Physica D 240 (2011) 872–881

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# Dynamics of heterogeneous oscillator ensembles in terms of collective variables

Arkady Pikovsky, Michael Rosenblum *

*Department of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24/25, D-14476 Potsdam, Germany*

ARTICLE INFO

*Article history:*
Received 2 October 2009
Received in revised form
4 January 2011
Accepted 5 January 2011
Available online 12 January 2011
Communicated by S. Coombes

*Keywords:*
Coupled oscillators
Oscillator ensembles
Kuramoto model
Nonlinear coupling
Watanabe–Strogatz theory
Ott–Antonsen theory

ABSTRACT

We consider general heterogeneous ensembles of phase oscillators, sine coupled to arbitrary external fields. Starting with the infinitely large ensembles, we extend the Watanabe–Strogatz theory, valid for identical oscillators, to cover the case of an arbitrary parameter distribution. The obtained equations yield the description of the ensemble dynamics in terms of collective variables and constants of motion. As a particular case of the general setup we consider hierarchically organized ensembles, consisting of a finite number of subpopulations, whereas the number of elements in a subpopulation can be both finite or infinite. Next, we link the Watanabe–Strogatz and Ott–Antonsen theories and demonstrate that the latter one corresponds to a particular choice of constants of motion. The approach is applied to the standard Kuramoto–Sakaguchi model, to its extension for the case of nonlinear coupling, and to the description of two interacting subpopulations, exhibiting a chimera state. With these examples we illustrate that, although the asymptotic dynamics can be found within the framework of the Ott–Antonsen theory, the transients depend on the constants of motion. The most dramatic effect is the dependence of the basins of attraction of different synchronous regimes on the initial configuration of phases.



## 1. Introduction

A model of (infinitely) many coupled limit cycle oscillators explains a variety of natural phenomena in various branches of science. The applications range from the description of the collective dynamics of Josephson junctions [1], lasers [2], and electrochemical oscillators [3] to that of pedestrians on footbridges [4,5], applauding persons in a large audience [6], cells exhibiting glycolytic oscillations [7–9], neuronal populations [10], etc. Very often, when the oscillator network is not too sparse, it can be approximately considered as fully connected (globally coupled); it means that the oscillator population is treated in the mean field approximation. Externally forced or feedback controlled globally coupled ensemble or several interacting ensembles serve as models of circadian rhythms, normal and pathological brain activity, interaction of different brain regions, and many other problems [11–17]. Many aspects of the ensemble dynamics, especially those related to inhomogeneity [18–20] or nonlinearity of coupling [21,22], temporal dynamics of the collective mode [23,24], different frequency distributions [25,26], and clustering [27–29] remain in the focus of the current research activity.

Ensembles of weakly interacting units are successfully treated within the framework of phase approximation [30–34]. The most popular is the Kuramoto model of sine-coupled phase oscillators, or its extension, the Kuramoto–Sakaguchi model [35]. This model explains self-synchronization and appearance of a collective mode (mean field) in an ensemble of generally nonidentical elements; the transition to synchrony occurs at a certain critical value of the coupling constant that is roughly proportional to the width of the distribution of natural frequencies [30,31]. With the further increase of coupling, more and more oscillators join the synchronous cluster, so that the amplitude of the mean field grows as a square root of supercriticality. It is instructive to interpret this transition as follows: the non-zero mean field forces individual units and entrains at least a part of them; these entrained units become coherent, thus yielding a non-zero mean field. A quantitative consideration, based on this self-consistency argument and first performed by Kuramoto [30,31], provides the amplitude and frequency of the stationary solution. References to many further aspects of the Kuramoto model can be found in [36–38].

An extension of the Kuramoto model for the case of nonlinear coupling has been suggested in our recent publications [21,22]; see also [39,40]. Nonlinearity in this context means that the effect of the collective mode on an individual unit depends on the amplitude of this forcing, so that, e.g., the interaction of the field and of a unit can be attractive for weak forcing and repulsive for strong forcing. Mathematically this is represented by the dependence of the parameters of the Kuramoto–Sakaguchi model (the coupling strength and the phase shift) on the mean field amplitude. The model exhibits nontrivial effects like destruction of a completely synchronous state and appearance of partial synchrony in an ensemble of identical units. Moreover, in this

* Corresponding author.
*E-mail address:* mros@uni-potsdam.de (M. Rosenblum).

0167-2789/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2011.01.002

setup the frequencies of the collective mode and of oscillators can be different and incommensurate.

Analytical description of the dynamics of oscillator ensembles remains an important and interesting problem. Although sometimes the stationary solution for the mean field can be found with the help of the above mentioned self-consistency argument, a full analysis of the ensemble dynamics remains a challenge. Two approaches to this problem have been developed by Watanabe and Strogatz (WS) [41,42] and by Ott and Antonsen (OA) [23,24].

The WS theory is a power tool which provides a full dynamical description of ensembles of *identical* oscillators, sine coupled to a common external force. In particular, this force can be the mean field of the population, so that for the case of identical units the WS theory completely describes the Kuramoto–Sakaguchi and the nonlinear model (see [22]). This description is given in terms of three collective (macroscopic) variables, hereafter called the WS variables, plus constants of motion. The collective variables obey three WS equations (see also Appendix A); thus, the dynamics of an ensemble of identical elements is effectively three dimensional. The OA theory treats an infinitely large ensemble of oscillators that are generally heterogeneous, i.e. with a distribution of natural frequencies. In this theory one shows that a certain form of distribution of phases is invariant under the evolution, constituting the so-called OA reduced manifold. This ansatz includes only a subset of all the possible solutions of the problem, however, it is argued to be valid asymptotically for large times [24].

The main goal of this paper is to extend the WS theory to cover ensembles of nonidentical oscillators and to establish a link between WS and OA theories. In our recent brief communication [43] we approached this problem by treating the ensemble as a system of coupled subpopulations, each consisting of identical oscillators. Each subpopulation can be then described by three WS variables, whereas the full system is described by a system of coupled WS equations. A description of an ensemble with a continuous frequency distribution is then obtained by performing a thermodynamic limit. Here we begin with the continuity equation and, following the idea of WS for the case of identical units, directly derive the WS equations for a general inhomogeneous case. Next, we relate the WS and OA theories, showing that the OA reduced manifold corresponds to a particular choice of constants of motion in the WS theory. Our theory is illustrated by the examples of the Kuramoto model, nonlinear model, and the model of two coupled populations, exhibiting a chimera state.

The paper is organized as follows. In Section 2 we discuss the main model and provide an extension of the WS theory to the case of nonidentical oscillators. Here we also discuss a relation between the WS and OA theories (cf. a recent paper by Marvel et al. [44]). In Section 3 the general theory is applied to describe the dynamics of ensembles with linear and nonlinear mean field coupling and is illustrated by numerics. We summarize and discuss our results in Section 4.

## 2. Dynamics of heterogeneous oscillator populations: description via collective variables

In this section we first derive the Watanabe–Strogatz equations for a general heterogeneous population of phase oscillators. Our derivation is heavily based on the derivation given by WS in Ref. [42], where they treated the case of identical oscillators. Next, we discuss under which conditions the WS equations reduce to the OA equations.

### 2.1. WS reduction for a system with general continuous distribution of parameters

Our basic model is an infinitely large ensemble of generally nonidentical phase oscillators. Each oscillator has natural frequency $\omega(x,t)$ which depends on a continuous parameter $x$; generally $x$ can be a vector. Oscillators are driven by a complex field $H(x,t)$:

$$\frac{d\phi(x,t)}{dt} = v = \omega(x,t) + \mathrm{Im}\left[H(x,t)e^{-i\phi}\right]. \tag{1}$$

The state of the ensemble can be described by the distribution density $W(x,\phi,t)$; it is convenient to write it as

$$W(x,\phi,t) = n(x)w(x,\phi,t), \tag{2}$$

where the distribution density of the parameter $n(x)$ and conditional distribution density of oscillators $w(x,\phi,t)$ are normalized according to

$$\int n(x)dx = 1 \quad \text{and} \quad \int_{-\pi}^{\pi} w(x,\phi,t)d\phi = 1.$$

We start with the continuity equation which expresses the conservation of the number of oscillators:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \phi}(wv) = 0, \tag{3}$$

where the velocity $v$ is defined according to Eq. (1). Following the idea of Watanabe and Strogatz [42] we demonstrate that, with the transformation to the WS variables $\rho(x,t)$, $\Phi(x,t)$, $\Psi(x,t)$ and $\psi(x)$ according to

$$e^{i\phi} = e^{i\Phi}\frac{\rho + e^{i(\psi-\Psi)}}{\rho e^{i(\psi-\Psi)} + 1}, \tag{4}$$

the time-dependent density $w(x,\phi,t)$ is transformed into a stationary density $\sigma(x,\psi)$.[1] We perform the following variable substitution in the continuity equation:

$$t,\ \phi,\ x \quad\to\quad \tau = t,\ \psi = \psi(x,\phi,t),\ y = x.$$

The relation between the densities in old and new variables takes the form:

$$w(x,\phi,t) = \sigma(y,\psi,\tau)\frac{\partial(y,\psi,\tau)}{\partial(x,\phi,t)} = \sigma(x,\psi,\tau)\frac{\partial\psi}{\partial\phi}. \tag{5}$$

Writing the continuity equation in new coordinates (see Appendix B), we obtain:

$$0 = \frac{\partial w}{\partial t} + \frac{\partial}{\partial \phi}(wv) = \frac{\partial\sigma}{\partial\tau}\frac{\partial\psi}{\partial\phi} + \frac{\partial\sigma}{\partial\psi}\left\{\frac{\partial\psi}{\partial\phi}\left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial\phi}\right)\right\}$$
$$+\sigma\left\{\frac{\partial}{\partial\tau}\left(\frac{\partial\psi}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\frac{\partial\psi}{\partial\phi}\right)\left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial\phi}\right)\right.$$
$$\left.+\left(\frac{\partial\psi}{\partial\phi}\right)^2\frac{\partial v}{\partial\psi}\right\}. \tag{6}$$

In Appendix B we show that both expressions in curly brackets vanish provided $\Phi(x,t)$, $\Psi(x,t)$, $\rho(x,t)$ obey

$$\begin{aligned}
\frac{\partial\rho(x,t)}{\partial t} &= \frac{1-\rho^2}{2}\mathrm{Re}(H(x,t)e^{-i\Phi}),\\
\frac{\partial\Phi(x,t)}{\partial t} &= \omega(x,t) + \frac{1+\rho^2}{2\rho}\mathrm{Im}(H(x,t)e^{-i\Phi}),\\
\frac{\partial\Psi(x,t)}{\partial t} &= \frac{1-\rho^2}{2\rho}\mathrm{Im}(H(x,t)e^{-i\Phi}).
\end{aligned} \tag{7}$$

This implies that $\frac{\partial\sigma}{\partial\tau} = 0$ and, thus, $\sigma(x,\psi)$ is a stationary distribution, so that $\psi(x)$ are constants of motion.

