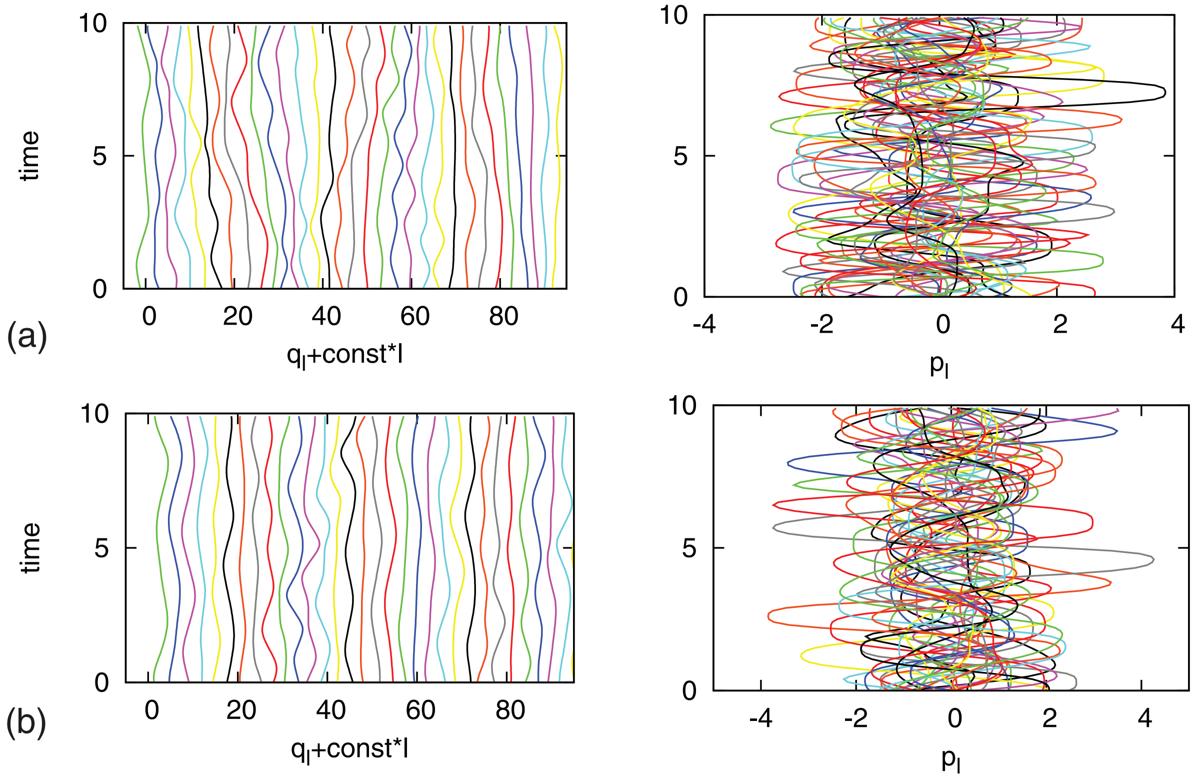

Figure 1. Examples of chaotic fields in the lattices of type A (a) and type B (b).

'optical'), for model A the frequency scales as $\Omega(k) \sim k$ ('acoustic' phonons) while the damping constant scales as $\gamma(k) \sim k^{5/3}$, as predicted by the theory developed in [24].

4. Second sound modes

In this section we proceed similarly to the investigation of sound modes above, but the spatially periodic perturbation is performed not in the displacement variable q, but in the local momentum p, which is modulated according to $p_l \to p_l(1+\varepsilon\cos(2\pi kl))^{1/2}$. This models spatially periodic modulation of the kinetic energy. Also the response $E(k,t)$ is calculated according to the corresponding mode of the local energy observable: $E(k,t) = \langle \sum_l \mathcal{E}_l \cos 2\pi kl \rangle$, where $\mathcal{E}_l$ is the local energy term (that includes both kinetic and potential energy) in the Hamilton function (i.e. for model B $\mathcal{E}_l = \frac{p_l^2}{2} + \beta_l \frac{q_l^4}{4} + \kappa_l \frac{(q_{l+1}-q_l)^4}{8} + \kappa_{l-1}\frac{(q_l-q_{l-1})^4}{8}$ and similar for model A). The results are more complex than in the case of first sound modes, and we consider them separately for models A and B.

