Optimal phase description of chaotic oscillators

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We introduce an optimal phase description of chaotic oscillations by generalizing the concept of isochrones. On chaotic attractors possessing a general phase description, we define the optimal isophases as Poincaré surfaces showing return times as constant as possible. The dynamics of the resultant optimal phase is maximally decoupled from the amplitude dynamics and provides a proper description of the phase response of chaotic oscillations. The method is illustrated with the Rössler and Lorenz systems.

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I. INTRODUCTION

Phase description lies in the base of the theory of self-sustained, autonomous oscillators [1–3]. A prudently defined phase variable yields a one-dimensional description of the oscillator, allowing one to characterize important aspects of its dynamics, such as regularity of oscillation, sensitivity to external forcing, etc. Moreover, the concept of phase is important in the data analysis of oscillatory processes in physics, chemistry, biology, and technical applications, where various approaches exist for extracting different variants of phase variables from oscillatory scalar time series.

On a very basic level, every phase description starts with the identification of those states of the oscillator that are in the same phase. For a good phase description, the identification must be done in an invariant way, independent of the variables and observables used, in order to make statements about the oscillator's phase dynamics nonarbitrary and comparable. The standard procedure of phase reduction is valid for periodic oscillators that possess a stable limit cycle. There, a certain family of Poincaré sections, called *isochrones* or *isophases*, is used for the identification of states with the same phase: Each isochrone consists of those states that are mapped onto each other after one oscillation period T and that converge to the corresponding state on the limit cycle for which the phase is defined unambiguously [4,5]. A limit cycle oscillator can be thought of as a perfect chronometer whose state can be used to measure time; this property is stressed by the term isochrone. For stable periodic oscillations isophases coincide with the isochrones, while for stochastic and chaotic oscillations one can hardly speak of isochrones but nevertheless can try to introduce isophases.

Even though chaotic oscillators do not possess a stable limit cycle, a phaselike variable has been used for their description. In this sense, the phase dynamics of chaotic systems has been initially discussed in relation to the diffusion properties of the phase [6,7] and to phase synchronization [8–10]. However, to describe these features, one does not need a *microscopic* definition of the phase that is precise on time scales smaller that the characteristic period T, because both diffusion and synchronization are defined macroscopically, i.e. for time scales much larger than T. On the other hand, in the theoretical description of phase synchronization a proper microscopic phase definition was presumed [11,12], although no practical algorithm for the construction of a phase variable with good properties has been presented. The main problem is that chaotic phase diffusion destroys the perfect chronometric properties of the oscillator because any two of its states that are thought to show the same phase will diverge as their respective phases diffuse. An attempt to define isochrones or isophases of chaotic oscillators as smooth hypersurfaces thus fails, leading to fractal sets, as we demonstrate below.

In this paper, we suggest a numerical technique for the phase description of chaotic oscillations. To overcome the fractality mentioned above, we construct isophases as Poincaré sections in a smoothness-constrained optimization problem. With examples we demonstrate how these *optimal isophases* yield phase dynamics with better properties compared with simpler definitions of phase. Specifically, we demonstrate an intimate relation between optimal isophases, chaotic phase diffusion, and unstable periodic orbits. Furthermore, we discuss the reduced phase dynamics of chaotic oscillations and the decoupling of the amplitudes from the phase dynamics. Next, we use the optimal phase to introduce a proper framework for the description of the phase response of chaotic oscillators.

Before proceeding, we would like to specify terminology that we use below (the exact meaning will be clear when the corresponding terms are introduced). Phaselike variable and protophase are basically the same; the former term is suitable in a theoretical consideration, while the latter one is suitable in data analysis. For periodic oscillations, a phaselike variable, or a protophase, is a 2π-periodic variable on the limit cycle in an arbitrary parametrization. For periodic oscillations, the genuine phase is defined as a reparametrized phaselike variable on the limit cycle that grows linearly with time. The extension of the genuine phase to a vicinity of the limit cycle (Sec. II A) allows one to define the sets of the same phase in this vicinity; these sets are called isochrones or isophases. Neither a genuine phase nor isochrones or isophases exist for general chaotic oscillators. Nevertheless, in Sec. II B we introduce protophases on the attractor as arbitrarily parametrized variables that increase by 2π in one oscillation. Then we "improve" these variables to ensure that they rotate as uniformly as possible, thus obtaining optimal phases (and optimal isophases as the sets of constant optimal phases).

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Starting with an outline of the standard phase definition for periodic oscillators via the isochrones, we introduce the generalized concept of isophases of chaotic oscillators in Sec. II. In Sec. III, certain dynamical properties of the optimal phase are highlighted by the example of the Rössler oscillator. Thereafter, the relation between optimal isophases and unstable periodic orbits is presented (Sec. IV). In Sec. V, certain aspects of the theory are presented for the Lorenz oscillator. In the last section we discuss our results.

II. ISOPHASES OF PERIODIC AND CHAOTIC OSCILLATORS

A. Periodic oscillators and their isochrones

Phase is a natural variable for the description of periodic motions in dynamical systems. It can be introduced in different ways, with different levels of mathematical rigor [1,2,4,5]. Here we outline an approach that is mostly suited for a generalization to the case of chaotic systems.

The consideration starts with a general dissipative dynamical system showing stable periodic oscillations; the system's state $\mathbf{x}(t)$ is thus attracted to the limit cycle $\mathbf{x}_0(t)$ having period T. In a vicinity of this periodic attractor the state space can be foliated by a nonintersecting family of Poincaré sections $J(\varphi)$ parametrized by a phaselike variable φ with period 2π. With $J(\varphi)$, a phaselike variable $\varphi(t)$ can be assigned to each state of the trajectory $\mathbf{x}(t) \in J(\varphi(t))$. Therefore, the family of isophases $J(\varphi)$ provides a precise definition of what is meant by an oscillation: The system completes one *oscillation* if the variable φ grows by 2π, i.e., if the trajectory returns to a chosen isophase, consequently passing through all sections in $J(\varphi)$. Introducing coordinates on the sections $J(\varphi)$, one can parametrize each point by a vector of amplitudes $\mathbf{a}$ and the phaselike variable φ.