[1] We notice here that our definition of the WS variables slightly differs from the original one [42]. The relation to original variables is given in Appendix A. There we also derive the transformation (4) from its original form [42].

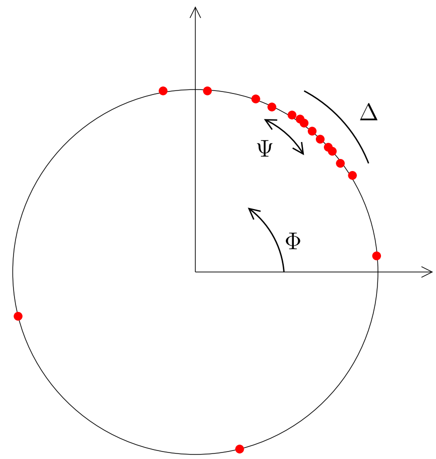


**Fig. 1.** Illustration of the meaning of the WS variables. Filled circles denote the phases of the oscillators having a certain value of $x$. The WS amplitude $\rho(x)$ is related to the width $\Delta$ of the distribution of phases; $\rho(x) = 0$, if this distribution is uniform and $\rho(x) = 1$, if it collapses to a $\delta$-distribution. Thus, $\rho(x)$ is roughly proportional to the local mean field amplitude $r(x)$. Angle variable $\Phi(x)$ describes the position of the hump in the distribution; therefore it roughly corresponds to the phase $\theta(x)$ of the local mean field. Angle variable $\Psi(x)$ characterizes the motion of individual oscillators with respect to the hump.

As discussed in detail in [42], transformation (4) determines $\rho, \Phi, \Psi$ in a unique way if $\psi(x)$ obey three additional constraints; they are

$$\int_{-\pi}^{\pi} \sigma(\psi, x) e^{i\psi} d\psi = 0 \tag{8}$$

and

$$\mathrm{Re} \int_{-\pi}^{\pi} \sigma(\psi, x) e^{i2\psi} d\psi = 0. \tag{9}$$

The choice of the constraint (8) fixes the relation between $\rho$ and mean field amplitude, as discussed in the next paragraph. Condition (9) relates $\psi$ and $\Psi$ and is rather arbitrary.

Now we discuss the physical meaning of the WS variables. For this goal we compare them with the complex *local mean field*, or the local Kuramoto order parameter

$$Z(x, t) = r(x, t) e^{i\theta(x,t)} = \int_{-\pi}^{\pi} e^{i\phi} w(x, \phi, t) d\phi, \tag{10}$$

where $r$ and $\theta$ are the amplitude and phase of the mean field, respectively. The WS amplitude variable $\rho(x, t)$ is roughly proportional to the mean field amplitude $r(x, t)$. Indeed, if $\rho(x, t) = 0$, then from Eq. (4) taking into account Eq. (8), we obtain $r(x, t) = 0$. Similarly, Eq. (4) shows that if $\rho(x, t) = 1$, then all $\phi(x, t)$ are equal, which yields $r(x, t) = 1$. For intermediate values $0 < \rho < 1$, the relation between $\rho$ and $r$ generally depends also on $\Psi$ and $\psi$.

The WS phase variable $\Phi$ characterizes the position of the maximum in the distribution of phases and is close to the phase of the mean field $\theta$. They coincide for $\rho = r = 1$; for $0 < \rho < 1$, $\Phi$ is shifted with respect to $\theta$ by a value, dependent on $\rho, \Psi$. The second WS phase variable $\Psi$ determines the shift of individual oscillators with respect to $\Phi$. See Fig. 1 for further illustration.

For the following it is convenient to introduce a combination of two WS variables $z = \rho e^{i\Phi}$. Introducing also the phase shift $\alpha = \Phi - \Psi$, we write the WS equation (7) in an equivalent form:

$$\frac{\partial z(x, t)}{\partial t} = i\omega(x, t) z + \frac{1}{2} H(x, t) - \frac{z^2}{2} H^*(x, t), \tag{11}$$

$$\frac{\partial \alpha(x, t)}{\partial t} = \omega(x, t) + \mathrm{Im}\left(z^* H(x, t)\right). \tag{12}$$

In the new variables, transformation (4) reads

$$e^{i\phi} = z \frac{1 + |z|^{-2} z^* e^{i(\psi+\alpha)}}{1 + z^* e^{i(\psi+\alpha)}}. \tag{13}$$

### 2.2. WS equations for the case of discrete distribution of parameters

Consider now the case when parameter $x$ takes only some discrete values. It means that the ensemble consists of groups (subpopulations), so that $\omega$ and $H$ are constants for each group. Thus, the group is a set of oscillators which are (i) identical and (ii) driven by a common force. In this case the system of PDEs (7) or Eqs. (11) and (12) simplifies to a system of ODEs.

If all oscillators are identical and subject to a common force, i.e. $\omega$ and $H$ do not depend on any parameter $x$, then Eq. (7) lose the $x$-dependence and we obtain the original WS equations [42] (in the new notation; see Appendix A). As shown in [42], these equations are also valid if the number of oscillators is finite (there are two limitations, discussed below). In this case the number of constants of motion $\psi$ is also finite and the only change is that the constraints Eqs. (8) and (9) are formulated in terms of sums, not integrals.

Consider now the case when an ensemble consists of a finite number of groups. (We call such an ensemble a hierarchically organized one.) Denoting the groups by index $a$ we obtain instead of Eqs. (11) and (12) a finite system for collective variables $z_a$ and $\alpha_a$:

$$\frac{dz_a}{dt} = i\omega_a z_a + \frac{1}{2} H_a - \frac{z_a^2}{2} H_a^*, \tag{14}$$

$$\frac{d\alpha_a}{dt} = \omega_a + \mathrm{Im}(z_a^* H_a). \tag{15}$$

Again, the number of oscillators in each group can be either infinite (then the group is characterized by the distribution function $\sigma_a(\psi)$) or finite (then it is characterized by constants $\psi_{a,k}$).

Remarkably, with the help of the formulation via Eqs. (14) and (15) we can overcome the limitation of the WS theory, namely that the number of identical oscillators should be larger than three and that an initial configuration of a subpopulation cannot have too large clusters of identical oscillators in identical states [42]. To this end we note that Eq. (1) for an individual oscillator satisfy Eqs. (14) and (15) if we set $z = \exp(i\phi)$ and $\alpha = \phi + \mathrm{const}$. Thus, an individual oscillator not belonging to a large group can be treated as a separate group, also described by Eqs. (14) and (15). The same idea helps us to treat large clusters inside groups, because each cluster can be considered as one oscillator. Correspondingly, for each cluster we introduce the collective variables and write the equations in the form of Eqs. (14) and (15), in addition to similar equations for the rest of the group.

Finally, we mention that generally an ensemble can be characterized by a parameter distribution which is continuous in some intervals and discrete in others. In this case the system is described by coupled systems of PDEs in the form of Eqs. (11) and (12) and of ODE in the form of Eqs. (14) and (15).

### 2.3. Linking the Watanabe–Strogatz and the Ott–Antonsen theories

In this section we relate WS variables to the complex mean field, or the Kuramoto order parameter, see Eq. (10), and to the generalized Daido order parameters. We demonstrate that particular solutions of the WS equations for the uniform distribution of constants of motion are equivalent to the solutions on the so-called reduced OA manifold [23,24]; see also [45,46]. Next, we discuss the properties of the OA solution for discrete and continuous distributions of oscillator frequencies. Note that a relation between the WS and OA theories has also been recently discussed by Marvel et al. [44].

*2.3.1. WS variables versus order parameters*

We recall the definition of the local mean field, or order parameter, given by Eq. (10). [For brevity of presentation we omit below the parameter $x$ (or index $a$, for the case of a hierarchical population).] By substituting $e^{i\phi}$ in Eq. (10) via Eq. (13), we obtain:

$$Z = re^{i\theta} = z\gamma(z, \alpha), \tag{16}$$

where

$$\gamma(z, \alpha) = \int_{-\pi}^{\pi} \frac{1 + |z|^{-2} z^* e^{i(\psi+\alpha)}}{1 + z^* e^{i(\psi+\alpha)}} \sigma(\psi) d\psi. \tag{17}$$

We see that in general a relation between the WS variables and the order parameter is rather complex and contains not only the macroscopic variables $z, \alpha$ but also depends heavily on the distribution $\sigma(\psi)$.