4.1. Model A

We present the results of the calculations of the response $E(k,\ t)$ in figure 4. One can see that this function is not simply a decaying cos-function. To reveal the structure of the response, we performed a cos-Fourier transform $F(k,\omega)$ of $E(k,\ t)$ with respect to variable t; the results are also presented in figure 4. To each exponentially decaying

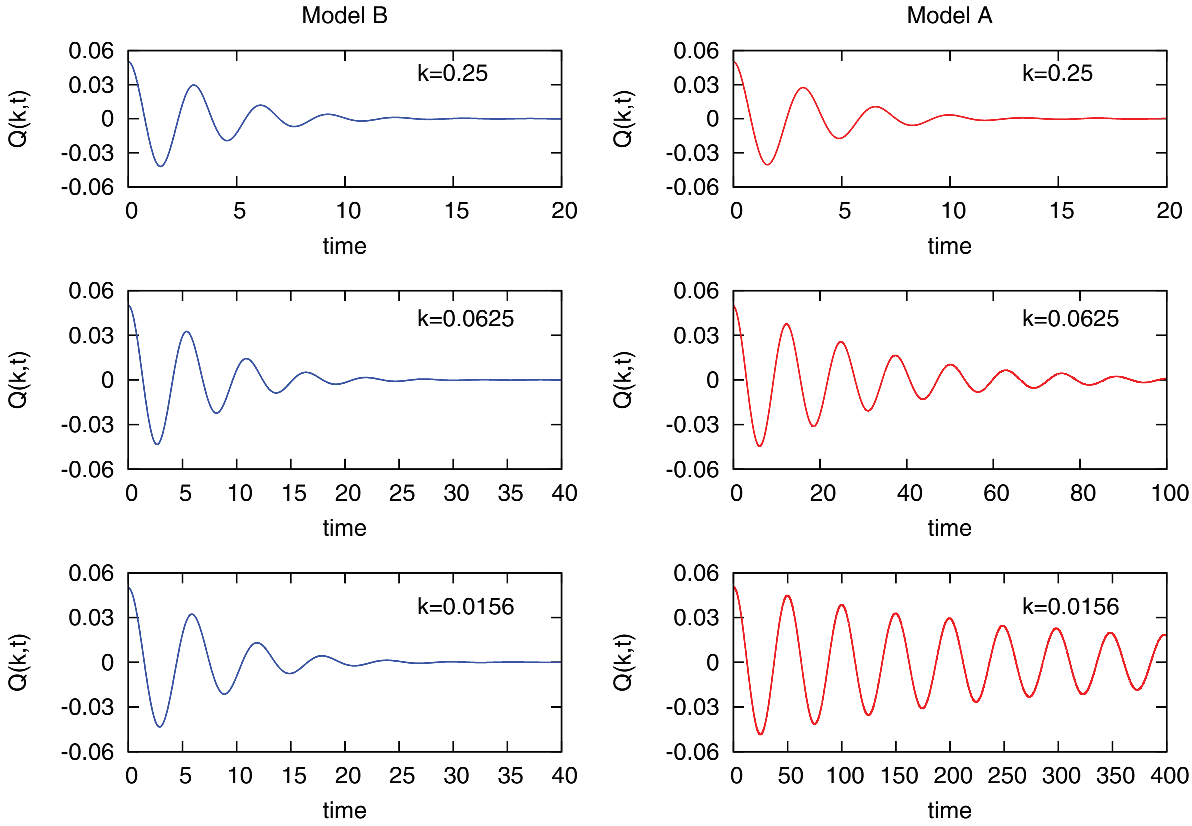

Figure 2. Evolution of the first sound mode amplitudes in models (A) and (B), for three different wave numbers.