There are various equivalent ways to foliate the state space in such a way that φ grows monotonically; for periodic oscillators with a period T, the optimal foliation does exist [4]. It can be introduced by considering the stroboscopic map $\mathbf{x}(t) \to \mathbf{x}(t+T)$. Clearly, all points on the limit cycle are stable fixed points of this map. Hence, for each fixed point $\mathbf{x}_0$ there exists a stable manifold which converges to $\mathbf{x}_0$ under the action of the stroboscopic map. These stable manifolds, called *isochrones*, constitute a special foliation of the neighborhood of the limit cycle, for which by construction the Poincaré map is the same as the stroboscopic map.

In this way one introduces the phase of oscillation so that its time evolution does not depend on the amplitudes $\mathbf{a}$. By virtue of a trivial reparametrization $\varphi \to \theta = \frac{2\pi}{T}\int \frac{dt}{d\varphi}d\varphi$ of this foliation, one can introduce the genuine phase θ, which grows strictly uniformly in time, with a constant instantaneous frequency $\dot{\theta} = \omega = 2\pi/T$. This phase, defined in the whole basin of attraction of the limit cycle, serves as a basis for a theoretical description of perturbed periodic oscillations [1]. In particular, one can easily formulate phase response properties in terms of this phase: If a state on the limit cycle $\mathbf{x}'$ is instantly perturbed to some other state (even outside of the limit cycle), $\mathbf{x}' \to \mathbf{x}''$, then the phase is reset by a value $\Delta\theta = \theta(\mathbf{x}'') - \theta(\mathbf{x}')$, which remains constant in the course of further evolution (see also Sec. III D).

It is worth noting that the extension of the phase to a vicinity of a periodic orbit can be defined for either a stable or unstable limit cycle. In the latter case, instead of using the stable manifold, one constructs the isophases by using the unstable manifolds of the fixed points of the stroboscopic map. However, for saddle limit cycles having both stable and unstable directions, this construction fails. Here one can construct isochrones on the stable and unstable manifolds separately, but not in the whole vicinity of the cycle. With this in mind, we use below for the chaotic case, where isochrones do not exist, the term "isophases" instead of the usual "isochrones."

B. Protophase for chaotic oscillators

We start the generalization of phase description to chaotic oscillators by discussing the construction of the protophase. For this purpose, we need the chaotic attractor to show the same property as a limit cycle, namely, that there exists a family of nonintersecting Poincaré sections $J(\varphi)$, monotonically parametrized by a protophase φ. The requirement includes periodicity, $J(\varphi + 2\pi) = J(\varphi)$, and it is also required that any trajectory on the attractor successively crosses each Poincaré section $J(\varphi)$ transversally. Of course, not all chaotic attractors possess such a family, but those that have such a foliation can be described in terms of phases and are the subject of further consideration here.

Let us consider as an example the Rössler oscillator [13]

$$\dot{x} = -y - z, \quad \dot{y} = x + 0.15y, \quad \dot{z} = 0.2 + z(x - 10), \tag{1}$$

and take a family of Poincaré sections $J(\varphi_1)$ defined via the cylindrical coordinates

$$\varphi_1 = \tan^{-1}\frac{y}{x}; \quad \mathbf{a} = (r,h) = (\sqrt{x^2+y^2},z). \tag{2}$$

This family of Poincaré sections with constant protophase φ_1 is shown in Fig. 1(a) with red solid lines. However, other families can be defined as well; an example of another foliation based on the protophase $\varphi_2 = \varphi_1 + 0.7\ln r$ is counterposed in Fig. 1(a) with blue dashed lines.

Because the difference of any two protophases is bounded, the asymptotic properties of their phase dynamics, such as the mean frequency and the diffusion constant of the phase rotations, do not depend on the definition of the protophase. However, local, microscopic properties of the dynamics for two protophases are different, which becomes apparent through the irregularly fluctuating phase difference $\varphi_1(t) - \varphi_2(t)$ shown in Fig. 1(b). The fluctuations show a bounded and irregular pattern that is specific to arbitrarily chosen variants of the Poincaré sections. In order to define a "genuine" phase, such as that of periodic oscillators, we need to define the "isophases" of chaotic attractors. (We recall that for chaotic oscillators the isochrones generally do not exist.) Because the phase of a chaotic system is, in fact, not as "genuine" and unique as in the periodic case (see discussion below), we will refer to it as the *optimal* phase, in the sense that it is obtained by means of an optimization procedure. Performing optimization, we minimize the variance of the return times (or, equivalently, minimize variations of

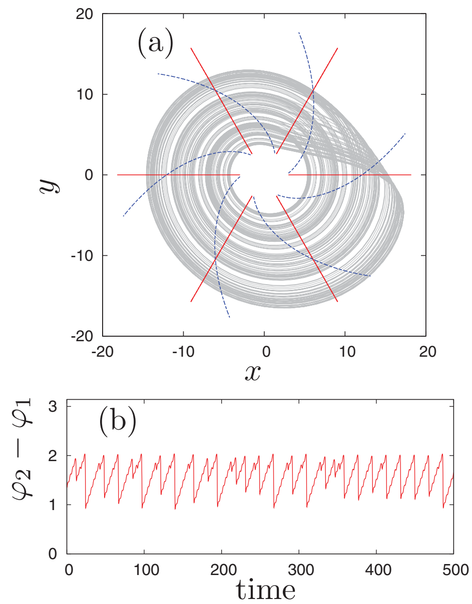

FIG. 1. (Color online) (a) Two different families of Poincaré sections of the Rössler system, ϕ_1 and ϕ_2, are shown by red solid and blue dashed lines, respectively. Both families yield a proper definition of an oscillation. (b) The corresponding protophases $\varphi_{1,2}$ are, however, different, so that $\varphi_2(t) - \varphi_1(t)$ shows irregular bounded fluctuations, specific to the particular shapes of the Poincaré surfaces.

the instantaneous frequency) be means of a smooth coordinate transformation and therefore obtain smooth isophases. (Notice that for the genuine phase the variance would be zero.)

C. Optimal isophases for chaotic oscillators

The genuine phase of periodic oscillators is defined by the basic property that there exist Poincaré sections where all return times are exactly equal to the period of oscillations; i.e., the corresponding Poincaré maps are stroboscopic maps as well. Naturally, such a situation does not occur for general chaotic oscillators. This is plausible because, on one hand, different periodic orbits embedded in chaos usually have different basic periods [total period divided by the number of intersections with a Poincaré surface, see Eq. (7)]. On the other hand, a coincidence of Poincaré and stroboscopic maps would also imply the absence of phase diffusion, which, however, is a degenerate, rarely observed situation [14].

