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Effects of nonresonant interaction in ensembles of phase oscillators

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We consider general properties of groups of interacting oscillators, for which the natural frequencies are not in resonance. Such groups interact via nonoscillating collective variables like the amplitudes of the order parameters defined for each group. We treat the phase dynamics of the groups using the Ott-Antonsen ansatz and reduce it to a system of coupled equations for the order parameters. We describe different regimes of cosynchrony in the groups. For a large number of groups, heteroclinic cycles, corresponding to a sequential synchronous activity of groups and chaotic states where the order parameters oscillate irregularly, are possible.

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I. INTRODUCTION

Models of coupled limit cycle oscillators are widely used to describe synchronization phenomena in various branches of science. The applications include physical systems like Josephson junctions [1], lasers [2], and electrochemical oscillators [3], but similar models are also used for neuronal ensembles [4], the dynamics of pedestrians on bridges [5,6], applauding persons [7], and so forth.

In many situations, a model of a fully connected (globally coupled) network is appropriate. In this case, each oscillator is driven by a mean field produced by all oscillators in the network. Ensembles of weakly interacting self-sustained oscillators are successfully handled in the framework of phase approximation [8–12]. Most popular are the Kuramoto model of sine-coupled phase oscillators, and its extension, the Kuramoto-Sakaguchi model [13]. This model describes self-synchronization and appearance of a collective mode (mean field) in an ensemble of generally nonidentical elements as a nonequilibrium phase transition. The basic assumptions behind the Kuramoto model are that of weak coupling (which allows one to consider pairwise interactions only) and of closeness of frequencies of oscillators; the latter results in the presence of resonant terms (depending on the phase differences) in the coupling function only (contrary to a general case where the coupling is a general function of two phases). References to detailed aspects of the Kuramoto model can be found in [14–16].

In many cases, the ensembles of oscillators are not homogeneous and can be considered as consisting of several subensembles (e.g., in the brain different groups of neurons can have different characteristic rhythms). If one still assumes that the frequencies of these subgroups are close (compared to the coupling), then a model of several resonantly interacting subpopulations [17–24] or of an ensemble having a bimodal (or a multimodal) distribution of frequencies [25–31] is adopted. Similarly, one can also model two ensembles, one of which consists of active and another of passive elements, which are coupled resonantly due to the closeness of their frequencies [32,33].

In this paper, we study the novel situation of *nonresonantly* coupled oscillator ensembles. We assume that there are several groups of oscillators and that the frequencies within each group are close to each other but are strongly different (compared to the coupling strength) between the groups. In this situation, the coupling within the group is resonant, like in usual Kuramoto-type models, but the coupling between the groups can be only nonresonant.[1] It means that the coupling can be via nonoscillating, slow variables only, that is, via the amplitudes of the mean fields. In the context of a single Kuramoto model, such a dependence of the parameters of the model on the amplitude of the mean field has been recently studied in [34–37]. Generally, both the amplitude and the phase of the coupling constant may depend on the mean field amplitude; in [34,35], such a dependence was termed *nonlinearity of coupling*. Nonlinearity in this context means that the effect of the collective mode on an individual unit depends on the amplitude of this mode, so that, for example, the interaction of the field and a unit can be attractive for a weak field and repulsive for a strong one. Mathematically, this is represented by the dependence of the parameters of the Kuramoto-Sakaguchi model (the coupling strength, the effective frequency spreading, and the phase shift) on the mean field amplitude. Here we generalize this approach to several ensembles, so that the parameters of the Kuramoto-Sakaguchi model describing each subgroup depend on the mean field amplitudes of other subgroups (e.g., resonant interactions within a group of oscillators can be attractive or repulsive depending on the amplitude of the order parameter of another group).