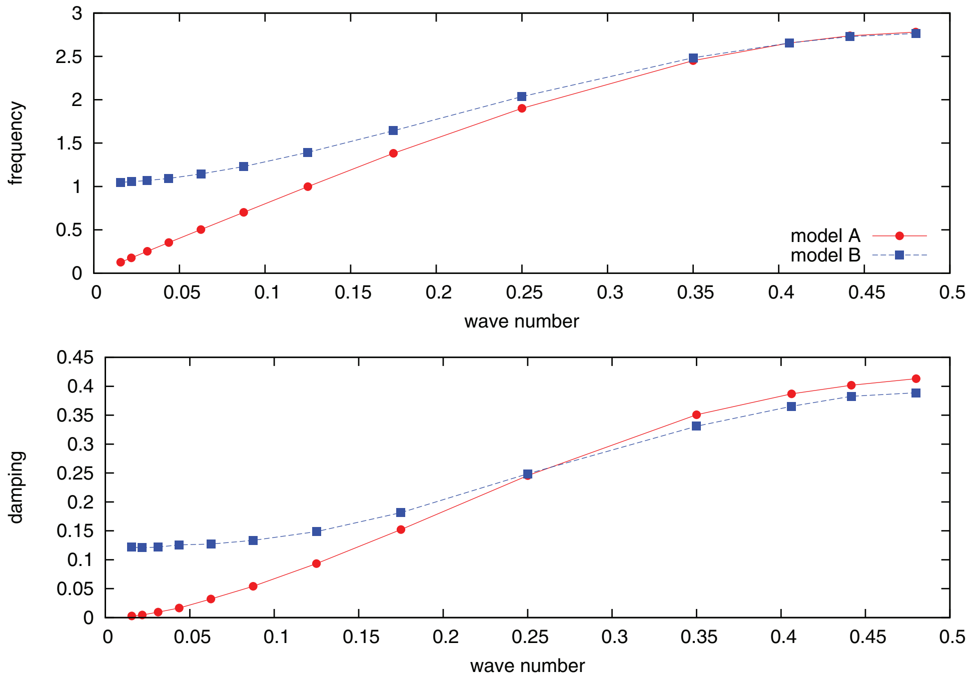

Figure 3. Dispersion properties of the first sound modes in models (A) (red circles) and (B) (blue squares). Top panel: $\Omega(k)$, bottom panel: $\gamma(k)$.