Since isophases of chaotic oscillators defined as sections with constant return times do not exist in the strict sense, we introduce *optimal isophases* that approximate the property above with some accuracy. Practically, we construct the optimal isophases as a smooth Poincaré section with a minimal (bounded by the smoothness) variation of return times. As this condition is not unambiguous, we describe below an algorithm that we practically use.

The starting point of our construction is a suitable vector time series of chaotic dynamics $\mathbf{x}(t)$ in a time interval $0 \leqslant t \leqslant t_{\text{end}}$ which can be obtained by numerical simulation or by embedding observed oscillations [15]. The first step is to introduce an arbitrary protophase φ as described above. Using it, we can estimate the average period of oscillations as

$$T = \frac{2\pi t_{\text{end}}}{\varphi(t_{\text{end}}) - \varphi(0)} .$$

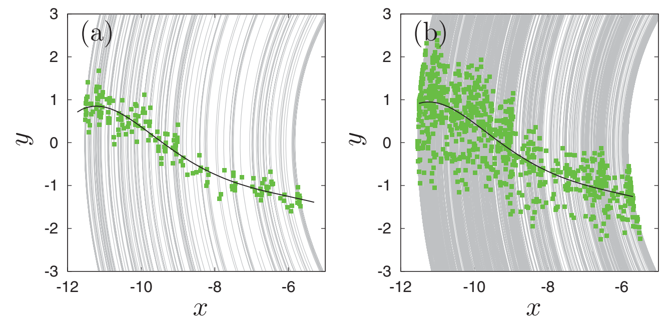

FIG. 2. (Color online) The stroboscopic sets Eq. (3) for the Rössler attractor for two lengths of the trajectory, (a) $t_{\text{end}} = 10^3$ and (b) $t_{\text{end}} = 5 \times 10^3$, are shown with squares. The trajectories are shown with gray lines, an the optimal local isophases obtained by fitting the set by a polynomial $\varphi(r)$ of order 4 are shown with black lines.

With this period, we define a family of *stroboscopic sets* for the trajectory $\mathbf{x}(t)$ as

$$\mathbf{x}_k(\tilde{\theta}) = \mathbf{x}\left(\frac{\tilde{\theta}}{2\pi}T + kT\right), \quad k = 0,1,2,\ldots,K_{\text{end}}. \qquad (3)$$

Here $\tilde{\theta} \in [0,2\pi)$ serves as a periodic parameter (neither phase nor protophase) parametrizing stroboscopic sets, and each set consists of K_{end} points. These sets are invariant under the stroboscopic map with time interval T but cannot serve as Poincaré maps as they do not form smooth curves because the rotation in chaotic systems is nonuniform. The larger the total time interval t_{end} is, the stronger the spreading of the points of the stroboscopic set is. We illustrate this in Fig. 2. We note that only in a degenerate case where the phase diffusion of the chaotic oscillator vanishes would these stroboscopic sets be smooth lines that can be used as Poincaré sections; such degenerate chaotic attractors (see an example in Ref. [16]) possess the same rigorous phase description as periodic oscillators.

In order to obtain a proper smooth Poincaré section, we fit the stroboscopic set, in the sense of least squares, by a polynomial $\varphi = \varphi(\mathbf{a})$ (we use a standard fitting procedure as described in Ref. [17]). The resulting curves shown in Fig. 2 are our optimal isophases, i.e., the curves of constant phase θ.

If we restrict ourselves to rather smooth isophases only, a good practical approximation can be achieved if one introduces a global phase correction function Δ according to smooth coordinate transformation:

$$\theta = \varphi + \Delta(\varphi,\mathbf{a}). \qquad (4)$$

Then, one represents Δ in terms of polynomial basis functions: For each of the amplitude components a_j we use the powers a_j^n, and for the phase variable φ we use trigonometric polynomials $\exp(i\varphi l)$. For example, for the Rössler system in $1+2$ dimensions, consisting of phase φ, radius r, and height h,

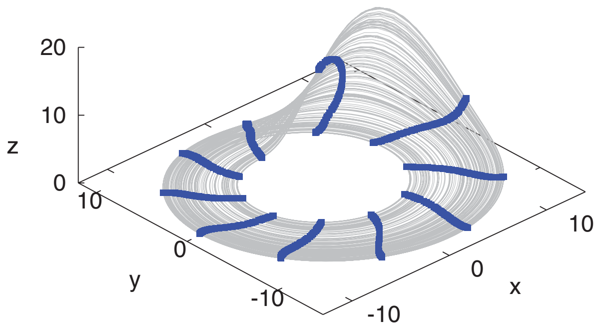

FIG. 3. (Color online) A global approximation of optimal isophases (blue dots, which look like thick lines) obtained for the Rössler attractor (gray) using approximation (5) with $N_\varphi = 4$, $N_r = 3$, and $N_h = 1$.

the phase correction is represented using a set of coefficients c_{mnl}:

$$\Delta(\varphi,r,h) = \sum_{m=0}^{N_r}\sum_{n=0}^{N_h}\sum_{l=0}^{N_\varphi} c_{mnl} r^m h^n e^{il\varphi}. \tag{5}$$

The coefficients can be computed by applying a linear least squares fit [17] to the stroboscopic sets. In this way it is easy to find an optimal phase globally, as a function of the state space coordinates $\mathbf{x}$. We illustrate the isophases obtained in this way in Fig. 3.

In Fig. 4 we compare the quality of the optimal isophases obtained via representation (5) with the results of the local fitting of stroboscopic sets as in Fig. 2. We compare the return times for these isophases with the return times of the Poincaré section $y = 0, x < 0$. One can see that globally defined smooth isophases in the form (5) give a quite good minimization of the variability of return times.

III. DYNAMICS OF THE OPTIMAL PHASE

In this section we discuss dynamical properties of the optimal phase introduced with the help of optimal isophases.

