Phase Description of Stochastic Oscillations

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We introduce an invariant phase description of stochastic oscillations by generalizing the concept of standard isophases. The average isophases are constructed as sections in the state space, having a constant mean first return time. The approach allows us to obtain a global phase variable of noisy oscillations, even in the cases where the phase is ill defined in the deterministic limit. A simple numerical method for finding the isophases is illustrated for noise-induced switching between two coexisting limit cycles, and for noise-induced oscillation in an excitable system. We also discuss how to determine isophases of observed irregular oscillations, providing a basis for a refined phase description in data analysis.

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Phase reduction is the basic tool in the characterization of self-sustained, autonomous oscillators. With a reasonably defined phase variable, one obtains a one-dimensional representation of the oscillator, allowing us to describe important aspects of its dynamics, such as regularity, sensitivity to forcing, and noise, et cetera [1–3].

The concept of phase is also substantial for the data analysis of oscillatory processes in physics, chemistry, biology, and technical applications, where events, such as a heart beat, are determined by recurrences of the process into a certain phase. In implementation of this concept, various approaches exist for extracting a phase variable from oscillatory time series [4–8].

To understand many properties of oscillating systems, such as their phase resetting and synchronizability, it is important to define the phases not only for the purely periodic motion, but for the whole state space. In the theory of deterministic oscillations, this is done via isochrones [9], i.e., isosurfaces of constant phase, each of which gathers those states that converge to the same state on the limit cycle. However, many applications, especially biological and geophysical ones [10], exhibit a nondeterminism that demands for an extension of existing nonlinear deterministic methods to incorporate stochasticity explicitly. For example, in the review on the neuronal dynamics Smeal *et al.* [11] raise the question: "Are neurons too noisy to be described by phase-response theory?"

In this Letter we extend the foundations of deterministic phase description to irregular, noisy oscillators. The main idea is based on the definition of the *isophases* by virtue of the mean first passage time concept. We will first apply our method to noise-perturbed deterministic oscillators for which the isophases can be compared with the deterministic isochrones. Because our theory is *nonperturbative* with respect to noise, it makes the phase-response theory generically applicable to strongly irregular oscillations, even for those where isochrones and oscillations do not exist in the deterministic limit. Furthermore, we will show, that application of our approach to noise-perturbed chaotic oscillations allows one to obtain well-defined isophases in this case as well. Finally, to highlight the relevance of the introduced concept in data analysis, we will demonstrate that isophases can be obtained from observed oscillatory signals.

We start by reminding the reader the standard definition of isophases in deterministic systems with a stable periodic orbit $\mathbf{x}_0(t) = \mathbf{x}_0(t + T)$ of period T. For these, isophases are also isochrones. First, one defines the phase on the orbit $\varphi(\mathbf{x}_0)$. When being observed stroboscopically with time interval T, all the points $\mathbf{x}$ that converge to a particular point on the orbit $\mathbf{x}_0$ have the phase $\varphi(\mathbf{x}_0)$. These points form a Poincaré surface of section $J(\varphi(\mathbf{x}_0))$ for the trajectories of the dynamical system, with the special property that the return time to this surface equals T for all points on it. Thus, finding an isophase surface is equivalent to finding a Poincaré surface of section with the constant return time T.

For a noisy system we define the isophase surface J as a Poincaré surface of section, for which the mean first return time $J \to J$, after performing one full oscillation, is a constant T, which can be interpreted as the average oscillation period. In order for isophases to be well defined, oscillations have to be well defined as well: for example in polar coordinates, the radius variable must never become zero, so that one can reliably recognize each "oscillation." Random processes for which this is not the case should be treated with care.