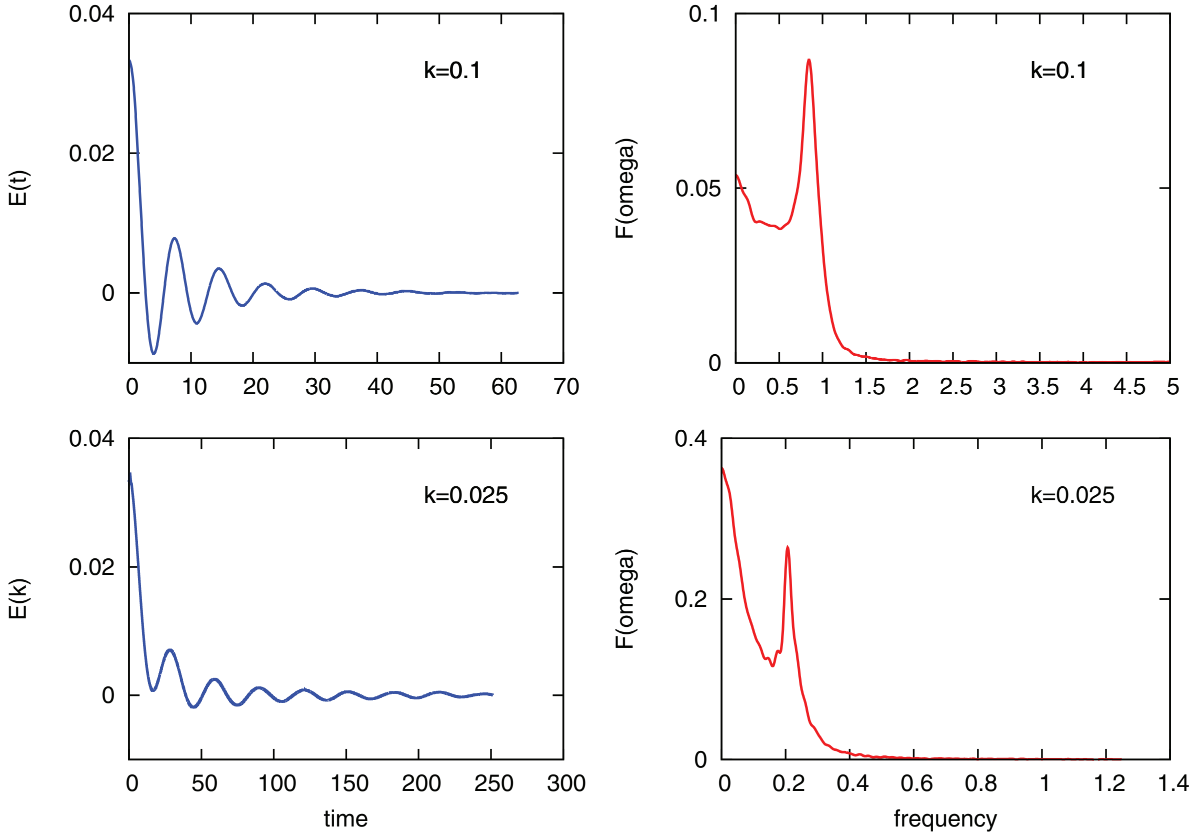

Figure 4. Time relaxation of the energy modes (left panel) and its cos-transform (right panel) for model (A) and two wave numbers.

cos-function corresponds a Lorentzian peak in the spectrum, and one can see two such peaks in $F(k,\omega)$, one at zero frequency and one at a non-zero one. This suggests the fit $E(k,t) \sim C_0 \exp[-\gamma_0 t] + C_1 \exp[-\gamma_1 t] \cos \Omega_1 t$. The obtained dependencies $\gamma_{1,2}(k)$ and $\Omega_1(k)$ are shown in figure 5. One can see that the frequency spectrum nearly overlaps with that of the first sound mode for model A, while the damping constants show a slightly different behavior. Both of them appear to scale like $\gamma_{0,1} \sim k^{5/4}$.

4.2. Model B

Response functions for the second sound perturbations in model B, presented in figure 6 appear to be more complex than in model A. One can see in the frequency dependence of the Fourier-transforms three peaks for large wave numbers, and two peaks for small wave numbers. Thus, we performed fitting according to the representation $E(k,t) \sim C_0 \exp[-\gamma_0 t] + C_1 \exp[-\gamma_1 t] \cos \Omega_1 t + C_2 \exp[-\gamma_2 t] \cos \Omega_2 t$, results of which are presented in figure 7. We plot here also the amplitudes of the modes C_0, C_1, C_2, to show that mode 2 exists for large wave numbers only $k \gtrsim 0.2$. Remarkably, the frequency of mode 1 looks like an acoustic spectrum with $\Omega \sim k$ for small wave numbers, although the first sound mode for model B does not have this property (see figure 3). Another interesting feature is that the dependence of the zero-frequency mode damping constant is the same as in model A: $\gamma_0 \sim k^{5/4}$, while for the first mode it is different: $\gamma_1 \sim k^{4/5}$.

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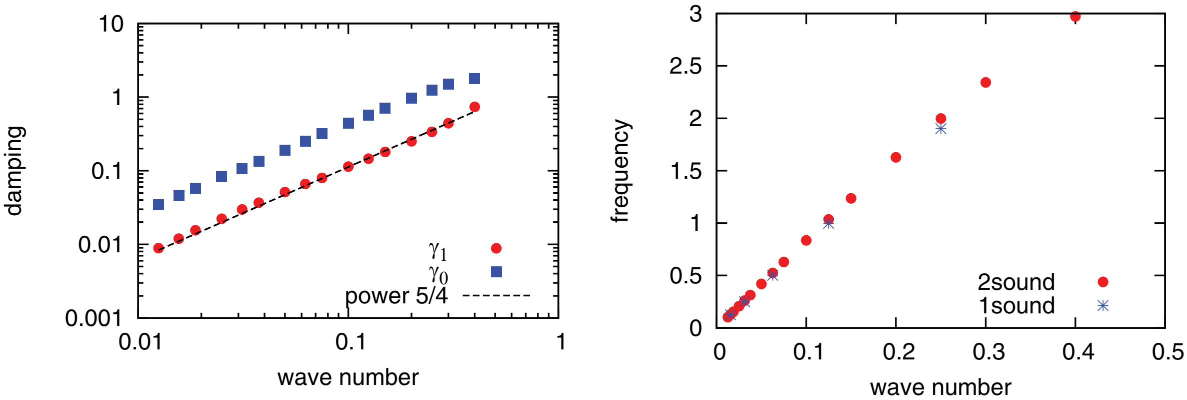