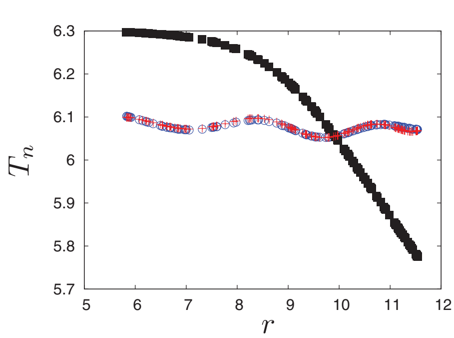

FIG. 4. (Color online) Return times T_n for the Rössler oscillator Eq. (1). Solid black squares correspond to an arbitrary Poincaré section $y = 0, x < 0$, where the spreading of the return times is large. Local (as in Fig. 2, blue open circles) and global (as in Fig. 3, red crosses) approximations (nearly coinciding on the figure) of the optimal isophases yield a strongly reduced variation of the return times.

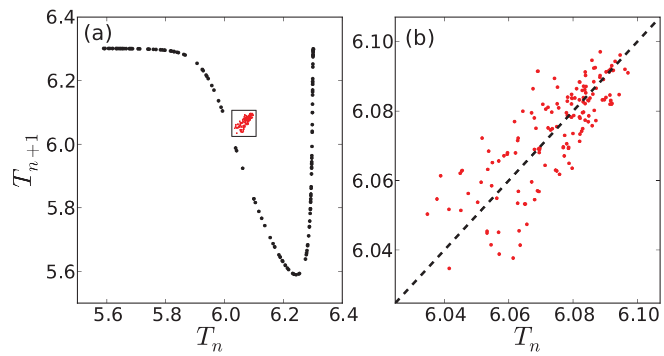

FIG. 5. (Color online) (a) The return time map Eq. (6) of the Rössler oscillator Eq. (1) for the cylindrical Poincaré section $\varphi = 4\pi/3$ [Eq. (2)] can be described as a one-dimensional chaotic map (black dots). (b) Using the optimal isophase, one obtains a map in a much smaller range [small box in (a) is enlarged]; this map corresponds to what we expect for a noisy limit cycle oscillator.

A. Return time map

A natural way to characterize the time intervals T_n between successive crossings of a Poincaré surface is to construct the return time map,

$$T_{n+1} = M(T_n). \tag{6}$$

In fact, because T_n is a function of the Poincaré map coordinate, it is just a scalar observable, and $M(T_n)$ is not a function but rather a one-dimensional projection of a Cantor set. Nevertheless, for nearly two-dimensional strange attractors the Poincaré map is nearly one-dimensional, and (6) looks like a curve [see Fig. 5(a)]. In Fig. 5 we demonstrate how this return time map changes if one uses an optimal isophase as a Poincaré surface. First, the range of variations of T_n drastically shrinks. Second, one can hardly recognize the one-dimensional structure of the map: because now T_n is a "bad" observable, it does not reproduce the nearly one-dimensional nature of the Poincaré map $\mathbf{a}_n \to \mathbf{a}_{n+1}$. This means that the dynamics of the new optimal phase looks like a random process even on a microscopic time scale of the order of the period T.

B. Uniformity of phase rotations

The basic property of the phase for a periodic oscillator is that it rotates uniformly. For the optimal phase of a chaotic oscillator we cannot expect pure uniformity, but nevertheless it should be considerably increased compared to any protophase that is typically used. We illustrate this in Fig. 6. Here we show the velocities of the protophase φ defined according to Eq. (2) and that of the optimal phase θ defined according to isophases shown in Fig. 3. While fluctuations in the protophase velocity $\dot\varphi$ heavily depend on φ, the fluctuations of $\dot\theta$ are almost uniformly distributed and, notably, in some regions are larger than those of the protophase. Similar results are reported in Ref. [18]. We conclude that optimal isophases not only eliminate the amplitude dependence of the phase velocity but also flatten the phase dependence of its velocity fluctuations.

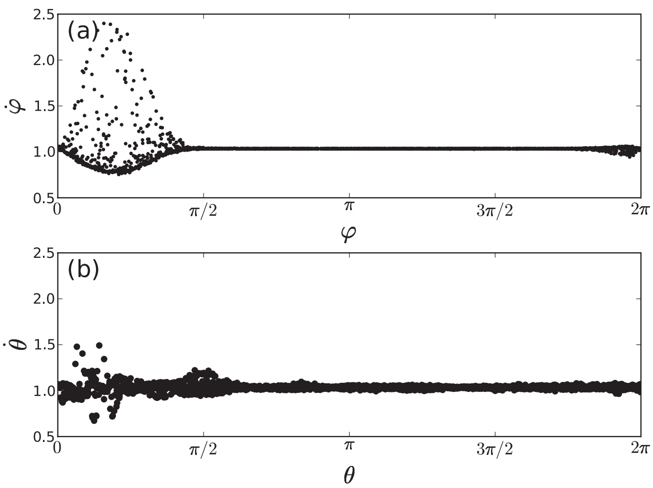

FIG. 6. The phase velocities for the Rössler system for the protophase defined (a) according to (2) and (b) according to isophases (Fig. 3).

C. Decoupling of amplitude and phase dynamics

One of the goals of introducing a phase is to decouple its dynamics from that of the amplitude variables. For periodic oscillators this decoupling is perfect, whereas for chaotic oscillators, it is only approximate. To illustrate how correlations of the phase dynamics with the amplitudes are reduced when the optimal phase is introduced, we performed a "mixing" experiment; the results are depicted in Fig. 7. We started an ensemble of initial conditions on a certain Poincaré surface and followed them for a time interval of length $5T$ (five average rotation periods). The trajectories starting at small, medium, and large amplitudes **a** are marked separately in Fig. 7. In Figs. 7(a) and 7(b), where the Poincaré surface ϕ_1 = const is used, we see that the states that started at small amplitudes lag behind, while those that started at large amplitudes are advanced. Contrary to this, using the optimal isophase as an initial condition, we see that after five rotations all points are mixed and one can hardly distinguish the points that had different amplitudes at the beginning. This is another illustration of the fact that the dynamics of the optimal phase is effectively decoupled from the amplitude.