Analytical calculations of the mean first return time (MFRT) are a complex problem in dimensions larger than one; therefore, below we apply a simple numerical algorithm for construction of the isophases: an initial Poincaré section is iteratively altered until all mean return times are approximately equal. In two-dimensional systems for which isophases are lines, we represent Poincaré sections by a linear interpolation in between a set of knots. For each knot x_j, the average return time T_j is computed via the Monte Carlo simulation. According to the mismatch of T_j and the mean period $\langle T\rangle$, the knot x_j is advanced or retarded. The procedure is repeated with all knots, until it converges and all return times T_j are nearly equal to $\langle T \rangle$.

Before demonstrating instructive examples of average isophases, we discuss the importance of knowing isophases for noisy oscillations. The first important application is that of phase resetting. Phase resetting curves determine how an oscillator responds to external kicks, which determines synchronization properties of the oscillator [12,13]. For deterministic oscillators the phase response curve is determined just from the isochrone to which the kick shifts the state of the system from the limit cycle. For irregular oscillators the proper definition of the phase response curve is based on first passage times [14]. Thus, to determine it, one has to find to which isophase the state of the system is shifted by the kick. The correct approach is to use the isophase defined above (whereas in the limit of small noise, perturbative calculations of phase dynamics have been shown to do the job as well [15,16]). The second application is in the analysis of experimental data of coupled oscillators (cf. [4–8]). There one needs to determine the phase dynamics from time series, this task is relatively simple to accomplish if the variations of the amplitudes are very small so that the definition of a phaselike variable along the observed limit cycle is unambiguous. However, in the presence of large irregular amplitude variations, the phase characterization of the oscillations is not unique (cf. Fig. 5 below). Proceeding according to the given above definition of the isophases as the lines on the two-dimensional embedding plane, for which the mean return times do not depend on the amplitudes, allows us to get rid of the ambiguity and to determine the phase in a consistent way.

We stress here that in our definition of isophases, we do not assume the process to be Markovian: if its dynamics is non-Markovian, then the definition of the MFRT includes averaging over the "prehistory" or hidden variables as well. To illustrate this we consider as the first example a simple Stuart-Landau oscillator (variables r, θ) perturbed by an Ornstein-Uhlenbeck noise $\zeta(t)$:

$$\begin{aligned} \dot{r} &= r(1-r^2) + \sigma r \zeta(t), \\ \dot{\theta} &= \omega - \kappa(r^2 - 1), \\ \gamma\dot{\zeta} &= -\zeta + \sqrt{\gamma}\xi(t), \end{aligned} \tag{1}$$

where $\xi(t)$ denotes a δ-correlated white noise, γ and σ are the correlation time and the noise intensity of the Ornstein-Uhlenbeck noise, ω is the frequency of the noise-free limit cycle, and κ is a nonisochronicity parameter. In the state space (r, θ, ζ) the process is Markovian, but on the two-dimensional plane (r, θ) it is not. Nevertheless, by the method described we obtain the numerical isophase for which the MFRT is nearly constant (Fig. 1). This isophase can be obtained also from an analytic approximation: first, we introduce a "corrected" phase variable $\psi = \theta - \kappa \ln r$ which obeys $\dot{\psi} = \omega + \sigma\kappa[\gamma\dot{\zeta} - \sqrt{\gamma}\xi(t)]$. Averaging this expression and identifying $\omega = \dot{\varphi}$ where φ is the correct

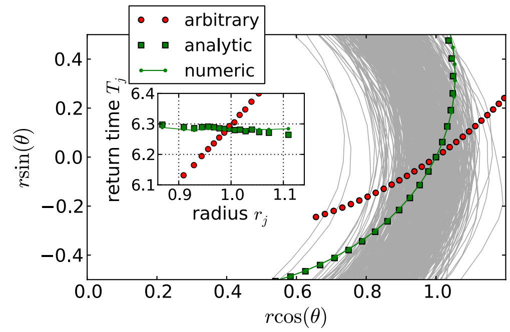

FIG. 1 (color online). The average isophase for a noise-driven oscillator [Eq. (1)] with $\omega = 1$, $\kappa = 1$, $\sigma = 0.15$, $\gamma = 1$. The inset compares its close to constant MFRTs with those of an arbitrary cross section. The numerically derived isophase shows minor differences to the analytic approximation [Eq. (2)].