Figure 5. Dispersion relations for the second sound modes in model (A). Right panel: $\Omega(k)$, here also the frequency of the first sound mode is shown. Left panel: damping constants $\gamma_0(k)$ and $\gamma_1(k)$, the dashed line has slope $5/4$.

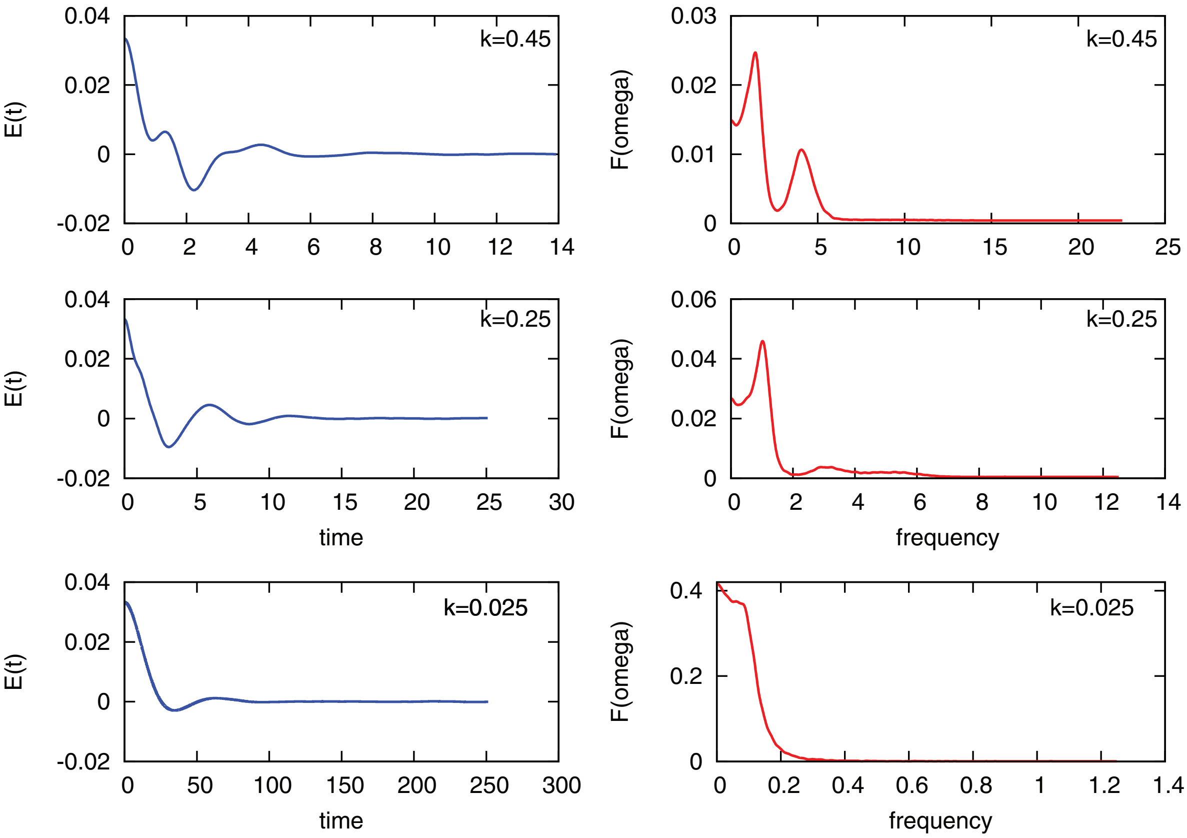

Figure 6. The same as figure 4, but for model (B).