Now we discuss an important particular case when the order parameter $Z$ can be expressed only through the complex WS variable $z$. For this purpose we use an expansion

$$(1 + z^* e^{i(\psi+\alpha)})^{-1} = \sum_{l=0}^{\infty} (-z^*)^l e^{il(\psi+\alpha)},$$

and re-write the function $\gamma$ as a series

$$\gamma(z, \alpha) = \sum_{l=0}^{\infty} C_l \left(-z^* e^{i\alpha}\right)^l - |z|^{-2} \sum_{l=0}^{\infty} C_{l+1} \left(-z^* e^{i\alpha}\right)^{l+1},$$

where the coefficients

$$C_l = \int_{-\pi}^{\pi} \sigma(\psi) e^{il\psi} \, d\psi \tag{18}$$

are the amplitudes of the Fourier harmonics of the distribution of constants of motion $\psi$. Using that $C_1 = 0$ due to Eq. (8), we finally obtain

$$\gamma = 1 + (1 - |z|^{-2}) \sum_{l=2}^{\infty} C_l (-z^* e^{i\alpha})^l. \tag{19}$$

The crucial observation is that Eq. (19) essentially simplifies and we obtain simply $\gamma = 1$ if all the amplitudes of the Fourier harmonics vanish, i.e. if $C_l = 0$ for $l \neq 0$. This happens if constants of motion are uniformly distributed, i.e. if $\sigma(\psi) = (2\pi)^{-1}$. Thus, for *the uniform distribution of constants of motion* the order parameter of a subpopulation is directly expressed via the WS variables:

$$Z(x, t) = z(x, t) \quad \text{or} \quad Z_a(t) = z_a(t). \tag{20}$$

As a result, the WS equations (11) and (14) become equations for the amplitude and phase of the local mean field. The system simplifies further if the forcing $H$ is independent of $\alpha$, then the WS equations (12) or (15) become irrelevant for the dynamics.

The Kuramoto order parameter $Z$ is an important quantity; however it does not provide a complete characterization of the oscillator population. Such a characterization is given by a set of *generalized Daido order parameters* [32,33,47,34], defined according to

$$Z_m = \int_0^{2\pi} w(\phi) e^{im\phi} d\phi. \tag{21}$$

Clearly, $Z_1$ is just the Kuramoto order parameter $Z$. The physical meaning of the parameters $Z_m$ is transparent: they are simply the Fourier harmonics of the distribution of the phases and thus completely characterize this distribution. To evaluate them, we have to insert (4) in (21):

$$Z_m = z^m \gamma_m(z, \alpha), \tag{22}$$

where

$$\gamma_m(z, \alpha) = \int_{-\pi}^{\pi} \left( \frac{1 + |z|^2 z^* e^{i(\psi+\alpha)}}{1 + z^* e^{i(\psi+\alpha)}} \right)^m \sigma(\psi) d\psi. \tag{23}$$

It can be shown that for uniform distribution of constants of motion, i.e. for $\sigma = (2\pi)^{-1}$, $\gamma_m = 1$ for all $m$. To verify this, one writes $\gamma_m$ as a series as above and observes that all terms except for one contain vanishing integrals of type $\int_{-\pi}^{\pi} e^{iL\psi} d\psi$. Thus, for the special case of uniformly distributed constants of motion, we obtain $\gamma_m = 1$ and

$$Z_m = z^m = Z^m. \tag{24}$$

*2.4. The Ott–Antonsen theory*

Ott and Antonsen [23] treated basic model (1) in the thermodynamic limit of infinite number of oscillators with the help of the continuity equation (3). Writing the density function $W(x, \phi, t)$ as a Fourier series[2]

$$W(x, \phi, t) = n(x) w(x) = \frac{n(x)}{2\pi} \left\{ 1 + \left[ \sum_{m=1}^{\infty} f_m(x, t) e^{-im\phi} + \text{c.c.} \right] \right\},$$

where c.c. denotes complex conjugate, Ott and Antonsen noticed that the continuity equation is fulfilled if the Fourier coefficients can be expressed as

$$f_m(x, t) = [F(x, t)]^m, \tag{25}$$

where $F(x, t)$ is the only unknown function. We denote this particular class of solutions of Eq. (1) as the OA reduced manifold. Recalling the definition of generalized order parameters, Eq. (21), we see that the quantities $f_m$ are exactly these order parameters and therefore the ansatz (25) is equivalent to (24). Thus, the OA reduced manifold *exactly corresponds* to the special case where the generalized order parameters are expressed via the powers of the WS variable $z$ (24). This holds if the distribution $\sigma(\psi)$ of the WS constants $\psi$ is uniform.

The idea of OA can be alternatively presented as follows. Let us consider a generalized order parameter $Z_m(x, t)$ of a subpopulation with the parameter $x$, see Eq. (21), and compute its time derivative

$$\dot{Z}_m = \int_0^{2\pi} \frac{\partial w(x, \phi, t)}{\partial t} e^{im\phi} d\phi = im \int_0^{2\pi} w(x, \phi, t) \dot{\phi} e^{im\phi} d\phi;$$

here we also used Eq. (3). Substituting $\dot{\phi} = \omega + (He^{-i\phi} - H^* e^{i\phi})/2i$ we obtain (cf. [26]):

$$\dot{Z}_m = i\omega m Z_m + \frac{m}{2}(HZ_{m-1} - H^* Z_{m+1}).$$

This (infinite-dimensional) system of ODEs reduces to a single equation if $Z_m = Z^m$; this case exactly corresponds to the OA manifold.

Important contribution has been made by Ott and Antonsen in [24]. In this publication they argued that the reduced manifold (25) is the only attractive one provided the parameter distribution $n(x)$ is continuous. This argumentation gave a justification for using this theory in a number of applications [45,46,20,48].

To conclude this section, we have demonstrated that the Ott–Antonsen manifold corresponds to a special case of uniformly distributed constants of motion in the Watanabe–Strogatz theory. As we demonstrate in the following section, the deviations of the OA dynamics from the exact one (given by WS equations) can be controlled by a proper choice of initial distribution of the oscillator phases.

[2] In fact, in [23] the natural frequency $\omega$ was used as a parameter $x$.

## 3. Ensembles with mean field coupling

### 3.1. Models with linear and nonlinear mean field coupling

Up until now we considered populations with general frequencies $\omega(x,t)$ and forcing terms $H(x,t)$. Now we specify the frequency distribution and the force, and consider several popular models as particular examples of the general approach. One popular assumption is that the oscillators differ only by their natural frequencies $\omega(x)$ but are subject to a common force $H$ that depends on the *global mean field* (order parameter) $Y$, computed over the whole population

$$Y = Re^{i\Theta} = \int dx\, n(x)Z(x) = \int dx\, n(x)\gamma(x)z(x). \qquad (26)$$

If the global coupling is *linear*, i.e. $H$ is simply proportional to $Y$,

$$H = \varepsilon e^{i\beta}Y, \qquad (27)$$

then from Eq. (1) we obtain the famous Kuramoto–Sakaguchi model

$$\frac{d\phi(x)}{dt} = \omega(x) + \varepsilon R\sin(\Theta - \phi(x) + \beta). \qquad (28)$$

Recently, this model has been generalized to the case of *nonlinear* coupling, when $H$ generally depends on $|Y|^2Y$, $|Y|^4Y$, etc. [21,22], so that

$$H = \varepsilon A(R,\varepsilon)e^{i\beta(R,\varepsilon)}Y.$$

For the purposes of this paper we concentrate on a particular case,

$$H = \varepsilon e^{i(\beta_0+\varepsilon^2R^2)}Y, \qquad (29)$$

that corresponds to the following microscopic equation:

$$\frac{d\phi(x)}{dt} = \omega(x) + \varepsilon R\sin\left(\Theta - \phi(x) + \beta_0 + \varepsilon^2R^2\right), \qquad (30)$$

where $\beta_0$ is some constant.

### 3.2. The Kuramoto–Sakaguchi model

Let us start with the standard Kuramoto–Sakaguchi model (28). For this model it is natural to identify the continuous variable $x$ with the frequency $\omega$. Substituting the force (27) into Eqs. (11) and (12) we obtain a closed system of WS equations

$$\frac{\partial z(\omega,t)}{\partial t} = i\omega z + \frac{\varepsilon e^{i\beta}}{2}Y - \frac{\varepsilon e^{-i\beta}}{2}z^2Y^*, \qquad (31)$$

$$\frac{\partial \alpha(\omega,t)}{\partial t} = \omega + \mathrm{Im}\left(z^*\varepsilon e^{i\beta}Y\right). \qquad (32)$$

Consider now the reduced set of solutions corresponding to the OA manifold, i.e. to the case $\gamma(\omega) = 1, z(\omega) = Z(\omega)$. The mean field

$$Y = \int dx\, n(x)z(x) \qquad (33)$$

becomes independent of $\alpha$, Eq. (32) decouples, and we are left with a closed system of Eqs. (31) and (33).