uniformly rotating phase, we obtain $\varphi = \psi - \sigma\kappa\gamma\zeta$. In this expression we have to account for correlations of ζ and r, to obtain the isophases on the plane (θ, r). Assuming that r follows $\zeta(t)$ adiabatically, we obtain $\sigma\zeta \approx r^2 - 1$, which leads to the following expression for the isophases

$$\varphi = \theta - \kappa \ln r - \kappa\gamma(r^2 - 1). \tag{2}$$

An isophase following from this formula is compared with a numerical one in Fig. 1. Interestingly, the noise-induced correction [last term in Eq. (2)] does not contain noise intensity σ, but the range where this correction is valid $|r - 1| \lesssim \sigma$ shrinks with the noise intensity.

While in the simplest example above the effect of noise is in the correction of the deterministic isochrones only, we consider now a situation where local isophases of different periodic motions are "mixed" by noise resulting in new, global isophases. To this end we analyze the following model of two coexisting stable limit cycles, driven by white noise:

$$\begin{aligned} \dot{r} &= r(1-r)(3-r)(c-r) + \sigma r \xi(t), \\ \dot{\theta} &= \omega + \delta(r-2) - (1-r)(3-r). \end{aligned} \tag{3}$$

Without noise, the system shows two limit cycles $r_{\mathrm{I}} = 1$, $r_{\mathrm{II}} = 3$ (which have the same frequency if $\delta = 0$), separated by an unstable cycle at $r = c$. Each of the stable cycles has its own isophases, which meet singularly (as infinitely rotating spirals) at the basin boundary $r = c$. With noise, trajectory switches between the basins, so that combined mixed-mode oscillations involving both cycles occur. By applying our method, we find the isophases of these oscillations in the whole range of radius r, as shown in Fig. 2. While for small noise intensity, a residue of the singularity at the basin boundary is clearly seen, for a strong noise the isophases are rather smooth curves.

Another example where otherwise singular isophases are smeared by noise is that of chaotic oscillations. Many chaotic attractors allow a representation in terms of

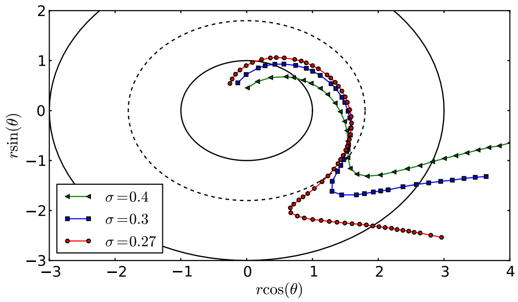

FIG. 2 (color online). Isophases of bistable oscillations mixed by noise, in model (3) with $\omega = 3$, $c = 1.8$, $\delta = 0.01$ and different noise intensities. Stable cycles (solid black curves), separated by an unstable cycle (dashed curve), have slightly different periods, leading to a "stretching" of isophases.

amplitudes and phases [3,17,18], but because the phase generally performs a chaos-induced diffusion, isophases in the strict sense do not exist. Recently, a description of chaotic oscillations in terms of approximate isophases has been suggested [19]. With noise, the return times to a Poincaré surface of a strange attractor can be defined in the averaged sense only, and in this respect there is no difference between chaotic and regular deterministic oscillators. Thus, the procedure of finding isophases based on the constancy of the MFRTs can be applied to chaotic systems as well, as is illustrated in Fig. 3 for the Roessler model

$$\begin{aligned} \dot{x} &= -y - z + \sigma\xi_1(t), \\ \dot{y} &= x + 0.16y + \sigma\xi_2(t), \\ \dot{z} &= 0.2 + z(x - 10), \end{aligned} \tag{4}$$

with uncorrelated Gaussian white noises in x and y components. Remarkably, including small noise not only allows us to give a good definition of isophases, which is absent for chaotic attractors in the noise-free limit, but provides additionally a stable numerical procedure for the isophases' calculations.