5. Conclusion

In this paper we reported on collective modes that are observed on top of a turbulent state of disordered, strongly nonlinear lattices. In fact, the most important property here is the turbulence: similar properties are observed for turbulent states in regular strongly nonlinear lattices as well; we expect also that usual nonlinear lattices will demonstrate

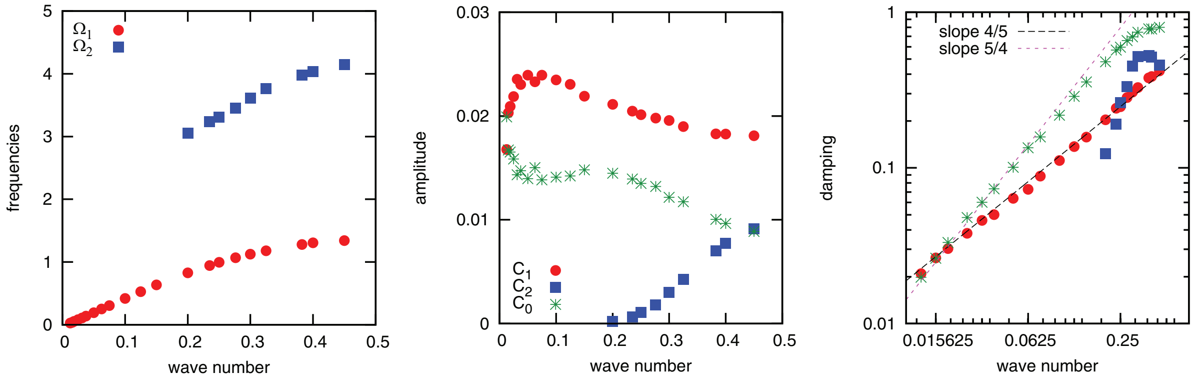

Figure 7. Dispersion relations for the second sound modes in model (B). Left panel: frequencies Ω_1 (red circles) and Ω_2 (blue squares). Middle panel: Amplitudes of the modes (asterisks denote zero-frequency mode). Right panel: damping constants $\gamma_0, \gamma_1, \gamma_2$ (asterisks, circles, and squares, respectively). Dashed black and dotted magenta lines have slopes 4/5 and 5/4.

similar features (at least for high energy densities, where strong chaos is observed), but this question deserves a separate study. Our approach is in the imposing of small spatially periodic perturbations of the underlying chaos, and in following the time evolution of the excited periodic response fields in time. It appears that the amplitude of the response fields can be represented as a sum of harmonic damping components $\sim e^{-\gamma t}\cos(\Omega t)$. The properties (and the number) of these components depend on the type of initial perturbation (first versus second sound) and on the lattice model used. In model A (a lattice without on-site potential) two modes are observed: a purely acoustic spectrum if the perturbation is in the displacement of particles, and additionally a zero-frequency mode if the perturbation is in the modulation of the kinetic energy. In model B (a lattice with an on-site potential) a perturbation in the displacement of particles produces 'optical phonons', i.e. a mode, the frequency of which tends to a finite value at a vanishing wave number. A perturbation in the kinetic energy produces three modes, one with zero frequency and another with an acoustic spectrum which exists for all wave numbers; an additional 'optical' mode exists for large enough wave numbers only.

Generally, the sound modes in a lattice could be studied in another setup, where not the wave number in an initial-value problem is imposed, but a frequency of a perturbation at a boundary is fixed by an external force. Studies of the first sound-type perturbation in such a model will be reported elsewhere. It is not clear, however, what type of boundary forcing corresponds to the second sound mode.

Recently, an important theoretical progress in the understanding of secondary modes in nonlinear lattices has been achieved in [25, 26], where it was shown that the Kardar–Parisi–Zhang universality holds for nonlinear lattices without local potential, like model A above, under some assumptions. The relation of the presented numerical findings to this theory should be explored in future studies.

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