For the following we fix the distribution of natural frequencies, choosing it to be a Lorentzian one: $n(\omega) = [\pi(\omega^2+1)]^{-1}$. As demonstrated by Ott and Antonsen [23], for this case, under an additional assumption that $z(\omega)$ is analytic in the upper half-plane, the integral in Eq. (33) can be calculated by the residue of the pole at $\omega = i$; this calculation yields $Y = z(i)$. Substituting this along with $\omega = i$ into Eq. (31) we obtain the OA equation for the time evolution of the Kuramoto mean field:

$$\frac{dY}{dt} = \left(-1 + \frac{\varepsilon e^{i\beta}}{2}\right)Y - \frac{\varepsilon e^{-i\beta}}{2}Y^2Y^*. \qquad (34)$$

This closed equation for the order parameter was first derived and solved in [23]; the solution with the initial condition $R(0) = R_0$ is

$$R(t) = \bar{R}\left\{1 + \left[\left(\frac{\bar{R}}{R_0}\right)^2 - 1\right]e^{(2-\varepsilon\cos\beta)t}\right\}^{-1/2}, \qquad (35)$$

where $\bar{R} = \sqrt{1 - 2/(\varepsilon\cos\beta)}$ (notice a misprint in Eq. (11) of [23]).

We emphasize that Eq. (35) represents only a particular solution of the full equation system (31), (32) and (26). Generally, the dynamics of system (28) can deviate from (35). We illustrate this important issue by the following numerical examples.

#### 3.2.0.1. Example 1: effect of analyticity of $z(\omega)$

We show that the mean field dynamics deviates from (35) if the analyticity assumption above does not hold. We perform a direct numerical simulation of the Kuramoto–Sakaguchi model with $N = 10^4$ oscillators and $\beta = 0$. The frequencies of the oscillators are all different and are chosen to approximate the Lorentzian distribution. We perform two runs with the same *macroscopic initial conditions* for the ensemble, choosing $Y(0) = R(0)e^{i\Theta(0)} = R_0 = 0.5$, but with different *microscopic initial conditions*, i.e. with different initial distributions of phases. Practically, we introduce an auxiliary angle variable $\varsigma$ which attains $N$ values, labeled by index $k$, uniformly distributed between $-\pi$ and $\pi$ (end points are excluded; $\varsigma_k$ grows monotonically with $k$). The frequencies of oscillators are then obtained as $\omega_k = \tan\frac{\varsigma_k}{2}$ and the initial phases as

$$\phi_k(0) = \pm 2\arctan\left[\frac{1-R_0}{1+R_0}\tan\left(\frac{\varsigma_k}{2}\right)\right]$$
$$= \pm 2\arctan\left[\frac{1-R_0}{1+R_0}\omega_k\right], \qquad (36)$$

cf. Eq. (A.3); here the plus and minus signs correspond to the first and to the second run, respectively. Notice that since we have only one oscillator at each frequency, $\rho(\omega_k) = 1$, and, hence, $\gamma(\omega_k) = 0$, so that the reduction of system (31), (32) and (26) to system (31) and (33) is valid. Using Eq. (A.4) we obtain for the WS variable

$$z(\omega_k, 0) = e^{\phi(\omega_k,0)} = \frac{R_0(1 \mp i\omega_k) + 1 \pm i\omega_k}{R_0(1 \pm i\omega_k) + 1 \mp i\omega_k}. \qquad (37)$$

Considering the obtained expression as an approximation of a continuous function $z(\omega)$, we find that the latter has a pole at $\omega = \mp i\frac{1+R_0}{1-R_0}$. Thus, the first run corresponds to the initial condition that is analytic in the upper half-plane, while the second run corresponds to the initial condition that is analytic in the lower half-plane. The results are shown in Fig. 2, with the curves *a* and *b* corresponding to the first and to the second run, respectively. One can see that the transient dynamics of the global mean field heavily depends on the microscopic initial conditions. We emphasize that the result for the first set of initial conditions agrees very well with the solution (35), while for the second set of initial conditions the transient dynamics is essentially different.

#### 3.2.0.2. Example 2: finite-size effects

Now we verify how well the dynamics of the finite system can be approximated by the solution on the OA manifold. To this end

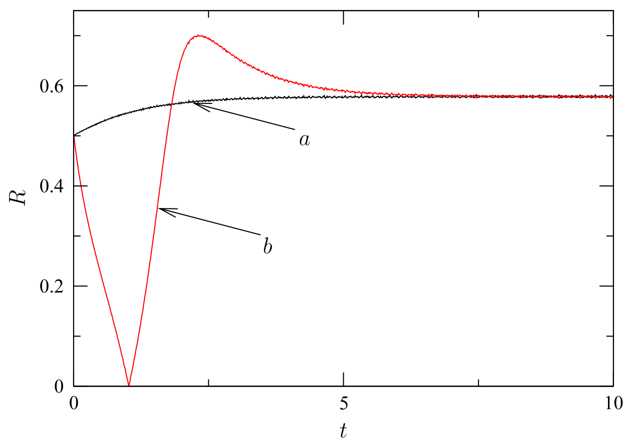


**Fig. 2.** (Color online) Mean field amplitude $R$ as a function of time, for the Kuramoto ensemble with the Lorentzian distribution of frequencies; the number of oscillators is $N = 10^4$, $\varepsilon = 3$. Curves (a, black) and (b, red) correspond to two different sets of initial phases, as described in the text. In the first case the evolution of the mean field follows theoretical solution given by Eq. (35), while for the second case the transient dynamics deviates significantly from this solution.

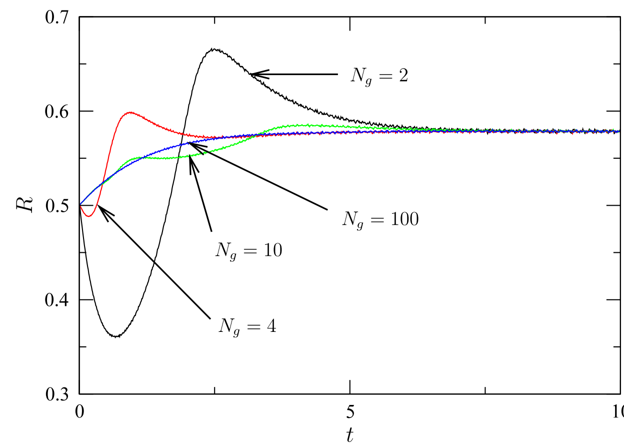


**Fig. 3.** Mean field amplitude $R$ as a function of time, for $\varepsilon = 3$, $M = 10^4$. In all simulations initial conditions have been chosen such that $Y(0) = 0.5$. For $N_g = 100$ the evolution of the mean field follows Eq. (35), while for smaller group sizes the transient deviates significantly from the OA manifold. In all runs the distribution of the constants $\psi_{a,k}$ inside each group was chosen to be uniform.

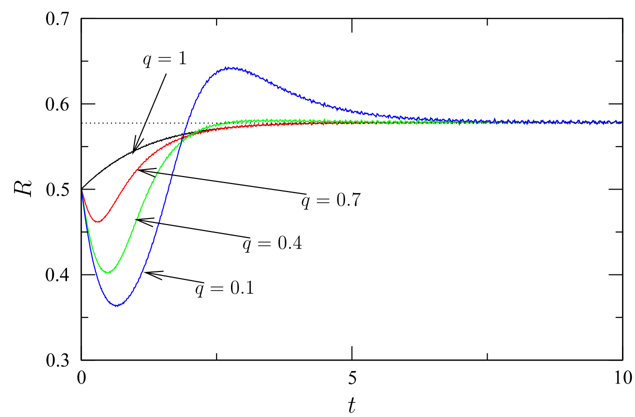


**Fig. 4.** The same as Fig. 3, but for different distributions of constants of motion $\psi_{a,k}$ and fixed group size $N_g = 200$. Dotted line shows the theoretical asymptotic value of $R$.

we approximate the continuous system with a Lorentzian distribution by a hierarchically organized population, consisting of $M = 10^4$ groups of identical elements. All groups have the same size $N_g$, so that the total number of oscillators is $N = M \cdot N_g$. Since $N_g$ is finite, each group is characterized by a discrete set of constants $\psi_{a,k}$, $k = 1, \ldots, N_g$. We assume first a uniform distribution of constants $\psi_{a,k} = \psi_{a,1} + 2\pi(k-1)/N_g$ and compute $\gamma_a$. Finite $N_g$ computation of $C_{a,l}$ according to $C_{a,l} = N_g^{-1}\sum_{k=1}^{N_g} \mathrm{e}^{\mathrm{i}l\psi_{a,k}}$ yields $|C_{a,l}| = 1$ and $\arg(C_{a,l}) = \psi_1 l$, for $l = N_g, 2N_g, \ldots$, and $C_{a,l} = 0$, otherwise. Eq. (19) becomes

$$\gamma_a = 1 + (1 - |z|^{-2}) \frac{\left[-z^* \mathrm{e}^{\mathrm{i}(\psi_1+\alpha)}\right]^{N_g}}{1 - \left[-z^* \mathrm{e}^{\mathrm{i}(\psi_1+\alpha)}\right]^{N_g}}, \quad (38)$$

and we see that the deviation of $\gamma_a$ from unity decreases with the size of the subpopulation and, therefore, can be neglected for large $N_g$. To illustrate this we perform numerical simulation for different $N_g$ with the macroscopic initial condition $Y(0) = 0.5$; the microscopic initial conditions are chosen in such a way that $z(\omega)$ is analytic in the upper half-plane. The results, shown in Fig. 3, confirm the theoretical prediction: with increase of $N_g$ the transient dynamics tends to the OA manifold and is nicely described by Eq. (35), while for a small number of oscillators in a group, the deviations from the OA solution are essential. If there is only one oscillator in a group, as in the previous example, the OA solution (35) is again valid, because for $N_g = 1$ we have $\rho_a = 1$ and $\gamma_a = 1$.