Our final example is noise-induced oscillations in an excitable system, which without noise has just a stable steady state, so deterministic isophases do not exist in

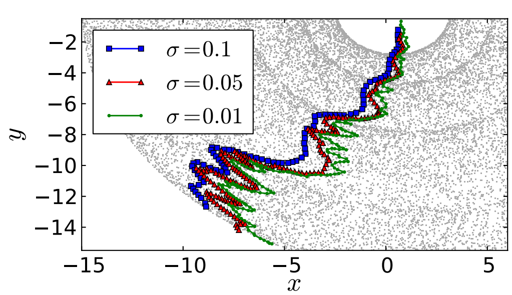

FIG. 3 (color online). Isophases of the noise-driven Roessler chaotic system [Eq. (4)], for different noise intensities σ. Smaller noise leads to less smooth curves. Gray dots show the deterministic Roessler attractor.

any sense. With noise, such a system demonstrates oscillations which may be quite regular in the case of coherence resonance [20]. To build the model, we modify the noisy Stuart-Landau oscillator, with y-polarized noise, to perform noise-induced oscillations:

$$\begin{aligned} \dot{r} &= r(1 - r^2) + \sigma r\cos\theta\xi(t), \\ \dot{\theta} &= \omega + r\cos\theta - \kappa r^2 + \sigma\sin\theta\xi(t). \end{aligned} \tag{5}$$

For $0 < \omega - \kappa < 1$, noise excites the stable state [$r = 1$, $\theta = \pi - \arccos(\omega - \kappa)$] beyond the unstable one [$r = 1$, $\theta = \pi + \arccos(\omega - \kappa)$] and induces oscillations. For strong excitability and small noise, the phase is well defined, and the isophases can be introduced as curves with constant MFRTs. We show ten isophases in Fig. 4 (examples Figs. 1 and 2 have been rotationally symmetric, so one drawing of one isophase was sufficient, here the rotational symmetry is broken). The effect of the noise intensity on the isophases is maximal at $\theta \approx \pi$, i.e., in the region of excitability where the oscillations spend most of the time; in the "deterministic" region $|\theta| < \pi/2$ the isochrones are less sensitive to noise.

A practical definition of the isophases, for which the MFRT is constant, is straightforward for numerical models of irregular oscillators as illustrated above, but it can be used for experimentally observed signals as well. For this purpose one needs a two-dimensional embedding of observed oscillations, which can be, e.g., achieved by using the Hilbert transform of the signal as the second variable. In Fig. 5 we present such a representation of measurements of human respiration, taken from the Physionet database [21]. One can see that the oscillations have a large amplitude variability, and defining the phase has a large degree of ambiguity—contrary to the situations with a nearly constant amplitude, where a similar embedding on the signal vs its Hilbert transform plane results in a very narrow band of trajectories. The initial phaselike angle variables and the isophases resulting from the iterative

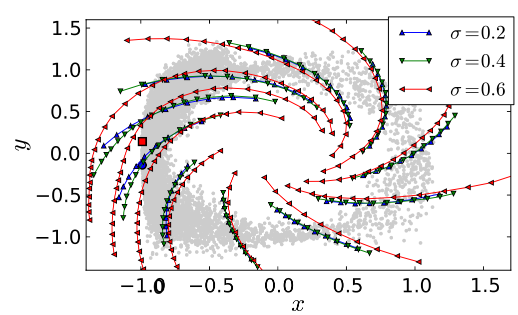

FIG. 4 (color online). Average isophases of Eq. (5) at $\omega = 1.99$ and $\kappa = 1$ vary with noise intensity σ as indicated. Larger noise intensity makes the system less isochronous, letting average isophases show stronger curvature. (Background trajectory of noise-induced oscillations (gray dots) corresponds to the strong noise case $\sigma = 0.6$.) Large markers show the stable (square) and the unstable (circle) steady states.