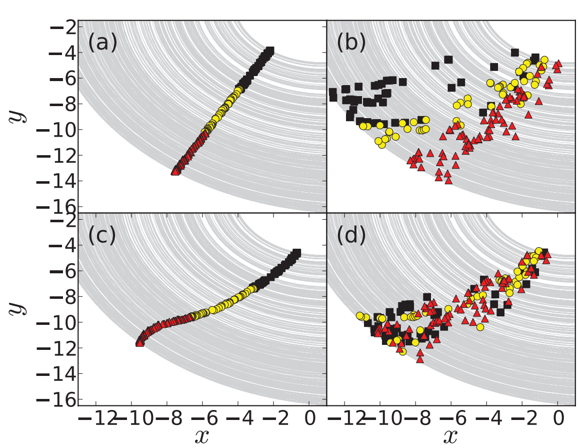

FIG. 7. (Color online) Two "mixing" setups where initial states of the Rössler oscillator Eq. (1) (marked as symbols) are chosen either (a) on the Poincaré section $\phi_1 = 4\pi/3$ [Eq. (2)] or (c) on the optimal isophase. (b) and (d) The same points at time $t = 5T$, where T is the average oscillation period. States on the optimal isophase show less diffusive broadening in the direction of the phase than the points on the arbitrary Poincaré section. Moreover, states of different amplitude become indistinguishable only for the optimal isophase, as seen by the mixing of symbols.

D. Phase response of chaotic oscillators

A basic application of the phase description of periodic oscillators is quantification of the system response to pulse stimulation by means of *phase response curves*. Given a state on the limit cycle $\mathbf{x}(\theta)$, one can determine the phase shift due to the change of the state $\mathbf{x}(\theta) \to \mathbf{x}' = \mathbf{x}(\theta) + \mathbf{k}$ simply by calculating $\theta' = \theta(\mathbf{x}')$. Because the phase rotates uniformly also outside of the limit cycle, the phase shift $\theta' - \theta$ remains invariant and characterizes the phase response (for noise-induced oscillations this notion can be also introduced [19]).

This approach has to be slightly modified when applied to chaotic oscillators. If both states $\mathbf{x}$ and $\mathbf{x} + \mathbf{k}$ lie on the attractor, then their optimal phases are well defined, and the phase shift can be simply calculated as $\theta(\mathbf{x} + \mathbf{k}) - \theta(\mathbf{x})$. However, generally state $\mathbf{x} + \mathbf{k}$ lies outside of the attractor, and we have to generalize the definition of the optimal phases from the attractor to its vicinity. This is ambiguous because the optimal isophases are not genuine isophases. They are not strictly invariant under time shifts, and we cannot define the phase of state $\mathbf{x} + \mathbf{k}$ by following its time evolution for arbitrarily large times. Instead, we have to fix the time interval after which the phase of state $\mathbf{x} + \mathbf{k}$ is defined. For the Rössler model we choose the mean period T as such an interval, as the relaxation time of approaching the attractor is typically smaller. So we define $\theta(\mathbf{x} + \mathbf{k}) = \theta[\hat{T}(\mathbf{x} + \mathbf{k})]$, where $\hat{T}$ is the operator of time evolution over the average period T (see Fig. 8). Applying now representation (5), we obtain the phase response plot (PRP) of the Rössler attractor $R(\mathbf{k},\mathbf{x}) = \theta[\hat{T}(\mathbf{x} + \mathbf{k})] - \theta(\mathbf{x})$, as shown in Fig. 9.

To stress the advantage of the optimal phase (compared to a nonoptimal protophase variable) for the description of phase response, we show in Fig. 10 the dynamics of the phase

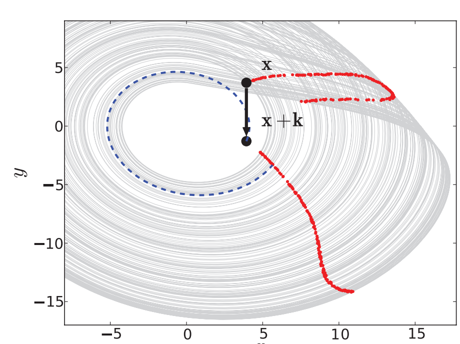

FIG. 8. (Color online) A state $\mathbf{x}$ of the Rössler attractor (gray trajectory) is kicked to state $\mathbf{x} + \mathbf{k}$ (black points with an arrow). After one period the perturbed trajectory (dashed line) returns to the attractor and now lies on the isophase $\theta[\hat{T}(\mathbf{x} + \mathbf{k})]$ (red dots). The kick's effect on the oscillator's phase is therefore given by the phase shift $R(\mathbf{k},\mathbf{x}) = \theta[\hat{T}(\mathbf{x} + \mathbf{k})] - \theta(\mathbf{x})$.

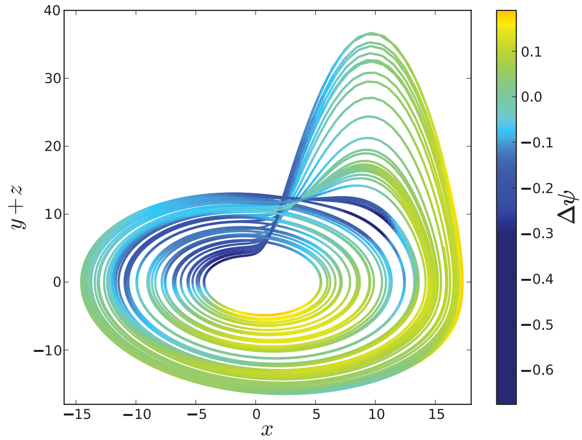

FIG. 9. (Color online) The phase response plot $R(\mathbf{k},\mathbf{x})$ for the states on the Rössler attractor, color (gray scale) coded, for $\mathbf{k} = (1,0,0)$.

difference $\delta\theta(t)$ and of the protophase difference $\delta\varphi$ [where φ is defined according to (2)]. The phase and protophase differences are defined for the initial state $\mathbf{x} = (-8,0,0)$ and kick $\mathbf{k} = (-2,0,0)$. Altogether 20 characteristic periods of oscillations after the kick are shown. In this time interval, $\delta\theta$ remains close to 0.12, compared with $\delta\varphi$, which widely oscillates around this value. Moreover, initially, the protophase difference $\delta\varphi$ is nearly zero, indicating a vanishing phase response, and only after half of a period does an effect of the kick on the protophase become visible.