*3.2.0.3. Example 3: nonuniform distribution of constants $\psi$*

The effect of a non-homogeneous distribution of the microscopic constants $\psi_{k,a}$ on the dynamics of the mean field, for a large group size is illustrated in Fig. 4. Again, we consider a hierarchically organized ensemble with $M$ groups of $N_g$ oscillators each. The parameters of the simulation are $\varepsilon = 3$, $M = 10^4$, $N_g = 200$, and $Y(0) = 0.5$. However, the microscopic initial conditions, given by the distribution of the constants of motion $\psi_{a,k}$, differ from run to run ($z(w)$ is kept analytic in the upper half-plane). In particular, we introduce a parameter $0 < q \le 1$ that quantifies the deviation of the distribution of $\psi_{a,k}$ from a uniform one; the value $q = 1$ corresponds to the uniform distribution. (In Appendix C we describe how one can choose different microscopic initial conditions while keeping the same macroscopic initial conditions.) From Fig. 4 we see that the deviations from the OA manifold become larger as this distribution becomes less and less uniform, i.e. as the parameter $q$ decreases.

*3.3. The model of a nonlinearly coupled ensemble*

Now we consider the model Eq. (30) and, as in the case of the Kuramoto–Sakaguchi model, we look for solutions on the reduced OA manifold. Again, we consider the Lorentzian distribution of natural frequencies $n(\omega) = [\pi(\omega^2+1)]^{-1}$ and assume that $z(\omega)$ is analytic in the upper half-plane. Proceeding in a way similar to the treatment of the Kuramoto–Sakaguchi model, we obtain an analog of Eq. (34), which we write here as two real equations for the amplitude $R$ and frequency $\Omega$ of the mean field:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -R + \frac{\varepsilon}{2} R(1-R^2)\cos(\beta_0 + \varepsilon^2 R^2), \quad (39)$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \Omega = \frac{\varepsilon}{2}(1+R^2)\sin(\beta_0 + \varepsilon^2 R^2). \quad (40)$$

Here $\beta_0 = \text{const}$ and we take $|\beta_0| < \pi/2$.

It is easy to see that the asynchronous solution $R = 0$ becomes unstable if the coupling strength exceeds the critical value $\varepsilon_{cr} = 2/\cos\beta_0$. For $\varepsilon > \varepsilon_{cr}$ we expect to find one or several synchronous solutions with the mean field amplitude $0 < R < 1$. Looking for solutions with $R = \text{const}$, we set in Eq. (39) $\dot{R} = 0$ and obtain:

$$\varepsilon(1-R^2)\cos(\beta_0 + \varepsilon^2 R^2) - 2 = f(R^2) = 0. \quad (41)$$

Obviously, $f(0) = (\varepsilon - \varepsilon_{cr})\cos\beta_0 > 0$ and $f(1) = -2$, hence, there always exists at least one solution of Eq. (41). Numerical

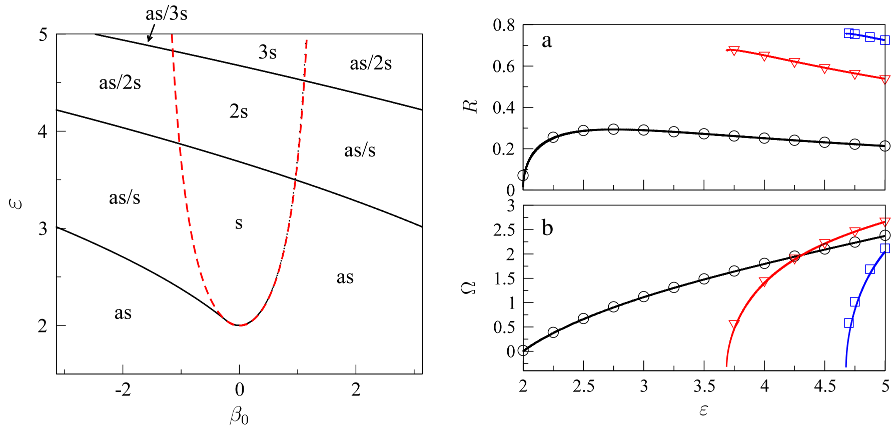


**Fig. 5.** Multistability in the nonlinearly coupled ensemble with $A = 1$, $\beta = \beta_0 + \varepsilon^2 R^2$. Left panel: different states on parameter plane. The red dashed line shows critical coupling $\varepsilon_{cr} = 2/\cos\beta_0$; inside the domain, determined by this curve, the asynchronous state is unstable. Labels *as*, *s*, and *ns* mean asynchrony (the state with $r = 0$ is stable), synchrony ($r > 0$), and coexistence of $n$ synchronous states, respectively. Label *as*/*ns* means coexistence of asynchronous and $n$ synchronous solutions. Right panel: illustration of the multistability for $\beta_0 = 0$. Three branches of the solution of Eq. (41) for the mean field amplitude are shown by different symbols in (a). Corresponding solutions for the frequency of the mean field, obtained from Eq. (40), are shown by the same colors in (b). Numerical results (see the text for details) are shown by symbols.

analysis shows that the number of its roots increases with $\varepsilon$. Thus, the system exhibits multistability. The corresponding bifurcation diagram in the parameter plane $\beta_0, \varepsilon$ is shown in Fig. 5. Here we also show the dependencies of the mean field amplitude and frequency on the coupling strength for $\beta_0 = 0$, compared with a direct numerical simulation of an ensemble with $M = 5000$ subpopulations of $N_g = 20$ oscillators each.

Finally, we demonstrate that although the asymptotic dynamics of the system can be described within the framework of the simplified OA theory by Eqs. (39) and (40), the transient dynamics depends on the initial distribution of constants $\psi$ and thus is not caught by Eqs. (39) and (40). In other words, the attractors of the multistable system can be obtained by assuming the uniform distribution of the constants of motion $\psi$, but their basins of attraction depend on the distributions of $\psi$. In numerical experiments we again take $M = 5000$ subpopulations of 20 oscillator each, for $\varepsilon = 4.5$, fix macroscopic initial conditions $R(0) = 0.52$ and $\Theta(0) = 0$, and vary parameter $q$. The results shown in Fig. 6 demonstrate that starting from the same macroscopic initial conditions, the system can evolve to different attractors, depending on the microscopic constants.

We conclude that for the ensemble with nonlinear coupling the Ott–Antonsen theory nicely describes asymptotic states (attractors), while the transient dynamics does not lie on the OA manifold. The most dramatic effect is the dependence of basins of attraction of different synchronous regimes on the microscopic initial conditions, as illustrated in Fig. 6.

### 3.4. Two coupled subpopulations

For the next example we concentrate on a model, recently studied by Abrams et al. [18]. They considered two identical subpopulations of the same size, i.e. $\omega_1 = \omega_2 = \omega$ (without loss of generality we set it to zero) and $N_1 = N_2$, but the coupling strength $\mu$ within a subgroup differs from the coupling strength $\nu$ between the subgroups. The equations are:

$$\frac{d\phi_k^{(1,2)}}{dt} = \omega + \mathrm{Im}\left[\left(\mu Z_{1,2} + \nu Z_{2,1}\right) e^{i(\beta - \phi_k^{(1,2)})}\right]. \quad (42)$$

The WS system (14) and (15) for this setup reads

$$\frac{dz_{1,2}}{dt} = \frac{1}{2}H_{1,2} - \frac{z_{1,2}^2}{2}H_{1,2}^*, \qquad \frac{d\alpha_{1,2}}{dt} = \mathrm{Im}(z_{1,2}^* H_{1,2}), \quad (43)$$

$$H_{1,2} = (\mu Z_{1,2} + \nu Z_{2,1})e^{i\beta}, \quad (44)$$

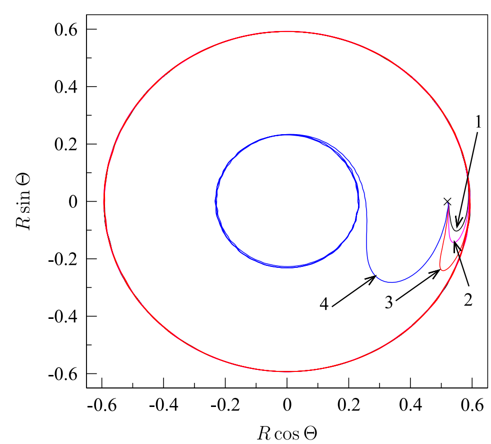


**Fig. 6.** Evolution of the mean field for same macroscopic initial conditions ($R(0) = 0.52$, $\Theta(0) = 0$, shown by a cross), but for different distributions of the constants of motions, parameterized by $q$ (see the text). Curve labels $1 - 4$ correspond to $q = 1$, $q = 0.95$, $q = 0.9$, and $q = 0.85$, respectively. Note that both attractors of the system (limit cycles with radii $\approx 0.23$ and $\approx 0.6$) correspond to the theory, developed under the assumption of uniform distribution of the constants of motion (cf. Fig. 5). However, transient dynamics and basins of attraction depend on the distribution of the constants of motion.

and the relation between $Z_{1,2}$ and $z_{1,2}$ is given by Eq. (16). By applying the OA ansatz, i.e. by setting $Z_{1,2} = z_{1,2}$ we obtain a set of equations, originally derived in Ref. [18]. Analyzing these equations, Abrams et al. have obtained an interesting solution where one subpopulation is completely synchronized, $|z_1| = 1$, while the other one is only partially synchronized, $|z_2| < 1$. Moreover, this partially synchronous state can be either steady, $z_2 = $ const, or time periodic, i.e. $z_2$ is a periodic function of time. These regimes are called *chimera* states.