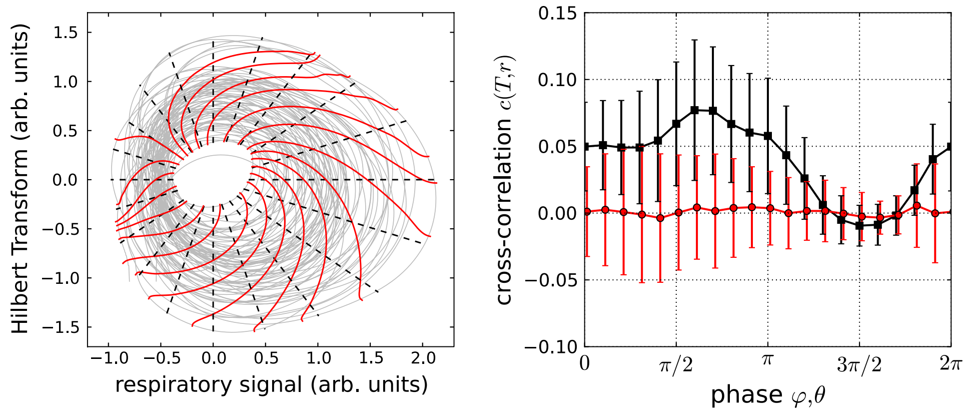

FIG. 5 (color online). Left panel: Arbitrary angle variables (black dashed lines) and isophases (red solid lines) of a respiration signal taken from the public database "Physionet." Right panel: For isophases the return times are independent of the radial variable (cross correlation vanishes).

procedure as described above are presented in Fig. 5. Application of the calculated isophases to determining the phase dynamics of the observed signals gives the mostly uniformly rotating phase, maximally uncorrelated from the amplitude variations.

In summary, we have introduced for irregular oscillations a concept of average isophases based on the constancy of the mean first return times. In the deterministic limit, average isophases reduce to the usual ones. It is noteworthy that the approach is not a perturbative one based on small noise approximation; its applicability is thus neither restricted by noise level, nor even by specific properties of the underlying deterministic dynamics (which may even be nonoscillatory). Of course, the general limitations of phase description apply: in order to define average isophases, "oscillations" should be well defined. Practically, this means that in some two-dimensional embedding the irregular trajectory should perform loops of finite radius, as the experimental data in Fig. 5, so that the return time to a cross section is well defined. Not all noisy oscillations fulfill this property; a similar restriction is also valid for the definition of an angle variable for deterministic chaotic oscillators where the existence of a well-defined Poincaré section is a prerequisite [22].

By applying a simple procedure [23], we determined these isophases in a unified way for different classes of noisy oscillators: (i) noise-perturbed periodic oscillators, which possess isophases also in the noise-free case; (ii) multistable oscillators which in the noise-free case possess different singular isophases, but the latter become well defined when different modes merge due to noise; (iii) chaotic attractors where in the purely deterministic case the isochrones are singular objects which become smooth and well defined due to noise; (iv) excitable systems, which do not oscillate without noise and therefore have no isophases, but the latter appear for the noise-induced dynamics. Furthermore, we have demonstrated applicability of the method to irregular experimental data [24].

Average isophases of noisy systems bear opportunities especially for data analysis and synchronization theory. In data analysis, determination of a phase from observations lies at the basis of correlation and synchronization analysis, with applications ranging from human physiology to climate variability [25]. In these applications, our approach will allow for an optimal distinction of phase and amplitude effects in subsequent analyses, that is especially useful if amplitude variations are large (e.g., in Fig. 5, angle and radius variable are correlated). In synchronization theory, consistently defined isophases allow us to determine the phase response of irregular oscillations to external kicks and thus to predict their synchronizability.

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