IV. ISOPHASES AND UNSTABLE PERIODIC ORBITS

In this section we discuss a relation between optimal isophases of a chaotic system and *unstable periodic orbits* (UPOs) $\mathbf{x}_0(t+\tau) = \mathbf{x}_0(t)$ embedded in chaos. For each UPO one can define the phase on this orbit just from the condition of uniform rotation. This approach is discussed in Sec. IV A. Similar to the construction discussed in Sec. II A, we can extend the notion of the phase for each periodic orbit to its stable or unstable manifold; these ideas are presented in Sec. IV B.

A. Approximation of orbit phase sets

For each UPO one can introduce a *topological period* (lap number) p as the number of intersections with a Poincaré section. With this number p and the total period τ, we define the *oscillation period*,

$$S = \frac{2\pi}{\nu} = \frac{\tau}{p}, \tag{7}$$

which is expected to be close, but not identical, to the mean period of chaotic oscillations (mean return time of the Poincaré map). Next, for the UPO we can introduce the phase $\tilde{\theta}$ that rotates uniformly with frequency $2\pi/\nu$ so that $\tilde{\theta}(\tau) = \tilde{\theta}(0) + 2\pi p$. With the help of this phase, a family of point sets $I(\tilde{\theta})$, called *orbit phase sets*, can be defined as points that are attained at constant time intervals, equal to the oscillation period S:

$$I(\tilde{\theta}) = \left\{\mathbf{x}_0\left(\frac{\tilde{\theta}}{2\pi}S + nS\right) |n = 0,\ldots,p-1\right\}, \tag{8}$$

with some arbitrary choice of the zero phase.

Let us now take a Poincaré surface that passes through the orbit phase set $I(\tilde{\theta})$. (Of course, there are many possibilities to draw such a surface; e.g., one can use splines.) Then it will be an approximation to an optimal isophase, as, at least on the orbit phase set, all the return times will be equal to S. We illustrate this in Fig. 11(a), where we show orbit phase sets of two UPOs, with topological periods $p = 10$ and $p = 9$, for the Rössler system (1). Since the orbits do not share any state, the zero phases can be chosen separately. Practically, the phase offsets have been chosen in a way that the orbit phase sets are mostly close to each other and approximate the same isophase, which is also drawn for comparison. One can see that the orbit phase states indeed can serve as approximations for the isophases.

This approximation is expected to work better for larger periods and for periodic orbits that are the most "typical." The probability for a trajectory to approach the orbit depends on the stability of the UPO, quantified by its unstable Floquet multiplier [20]. Therefore, it is expected that the correspondence between isophases and the orbit phase sets will be better for UPOs that are visited more often because they are less unstable. To check this for the Rössler system,

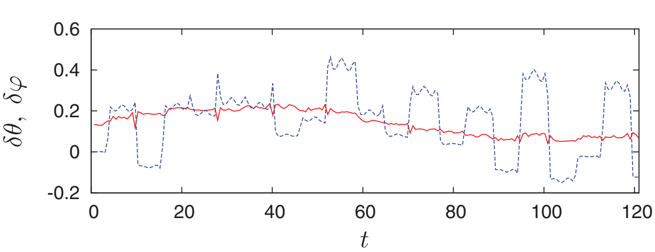

FIG. 10. (Color online) Evolution of phase (red solid line) and protophase (blue dashed line) differences (see text for details).

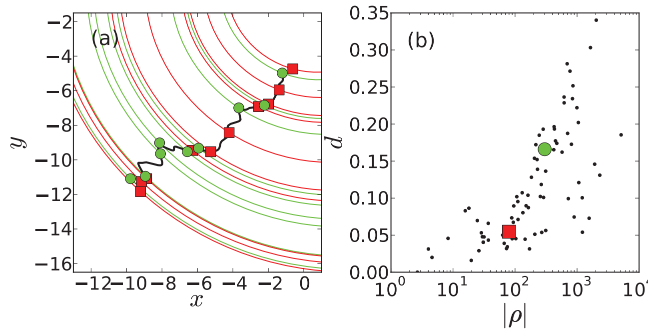

FIG. 11. (Color online) (a) The optimal isophase (black line, obtained using a fit with a high-degree polynomial) of the Rössler oscillator [Eq. (1)] overlapped with orbit phase sets [Eq. (8)] of two orbits having topological periods nine (green circles) and ten (red squares). (b) A distance measure d [Eq. (9)] quantifies how close the orbit phase set is to the isophase, here shown for 80 p orbits with $p \leqslant 10$. One can see weak correlations to the instability of the UPOs measured by the Floquet multiplier $|\rho|$.

we introduce a measure d of the distance of a p-orbit $\mathbf{y}$ to the optimal isophase shown in Fig. 2(a) with a thick black line as

$$d = \sqrt{p^{-1}\sum_{k=0}^{p-1}\left\|\mathbf{y}_k^J - \mathbf{y}_k\right\|^2}\,, \tag{9}$$

where $\mathbf{y}_k^J$ are coordinates of the orbit phase set (for which we also optimized the zero phase to achieve a minimum of d) and $\mathbf{y}_k$ are the crossings of the periodic orbits with the isophase. This measure was calculated for the 80 available UPOs together with their Floquet multipliers. It was found that orbits showing a larger distance had a tendency to be less stable [cf. Fig. 11(b)].

B. Orbit isophase

As described in Sec. II A, after the phase on a periodic orbit is introduced, the isophases in its vicinity can be defined separately on the stable and unstable manifolds of the orbit as the stable and the unstable manifolds of the fixed points of the stroboscopic (with the period of the orbit) map. This definition can be applied to the UPOs in chaos, where the unstable manifolds are especially interesting as they lie in the attractor.

Let us consider the simplest UPO of the Rössler oscillator that has topological period $p = 1$. Its oscillation period is $S \approx 6.024$, whereas the mean period of a typical trajectory is $T \approx 6.073$. Numerically, we calculated the isophase on the unstable manifold of this orbit using the oscillation period S for the stroboscopic map and obtained the blue line in Fig. 12. This isophase becomes folded together with the unstable manifold and is not close to the optimal isophases obtained by methods above.