The model of Abrams et al. serves as a good illustration of the usefulness of the above described approach based on the *exact* WS theory. A complete description of the dynamics for arbitrary initial conditions is given not by the OA equations, but by system (43), (44) and (16). Correspondingly, the additional equations generally lead to an additional time periodicity for chimera states [43]: a steady-state solution becomes time periodic (Fig. 7), and a time-periodic state becomes quasiperiodic (Fig. 8). We notice that in this case the solutions do not evolve towards the OA manifolds, because the distribution of the oscillators' parameters is not continuous.

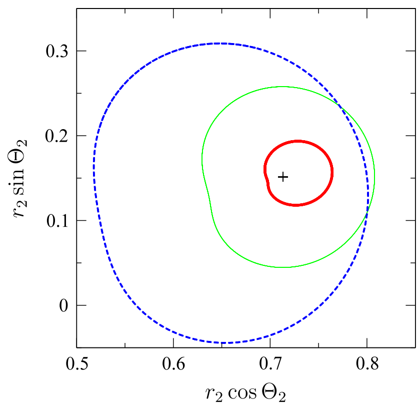


**Fig. 7.** (Color online) Simulation of ensemble (42) for $N = 64$, $\beta = \pi/2 - 0.1$, $\mu = 0.6$, $\nu = 1 - \mu = 0.4$, and different distributions of the microscopic constants $\psi_k^{(2)}$, controlled by the parameter $q$ (see Appendix C). Note that the distribution of constants $\psi_{1,k}$ is irrelevant since the first subpopulation is completely synchronized. The case $q = 1$ (marked by plus) corresponds to the OA manifold, here the mean field is constant. For $q = 0.9$, $q = 0.7$, and $q = 0.5$ one observes time-periodic states represented by limit cycles in the complex plane $Z_2$ (red bold, green solid, and blue dotted curves, respectively).

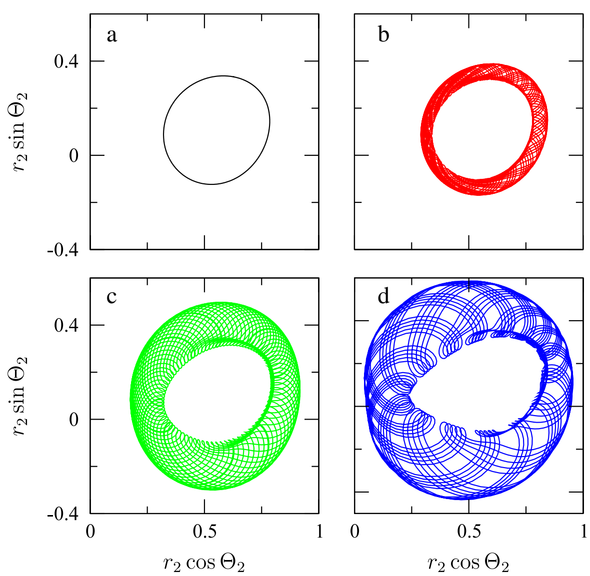


**Fig. 8.** The same as in Fig. 7, but for $\mu = 0.65$ and $\nu = 1 - \mu = 0.35$. In this case on the OA manifold, i.e. for $q = 1$, the dynamics is periodic (a), while for more general initial conditions parameterized by $q = 0.9$ (b), $q = 0.7$ (c), and $q = 0.5$ (d) the dynamics is quasiperiodic.

## 4. Conclusions and outlook

The main goal of this paper was to provide a generalization of the powerful Watanabe–Strogatz theory for the case of heterogeneous populations of phase oscillators. Beginning with the continuity equation, we have formulated the WS equations for a general inhomogeneous infinite population. As a particular case we obtained the description of the hierarchically organized ensembles, i.e. when the system can be treated as a collection of subpopulations of identical units. The derived equations provide an exact reduction of the dynamics; for hierarchical ensembles the problem under consideration becomes finite dimensional. We have analyzed the derived equations for the important case of mean field coupling. It is noteworthy that the reduced equations are valid both for linear and nonlinear coupling. In the latter case we were able to describe the dynamics of an inhomogeneous population with the Lorentzian distribution of frequencies; the main result here is appearance of the multistability of synchronous dynamics. (See [49] for the analysis of the population with nonlinear coupling and uniform distribution of natural frequencies.)

Next, we have thoroughly studied a relation between the Watanabe–Strogatz and the Ott–Antonsen theories and have demonstrated that the latter corresponds to a particular choice of initial conditions for the ensemble. To be exact, the OA reduced manifold corresponds to the case when the constants of motion in the WS framework are uniformly distributed. This was illustrated by the analysis of the chimera model by Abrams et al. [18], consisting of two interacting populations of identical oscillators. Here the solution of the four-dimensional OA equation system yields only a subset of the solutions of the full six-dimensional WS system. So, the latter one describes quasiperiodic chimera states, not possible within a framework, based on the OA theory.

As argued by Ott and Antonsen in Ref. [24], solutions of the full problem asymptotically tend to the OA reduced manifold (in a weak sense, i.e. the averages converge to the values on the OA manifold), if the elements of the ensemble have a continuous frequency distribution, which satisfies an analyticity condition. Our simulations of the linear and the nonlinear globally coupled ensemble confirm this statement. Moreover, we have explicitly demonstrated that the OA equations provide only asymptotic solutions, whereas the transient dynamics and the basins of attraction of these solutions depend on the choice of initial conditions and cannot be treated within the OA theory. (See [50] for another example of nontrivial transient dynamics of the OA manifold.) An important issue is that, possibly, the weak convergence to the OA manifold occurs not only for oscillators having a distribution of frequencies, but also for populations of oscillators with identical frequencies and a distribution of some other parameter. For example, consider a popular setup of a linear chain of oscillators with a long-range coupling; an interesting phenomenon here is the appearance of a chimera state [51,52]. In this state the amplitude of the force, acting on the units, is continuously distributed in a certain range. Preliminary theoretical and numerical treatments [53] confirm the validity of the OA approximation in this setup, at least for the case of the harmonic forcing field.

Finally, we would like to mention that the presented approach opens new perspectives in analysis of such long-standing problems as finite-size effects and the effects of a common external noise on oscillator ensembles. Some results in this direction have been already presented in this paper—we have demonstrated how the accuracy of the OA theory depends on the size of the oscillator groups. Also, application of the approach to systems with delayed coupling appears promising.

## Acknowledgements

We thank G. Bordyugov, A. Politi, and E. Ott for useful discussions, and E. Ott and S. Strogatz for communicating their works prior to publication. The work was supported by DFG (SFB 555).

## Appendix A. Watanabe–Strogatz variables

According to Watanabe and Strogatz [41,42], the system of $N > 3$ identical sine-coupled phase oscillators, subject to an arbitrary common force (see Eq. (1)) admits a low-dimensional description. For arbitrary functions of time $\omega(t)$ and $H(t)$, this $N$-dimensional system is completely described by the "global phases" $\tilde{\Psi}$ and $\tilde{\Phi}$ and the global "amplitude" $\tilde{\rho}$, $0 \le \tilde{\rho} \le 1$, plus constants of motion $\psi_k$, $k = 1, \ldots, N$, which obey three additional constraints, so that $N - 3$ of them are independent. The global WS variables $\tilde{\Psi}$, $\tilde{\Phi}$, and $\tilde{\rho}$ obey the WS equations; their original form can be found in

[41,42]. The solution of the original system (1) can be recovered via the following transformation:

$$\tan\left(\frac{\phi_k-\tilde{\Phi}}{2}\right)=\sqrt{\frac{1+\tilde{\rho}}{1-\tilde{\rho}}}\tan\left(\frac{\psi_k-\tilde{\Psi}}{2}\right). \tag{A.1}$$

We perform the variable substitution $\tilde{\rho},\tilde{\Psi},\tilde{\Phi}\ \rightarrow\ \rho,\Psi,\Phi$ according to

$$\tilde{\rho}=\frac{2\rho}{1+\rho^2},\qquad \tilde{\Psi}=\Psi+\pi,\qquad \tilde{\Phi}=\Phi+\pi. \tag{A.2}$$

The new variables admit a clear physical interpretation; see Fig. 1 and its discussion. The transformation (A.1) now takes the form

$$\tan\left(\frac{\phi_k-\Phi}{2}\right)=\frac{1-\rho}{1+\rho}\tan\left(\frac{\psi_k-\Psi}{2}\right). \tag{A.3}$$

It is convenient to re-write this transformation in the exponential form [43,44], using the following identity:

$$\begin{aligned}e^{i\alpha}&=\frac{1+i\tan(\alpha/2)}{1-i\tan(\alpha/2)}=\frac{[1+i\tan(\alpha/2)]^2}{1+\tan^2(\alpha/2)}\\&=\cos^2\frac{\alpha}{2}\cdot\left(1-\tan^2\frac{\alpha}{2}+2i\tan\frac{\alpha}{2}\right)=\cos\alpha+i\sin\alpha.\end{aligned} \tag{A.4}$$

With the help of this identity we write:

$$\begin{aligned}e^{i(\phi_k-\Phi)}&=\frac{1+i\tan\frac{\phi_k-\Phi}{2}}{1-i\tan\frac{\phi_k-\Phi}{2}}=\frac{1+i\frac{1-\rho}{1+\rho}\tan\frac{\psi_k-\Psi}{2}}{1-i\frac{1-\rho}{1+\rho}\tan\frac{\psi_k-\Psi}{2}}\\&=\frac{(1+\rho)\cos\frac{\psi_k-\Psi}{2}+i(1-\rho)\sin\frac{\psi_k-\Psi}{2}}{(1+\rho)\cos\frac{\psi_k-\Psi}{2}+i(\rho-1)\sin\frac{\psi_k-\Psi}{2}}\\&=\frac{\rho e^{-i(\psi_k-\Psi)/2}+e^{i(\psi_k-\Psi)/2}}{\rho e^{i(\psi_k-\Psi)/2}+e^{-i(\psi_k-\Psi)/2}},\end{aligned}$$

which yields Eq. (4). Note that the obtained transformation is known as the Möbius transformation (see [44]).