It is instructive to try to construct the isophase on the unstable manifold of the UPO using not its period S but the mean period T. It is clear that such an isophase cannot exist,

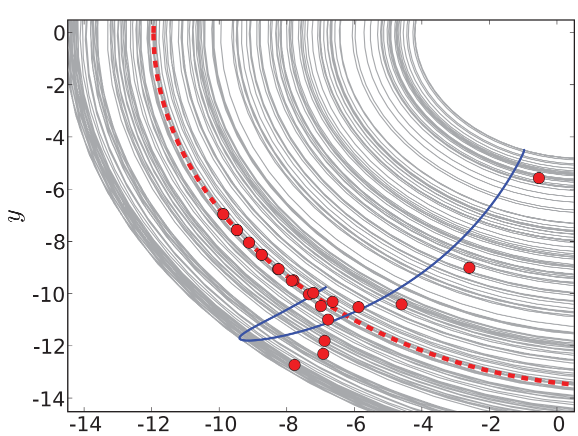

FIG. 12. (Color online) For a UPO of the Rössler oscillator (dashed line), the orbit phase set Eq. (8) can be extended to the unstable manifold in two ways. The blue solid line shows the extension where the orbit's period S is used. Red circles depict the extension based on the mean period of the chaotic attractor. The singularity of the latter curve indicates a divergence of the phase correction (see the Appendix for the analytic form of this correction for the unstable Stuart-Landau oscillator.)

but trying to approximate it (see the Appendix for details) we obtain a singular curve (Fig. 12). This is another representation of the nonsmoothness of stroboscopic sets that appears in the algorithm described in Sec. II C due to the nonexistence of true isophases. In fact, when one tries to construct an isophase, such a singularity will appear for every periodic orbit, and the procedure should be constrained by the requirement that the isophase should be sufficiently smooth.

V. PHASE OF THE LORENZ SYSTEM

In our presentation above we have used the Rössler model Eq. (1) as the basic example. Here we discuss how the approach works for the Lorenz system,

$$\dot{x} = 10(y - x), \quad \dot{y} = 28x - y - xz, \quad \dot{z} = -\tfrac{8}{3}z + xy. \tag{10}$$

Chaotic phase diffusion of the Lorenz system is orders of magnitude stronger than that of the Rössler oscillator, Eq. (1); thus introducing its phase is a more challenging task. The main difficulty lies in the unboundedness of the return times of the Poincaré map due to the presence of the saddle steady state at the origin ($x = y = z = 0$). Due to this, the stroboscopic sets are spread over the attractor and cannot serve as a basis for the construction of isophases as described above. Therefore we applied the following iterative procedure for obtaining smooth optimal isophases. First, we use projections of the trajectory onto the plane ($u = \sqrt{x^2 + y^2}, z$). On this plane the trajectory rotates around a center approximately at (12,27), and the protophases can be easily defined (cf. [21]). We choose a Poincaré surface and find the points of the trajectory at the intersection with this surface; these are $x(t_k), y(t_k), z(t_k)$, $k = 1,2,\ldots$ Of course, the times t_k are not equidistant because the Poincaré map is far from the stroboscopic one. We adjust the times t_k, trying to make them equal by introducing a parameter s on which these times depend and letting them evolve according to

$$\frac{dt_k}{ds} = -\frac{\partial V(t_1,t_2,\ldots)}{\partial t_k}, \quad V = \frac{1}{2}\sum_k (t_{k+1} - t_k - T)^2\,, \tag{11}$$

where T is the average period. One can easily see that the "evolution" of t_k according to Eq. (11) leads to equalization of the intervals $t_{k+1} - t_k$ because of the minimization of the

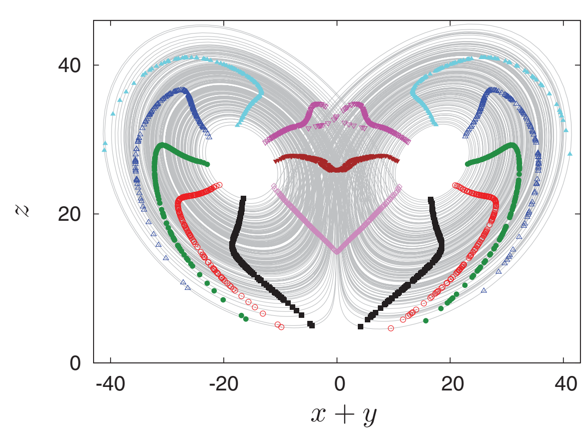

FIG. 13. (Color online) Optimal isophases (depicted by different symbols or colors) of the Lorenz attractor Eq. (10) (gray line).

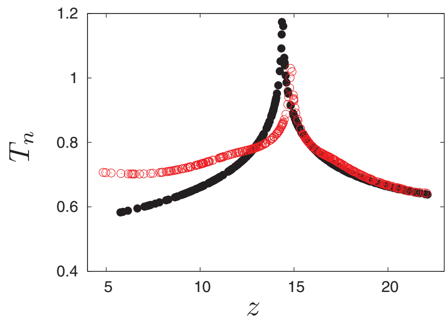

FIG. 14. (Color online) Return times for the Poincaré section $u = 12, z < 27$ (solid circles) and for the optimal isophase (open circles) resulting from it iterations (this isophase is shown in Fig. 13 with solid black squares). The variations of the return times only slightly decrease.

Lyapunov function V. However, we "evolve" the times t_k only for a finite interval of s and obtain new times $\tilde{t}_k = t_k(s)$. The new points $\tilde{x}(\tilde{t}_k), \tilde{y}(\tilde{t}_k), \tilde{z}(\tilde{t}_k)$ form a new distorted and singular Poincaré section. We smoothen this set by applying a kernel technique [22] and obtain a smooth new Poincaré section with more equidistant time intervals. We make several iterations of this procedure and finally obtain the approximate smooth isophases as depicted in Fig. 13.

To characterize the quality of the introduced isophases for the Lorenz system, we plot the return times for an initial arbitrary Poincaré section and for the obtained isophase in Fig. 14. We see that the variations of the return times decrease only slightly, and the singularity (corresponding to the stable manifold of the origin) remains.

In Fig. 15 we use orbit phase sets of UPOs of the Lorenz system to approximate isophases. Nine periodic orbits of the Lorenz system with topological period 6 are shown with gray line. By manually adjusting phase shifts of these orbits, it is possible to arrange the isophase sets for each orbit (different markers) to build a set close to a curve (drawn manually as a black line) that can serve as an optimal isophase. The form of this curve is close to one of the optimal isophases presented in Fig. 13.

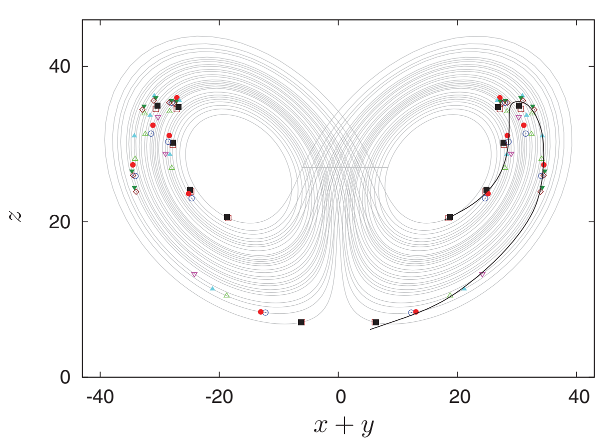

FIG. 15. (Color online) Building an isophase using nine UPOs of the Lorenz system with $p = 6$ (see text for details).

VI. CONCLUSION

In summary, we have proposed a method of phase description of chaotic oscillators by generalizing the concept of standard isophases (isochrones) of periodic oscillators. In the absence of a stable limit cycle, the definition of optimal isophases of chaotic oscillations is solely based on their return times. Because of nonvanishing chaotic diffusion and embedded unstable periodic orbits with different periods, isophases could only be obtained in an optimal, approximate way constrained by certain smoothness conditions. In the case of the Rössler attractor, where the phase diffusion is relatively small, we obtain the optimal isophases by smoothing the stroboscopic sets of a chaotic trajectory. For the Lorenz attractor, where phase diffusion is large, an iterative numerical scheme was proposed. Using the Rössler oscillator as an example, we have presented different aspects of the phase dynamics. Specifically, the decoupling of the phase dynamics from the amplitudes and a way to describe the phase response of chaotic oscillators have been outlined. The optimal phase has also other advantages when compared to arbitrary phase-like variables. We have demonstrated that, while the optimal phase yields a proper description of the phase response, phaselike variables do not provide reasonable phase response plots. Furthermore, the optimal phase, contrary to phaselike variables, is directly related to the phase of unstable periodic orbits inside chaos.

The theory of optimal isophases can possibly provide a refined understanding of emergent behavior of weakly coupled oscillating systems. For example, a theoretical phase description of weakly coupled limit cycle oscillators can be extended to ones of greater complexity, such as stochastic or chaotic oscillators (cf. [11,12]). In this way, one can treat more realistic models of natural systems. Furthermore, the theory can easily be utilized in the analysis of observed chaotic oscillations, where the numerical scheme described above can be used to refine a preliminary phase description. This can help to reduce certain systematic errors that may be present in phase-related quantities such as coupling strengths.

The theory is easily adaptable for the analysis of nonlinear oscillations with a random component (for theoretical approaches, see, e.g., [23–25]). Here the return times to optimal isophases have to be interpreted in an average sense. The corresponding results will be presented elsewhere.

In the present form, the approach is applicable to relatively simple chaotic and irregular systems, described by a single phase variable. The next step would be a generalization to systems with two time scales, generating mixed mode regular or chaotic oscillations (e.g., bursting regimes of neuron models). Such a step requires a substantial refinement of the present approach, and it remains a challenging problem for the future.

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APPENDIX: ISOPHASES OF THE UNSTABLE STUART-LANDAU OSCILLATOR

To give an analytically tractable example of isophases of UPOs on their unstable manifold, let us consider the *unstable Stuart-Landau oscillator* governed by

$$\dot{r} = r(r^2 - 1); \quad \dot{\varphi} = \alpha - \kappa r^2. \tag{A1}$$

It is exactly solvable: For the initial conditions $r(0) = R$ and $\varphi(0) = \Phi$, it has the well-known solution

$$r(t) = \left[1 + \frac{1 - R^2}{R^2} e^{2t}\right]^{-1/2}, \tag{A2}$$

$$\varphi(t) = (\alpha - \kappa)t - \kappa \ln r(t) + \Phi + \kappa \ln R. \tag{A3}$$

Oscillator (A1) shows an unstable periodic orbit (UPO) with frequency $\omega = \alpha - \kappa$. Depending on the initial conditions, it either performs decaying oscillations (for $R < 1$) or diverges in finite time (for $R > 1$).

As the characteristic period we first choose that of the UPO: $S = \frac{2\pi}{\omega}$. In order to obtain a phase that rotates independently of r, we set $\theta = \omega t + \Phi + \kappa \ln R$. Inserting θ into Eq. (A3), we find that optimal isophases $I(\theta)$ are solutions of the equation

$$\theta = \varphi + \kappa \ln r. \tag{A4}$$

For each $(\Phi,R) \in I(\theta)$ the return time for $\theta \to \theta + 2\pi$ is equal to S because $\dot{\theta} = \omega$. This is the standard definition of the isophases.

Alternatively, one may think of the unstable Stuart-Landau oscillator as a rarely visited part of the state space of a bigger chaotic system that has a different characteristic frequency $\frac{2\pi}{T} = \omega_0 = \omega + \Delta\omega$. This means that average period T is different from the period S of the UPO. Therefore, the condition that the return time for a Poincaré surface is equal to T cannot be fulfilled on the orbit. To fulfill the condition for states off the periodic orbit, we now seek a phase with the dynamics $\dot{\theta} = \omega_0$. Therefore, we rewrite Eq. (A3) in terms of $\omega_0 t$:

$$\varphi(t) = \omega_0 t + \Phi + \kappa \ln R - \kappa \ln r - \Delta\omega t(r). \tag{A5}$$

Here we need to rewrite time as a function of the radius, using (A2). We get

$$t(r) = \frac{1}{2} \ln |r^2 - 1| - \ln r + \ln \frac{R}{\sqrt{1 - R^2}}. \tag{A6}$$

After the substitution, a uniformly rotating phase is given by $\theta = \omega_0 t + \Phi + \kappa \ln R + \Delta\omega \ln(\sqrt{1 - R^2}/R)$. Comparing the result with Eq. (4), we obtain the phase correction as

$$\Delta(r) = -(\kappa - \Delta\omega) \ln r - \frac{\Delta\omega}{2} \ln |r^2 - 1|. \tag{A7}$$

While the return time is equal to T off the periodic orbit, the phase correction diverges as $\ln |1 - r|$ in the limit $r \to 1$. Thus, the "isophase" is singular and winds itself infinitely often around the limit cycle.

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