## Appendix B. Variable transformation for continuity equation

We perform transformation of variables in Eq. (3), using Eq. (5):

$$\begin{aligned}0&=\frac{\partial w}{\partial t}+\frac{\partial}{\partial\phi}(wv)=\frac{\partial w}{\partial\tau}\frac{\partial\tau}{\partial t}+\frac{\partial w}{\partial\psi}\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial\psi}(wv)\frac{\partial\psi}{\partial\phi}\\&=\frac{\partial}{\partial\tau}\left(\sigma\frac{\partial\psi}{\partial\phi}\right)+\frac{\partial}{\partial\psi}\left(\sigma\frac{\partial\psi}{\partial\phi}\right)\cdot\frac{\partial\psi}{\partial t}\\&\quad+\left[\frac{\partial}{\partial\psi}\left(\sigma\frac{\partial\psi}{\partial\phi}\right)v+\left(\sigma\frac{\partial\psi}{\partial\phi}\right)\frac{\partial v}{\partial\psi}\right]\frac{\partial\psi}{\partial\phi}\\&=\frac{\partial\sigma}{\partial\tau}\frac{\partial\psi}{\partial\phi}+\sigma\frac{\partial}{\partial\tau}\left(\frac{\partial\psi}{\partial\phi}\right)+\left[\frac{\partial\sigma}{\partial\psi}\frac{\partial\psi}{\partial\phi}+\sigma\frac{\partial}{\partial\psi}\left(\frac{\partial\psi}{\partial\phi}\right)\right]\frac{\partial\psi}{\partial t}\\&\quad+\left\{\left[\frac{\partial\sigma}{\partial\psi}\frac{\partial\psi}{\partial\phi}+\sigma\frac{\partial}{\partial\psi}\left(\frac{\partial\psi}{\partial\phi}\right)\right]v+\sigma\frac{\partial\psi}{\partial\phi}\frac{\partial v}{\partial\psi}\right\}\frac{\partial\psi}{\partial\phi}\\&=\frac{\partial\sigma}{\partial\tau}\frac{\partial\psi}{\partial\phi}+\sigma\left\{\frac{\partial}{\partial\tau}\left(\frac{\partial\psi}{\partial\phi}\right)+\frac{\partial}{\partial\psi}\left(\frac{\partial\psi}{\partial\phi}\right)\left(\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial\phi}\right)\right.\\&\quad\left.+\left(\frac{\partial\psi}{\partial\phi}\right)^2\frac{\partial v}{\partial\psi}\right\}+\frac{\partial\sigma}{\partial\psi}\left\{\frac{\partial\psi}{\partial\phi}\left(\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial\phi}\right)\right\}.\end{aligned} \tag{B.1}$$

Let us demonstrate that the coefficients at $\sigma$ and $\frac{\partial\sigma}{\partial\psi}$ vanish if $\rho,\Phi$, and $\Psi$ obey the WS equations. For this goal we first compute $\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial\phi}$. It is convenient to use the notations $f=e^{i(\phi-\Phi)}$, $c=e^{i(\psi-\Psi)}$. Resolving Eq. (4) with respect to $\psi$, we obtain

$$\psi=\Psi-i\ln(f-\rho)+i\ln(1-\rho f). \tag{B.2}$$

Taking the derivative and re-arranging the terms, we obtain

$$\frac{\partial\psi}{\partial t}(\phi)=\dot{\Psi}-f\frac{1-\rho^2}{(f-\rho)(1-f\rho)}\dot{\Phi}+i\frac{1-f^2}{(f-\rho)(1-f\rho)}\dot{\rho}. \tag{B.3}$$

Using $e^{-i\phi}=e^{-i\Phi}/f=e^{-i\Phi}f^*$, we obtain in new variables:

$$v=\omega+\mathrm{Im}\left[He^{-i\Phi}f^*\right]. \tag{B.4}$$

Next, from Eq. (B.2) we compute, using $\frac{\partial f}{\partial\phi}=if$:

$$\frac{\partial\psi}{\partial\phi}(\phi)=\frac{(1-\rho^2)f}{(f-\rho)(1-\rho f)}. \tag{B.5}$$

Substituting into Eq. (B.3) the derivatives via the r.h.s. of the WS equations and using Eqs. (B.4) and (B.5), we obtain after tedious but straightforward algebra

$$\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial\phi}=0. \tag{B.6}$$

Hence the coefficient at $\frac{\partial\sigma}{\partial\psi}=0$ and the coefficient at $\sigma$ reduces to

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\psi}{\partial\phi}\right)+\left(\frac{\partial\psi}{\partial\phi}\right)^2\frac{\partial v}{\partial\psi}=Q.$$

To compute $Q$, we first substitute in Eq. (B.5) $f=\frac{\rho+c}{\rho c+1}$ from Eq. (4) and obtain, after straightforward manipulations,

$$\frac{\partial\psi}{\partial\phi}(\psi)=\frac{(\rho+c)(\rho+c^*)}{1-\rho^2}=\frac{\rho c+\rho c^*+2}{1-\rho^2}-1. \tag{B.7}$$

Derivation with respect to time yields

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(\frac{\partial\psi}{\partial\phi}\right)&=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\phi}\right)\\&=\frac{i\rho(c^*-c)}{1-\rho^2}\dot{\Psi}+\frac{(1+\rho^2)(c+c^*)+4\rho}{(1-\rho^2)^2}\dot{\rho}.\end{aligned} \tag{B.8}$$

Here we used $\frac{\partial c}{\partial t}=-ic\dot{\Psi}$, $\frac{\partial c^*}{\partial t}=ic^*\dot{\Psi}$. Next, we compute

$$\frac{\partial v}{\partial\psi}=\mathrm{Im}\left[He^{-i\Phi}\frac{\partial}{\partial\psi}\frac{\rho c+1}{\rho+c}\right]=(\rho^2-1)\mathrm{Re}\left[\frac{cHe^{-i\Phi}}{(\rho+c)^2}\right]. \tag{B.9}$$

Using the obtained expressions (B.7)–(B.9), we show, after tedious but straightforward manipulations, that $Q=0$ if $\dot{\Psi}$ and $\dot{\rho}$ obey the WS equations.

Thus, we demonstrate that the r.h.s. of the continuity equation Eq. (B.1) simplifies to $\frac{\partial\sigma}{\partial\tau}\frac{\partial\psi}{\partial\phi}$ and is therefore valid if $\sigma(\omega,\psi)$ is a stationary distribution.

## Appendix C. Choice of initial conditions for simulation of hierarchical populations

Our goal is to choose different microscopic initial conditions, i.e. initial values for oscillator phases, but keep the same macroscopic initial conditions, i.e. the amplitude of the mean field. For this goal we proceed as follows. For each subpopulation with the frequency $\omega_a$ we take $\psi_a$ uniformly distributed along the arcs $\left[(1-q)\frac{\pi}{2},(1+q)\frac{\pi}{2}\right]$ and $\left[-(1+q)\frac{\pi}{2},-(1-q)\frac{\pi}{2}\right]$, as shown in Fig. C.1. Here $0<q\le 1$ is a parameter quantifying deviation of the distribution from a uniform one; $q=1$ corresponds to a uniform distribution, with $q\to 0$ the distribution collapses to two points. For this construction, the subpopulation size $N_a$ should be an even number. Note that this choice of $\psi_{a,k}$ satisfies constraints (8) and $\sum\psi_{a,k}=0$. The initial values of the oscillator phases $\phi_{a,k}(0)$ are obtained from $\psi_{a,k}$ according to Eq. (A.3).

Now we show that with a special choice of the initial values of the WS variables we can ensure the same initial value of the

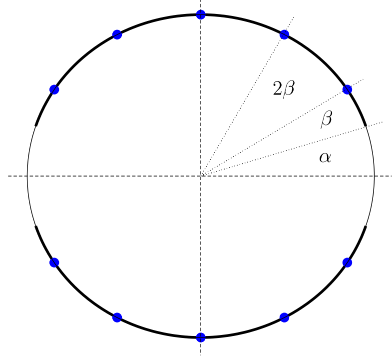


**Fig. C.1.** Illustration of the special choice of the constants of motion $\psi_k$, here for $q = 0.8$ and $N_a = 10$. The points are distributed along two arcs of length $q\pi$ each; angle $\alpha = \frac{\pi}{2}(1-q)$, angle $\beta = \frac{q\pi}{N_a}$.

mean field, independently of the parameter $q$. These special values are $\Phi_a = 0$, $\rho_a = \rho_0$, and $\Psi_a = \frac{2\pi}{M}a$. In order to compute the initial value of the Kuramoto mean field $Y(0) = r_0 e^{i\Theta_0}$ we write the discrete version of Eq. (26) for $t = 0$:

$$
\begin{aligned}
r_0 e^{i\Theta_0} &= \rho_0 \sum_{a=1}^{M} N_a \gamma_a \\
&= \rho_0 \left( \sum_{a=1}^{M} N_a + (1-\rho_0^{-2}) \sum_{a=1}^{M} N_a \sum_{l=2}^{\infty} C_l (-\rho_0)^l e^{-i\frac{2\pi a l}{M}} \right) \\
&= \rho_0,
\end{aligned}
$$

taking into account that all groups are of equal size, $N_a = N_g$. Thus, taking different values of the parameter $q$ and fixing other parameters we obtain the same macroscopic initial conditions (i.e. for the mean field), whereas the initial conditions for individual oscillators are different.

## References


[1] K. Wiesenfeld, J.W. Swift, Averaged equations for Josephson junction series arrays, Phys. Rev. E 51 (2) (1995) 1020–1025.
[2] A.F. Glova, Phase locking of optically coupled lasers, Quantum Electron. 33 (4) (2003) 283–306.
[3] I.Z. Kiss, Y. Zhai, J.L. Hudson, Emerging coherence in a population of chemical oscillators, Science 296 (2002) 1676–1678.
[4] S.H. Strogatz, D.M. Abrams, A. McRobie, B. Eckhardt, E. Ott, Theoretical mechanics: crowd synchrony on the millennium bridge, Nature 438 (2005) 43–44.
[5] B. Eckhardt, E. Ott, S.H. Strogatz, D.M. Abrams, A. McRobie, Modeling walker synchronization on the millennium bridge, Phys. Rev. E 75 (2007) 021110.
[6] Z. Néda, E. Ravasz, Y. Brechet, T. Vicsek, A.-L. Barabási, Tumultuous applause can transform itself into waves of synchronized clapping, Nature 403 (6772) (2000) 849–850.
[7] P. Richard, B.M. Bakker, B. Teusink, K. Van Dam, H.V. Westerhoff, Acetaldehyde mediates the synchronization of sustained glycolytic oscillations in population of yeast cells, Eur. J. Biochem. 235 (1996) 238–241.
[8] S. Dano, P.G. Sorensen, F. Hynne, Sustained oscillations in living cells, Nature 402 (6759) (1999) 320–322.
[9] D. Gonze, N. Markadieu, A. Goldbeter, Selection of in-phase or out-of-phase synchronization in a model based on global coupling of cells undergoing metabolic oscillations, Chaos 18 (3) (2008) 037127.
[10] D. Golomb, D. Hansel, G. Mato, Mechanisms of synchrony of neural activity in large networks, in: F. Moss, S. Gielen (Eds.), Neuro-Informatics and Neural Modeling, in: Handbook of Biological Physics, vol. 4, Elsevier, Amsterdam, 2001, pp. 887–968.
[11] H. Sakaguchi, Cooperative phenomena in coupled oscillator systems under external fields, Progr. Theoret. Phys. 79 (1) (1988) 39–46.
[12] T.M. Antonsen Jr., R.T. Faghih, M. Girvan, E. Ott, J. Platig, External periodic driving of large systems of globally coupled phase oscillators, Chaos 18 (3) (2008) 037112.
[13] P.A. Tass, Phase Resetting in Medicine and Biology. Stochastic Modelling and Data Analysis, Springer-Verlag, Berlin, 1999.
[14] M. Rosenblum, A. Pikovsky, Controlling synchronization in an ensemble of globally coupled oscillators, Phys. Rev. Lett. 92 (11) (2004) 114102.
[15] M. Rosenblum, A. Pikovsky, Delayed feedback control of collective synchrony: an approach to suppression of pathological brain rhythms, Phys. Rev. E 70 (2004) 041904.
[16] L.M. Childs, S.H. Strogatz, Stability diagram for the forced Kuramoto model, Chaos 18 (4) (2008) 043128.
[17] E.A. Martens, E. Barreto, S.H. Strogatz, E. Ott, P. So, T.M. Antonsen, Exact results for the Kuramoto model with a bimodal frequency distribution, Phys. Rev. E 79 (2) (2009) 026204.
[18] D.M. Abrams, R. Mirollo, S.H. Strogatz, D.A. Wiley, Solvable model for chimera states of coupled oscillators, Phys. Rev. Lett. 101 (2008) 084103.
[19] O.E. Omel'chenko, Yu.L. Maistrenko, P.A. Tass, Chimera states: the natural link between coherence and incoherence, Phys. Rev. Lett. 100 (4) (2008) 044105.
[20] C.R. Laing, The dynamics of chimera states in heterogeneous Kuramoto networks, Physica D 238 (16) (2009) 1569–1588.
[21] M. Rosenblum, A. Pikovsky, Self-organized quasiperiodicity in oscillator ensembles with global nonlinear coupling, Phys. Rev. Lett. 98 (2007) 064101.
[22] A. Pikovsky, M. Rosenblum, Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators, Physica D 238 (1) (2009) 27–37.
[23] E. Ott, Th.M. Antonsen, Low dimensional behavior of large systems of globally coupled oscillators, Chaos 18 (3) (2008) 037113.
[24] E. Ott, Th.M. Antonsen, Long time evolution of phase oscillator systems, Chaos 19 (2) (2009) 023117.
[25] D. Paźo, Thermodynamic limit of the first-order phase transition in the Kuramoto model, Phys. Rev. E 72 (4, Part 2) (2005) 046211.
[26] D. Pazó, E. Montbrió, Existence of hysteresis in the Kuramoto model with bimodal frequency distributions, Phys. Rev. E 80 (4) (2009) 046215.
[27] H. Kori, Y. Kuramoto, Slow switching in globally coupled oscillators: robustness and occurrence through delayed coupling, Phys. Rev. E 63 (2001) 046214.
[28] Z. Liu, Y.-C. Lai, F.C. Hoppensteadt, Phase clustering and transition to phase synchronization in a large number of coupled nonlinear oscillators, Phys. Rev. E 63 (5) (2001) 055201.
[29] Yu. Maistrenko, O. Popovych, O. Burylko, P.A. Tass, Mechanism of desynchronization in the finite-dimensional Kuramoto model, Phys. Rev. Lett. 93 (8) (2004) 084102.
[30] Y. Kuramoto, Self-entrainment of a population of coupled nonlinear oscillators, in: H. Araki (Ed.), International Symposium on Mathematical Problems in Theoretical Physics, in: Lecture Notes Phys., vol. 39, Springer, New York, 1975, p. 420.
[31] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence, Springer, Berlin, 1984.
[32] H. Daido, Order function and macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators, Progr. Theoret. Phys. 88 (6) (1992) 1213–1218.
[33] H. Daido, Critical conditions of macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators, Progr. Theoret. Phys. 89 (4) (1993) 929–934.
[34] H. Daido, Onset of cooperative entrainment in limit-cycle oscillators with uniform all-to-all interactions: bifurcation of the order function, Physica D 91 (1996) 24–66.
[35] H. Sakaguchi, Y. Kuramoto, A soluble active rotator model showing phase transition via mutual entrainment, Progr. Theoret. Phys. 76 (3) (1986) 576–581.
[36] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization. A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001.
[37] J.A. Acebron, L.L. Bonilla, C.J. Perez Vicente, F. Ritort, R. Spigler, The Kuramoto model: a simple paradigm for synchronization phenomena, Rev. Modern Phys. 77 (1) (2005) 137–175.
[38] S.H. Strogatz, From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators, Physica D 143 (1–4) (2000) 1–20.
[39] G. Filatrella, N.F. Pedersen, K. Wiesenfeld, Generalized coupling in the Kuramoto model, Phys. Rev. E 75 (2007) 017201.
[40] F. Giannuzzi, D. Marinazzo, G. Nardulli, M. Pellicoro, S. Stramaglia, Phase diagram of a generalized winfree model, Phys. Rev. E 75 (2007) 051104.
[41] S. Watanabe, S.H. Strogatz, Integrability of a globally coupled oscillator array, Phys. Rev. Lett. 70 (16) (1993) 2391–2394.
[42] S. Watanabe, S.H. Strogatz, Constants of motion for superconducting Josephson arrays, Physica D 74 (1994) 197–253.
[43] A. Pikovsky, M. Rosenblum, Partially integrable dynamics of hierarchical populations of coupled oscillators, Phys. Rev. Lett. 101 (2008) 2264103.
[44] S.A. Marvel, R.E. Mirollo, S.H. Strogatz, Phase oscillators with global sinusoidal coupling evolve by Mobius group action, Chaos 19 (2009) 043104.
[45] W.S. Lee, E. Ott, T.M. Antonsen, Large coupled oscillator systems with heterogeneous interaction delays, Phys. Rev. Lett. 103 (4) (2009) 044101.
[46] M.M. Abdulrehem, E. Ott, Low dimensional description of pedestrian-induced oscillation of the millennium bridge, Chaos 19 (2009) 013129.
[47] H. Daido, Multi-branch entrainment and multi-peaked order-functions in a phase model of limit-cycle oscillators with uniform all-to-all coupling, J. Phys. A: Math. Gen. 28 (1995) L151–L157.
[48] S.A. Marvel, S.H. Strogatz, Invariant submanifold for series arrays of Josephson junctions, Chaos 19 (2009) 013132.
[49] Y. Baibolatov, M. Rosenblum, Z.Zh. Zhanabaev, A. Pikovsky, Complex dynamics of an oscillator ensemble with uniformly distributed natural frequencies and global nonlinear coupling, Phys. Rev. E 82 (1) (2010) 016212.
[50] E. Ott, J.H. Platig, T.M. Antonsen, M. Girvan, Echo phenomena in large systems of coupled oscillators, Chaos 18 (3) (2008) 037115.
[51] Y. Kuramoto, D. Battogtokh, Coexistence of coherence and incoherence in nonlocally coupled phase oscillators, Nonlinear Phenom. Complex Syst. 5 (2002) 380–385.
[52] D.M. Abrams, S.H. Strogatz, Chimera states for coupled oscillators, Phys. Rev. Lett. 93 (17) (2004) 174102.
[53] G. Bordyugov, A. Pikovsky, M. Rosenblum, Self-emerging and turbulent chimeras in oscillator chains, Phys. Rev. E 82 (3) (2010) 035205.