A general derivation of such a coupling appears in Appendix A; we outline here a simple physical situation where such an interaction appears. A well-known example of a physical system that can be described by the Kuramoto-Sakaguchi model is an array of Josephson junctions with a common inductor-capacitor-resistor (LCR) load [1]. In this setup, a linear LCR circuit is fed by the sum of voltages of the junctions, and each junction is driven by the current of the LCR circuit. It is quite natural to consider a *nonlinear* circuit as a

[1] Another novel type of interaction appears if the frequencies of two groups are in a high-order resonance like 2:1; see [60].

common load. Then the relation between the force that drives each junction and the mean field (sum of voltages from the junctions) will be nonlinear—there will be terms proportional to the mean field, to the square of the mean field, to the third power of the mean field, and so forth. In the language of individual interactions between the junctions, this will produce two-phase coupling terms, three-phase coupling terms, and so on. Next, if there are two groups of Josephson junctions having different natural frequencies, their interaction via a common nonlinear load will belong to the class we consider in this paper.

In Sec. II, we introduce the basic model of nonresonantly interacting ensembles. We also formulate the equations for the mean fields of the ensembles following the Ott-Antonsen theory [38,39]. The simplest situation of two interacting ensembles is studied in Sec. III. In Sec. IV, we describe three and several interacting ensembles, focusing on nontrivial regimes of sequential synchronous activity following a heteroclinic cycle and on chaotic dynamics.

II. BASIC MODEL OF NONRESONANTLY INTERACTING OSCILLATOR ENSEMBLES

A. Kuramoto-Sakaguchi model and Ott-Antonsen equations for its dynamics

A popular model describing *resonant* interactions in an ensemble of oscillators having close frequencies is due to Kuramoto and Sakaguchi [13]:

$$\dot\phi_k = \omega_k + \mathrm{Im}(KZe^{-i\phi_k}),\quad Z=\frac{1}{N}\sum_{l=1}^{N} e^{i\phi_l},\qquad k=1,\ldots,N. \tag{1}$$

Here ϕ_k is the oscillator's phase, Z is the complex order parameter (mean field) that also serves as a measure for synchrony in the ensemble, ω_k are natural frequencies of oscillators, and $K=2a+2ib$ is a complex coupling constant. The argument of this constant gives the phase shift in the coupling; for $b=0$ the model reduces to the standard Kuramoto one.

Recently, Ott and Antonsen [38,39] have demonstrated that in the thermodynamic limit $N\to\infty$, and asymptotically for large times the evolution of the order parameter Z in the case of a Cauchy (Lorentzian) distribution of natural frequencies $g(\omega)=\Delta[\pi((\omega-\omega_0)^2+\Delta^2)]^{-1}$ around the central frequency ω_0, is governed by a simple ordinary differential equation:

$$\dot Z = (i\omega_0-\Delta)Z + \tfrac{1}{2}(K-K^*|Z|^2)Z. \tag{2}$$

Written for the amplitude and the phase of the order parameter defined according to $Z=\rho e^{i\Phi}$, the Ott-Antonsen equations

$$\dot\rho = -\Delta\rho + a(1-\rho^2)\rho, \tag{3}$$

$$\dot\Phi = \omega_0 + 2b\rho^2, \tag{4}$$

are easy to study: Eq. (3) defines the stationary amplitude of the mean field (which is nonzero above the synchronization threshold $a_c=\Delta$), while Eq. (4) yields the frequency of the mean field.

The Ott-Antonsen approach has been successfully applied to a description of coupled populations of oscillators with resonant couplings [19,24,35,40]; here we generalize these results to the case of nonresonant coupling.

B. Nonresonantly interacting ensembles

We consider several ensembles of oscillators, each characterized by its own parameters ω_0,Δ,a,b. The main assumption is that the central frequencies ω_0 of different populations are not close to each other, and also high-order resonances between them are not present. Such a situation appears typical for neural ensembles, where different areas of brain demonstrate oscillations in a very broad range of frequencies, from α to γ rhythms. Because there is no resonant interaction between the oscillators in different ensembles, they can interact only nonresonantly, via the absolute values of the mean fields. For a more detailed derivation, we refer to Appendix A; here we present only the qualitative arguments yielding the form of the basic equations. Assuming that only Kuramoto order parameters (1) (but not higher order Daido order parameters $Z_m=\langle e^{im\phi}\rangle$) enter the coupling, a general nonresonant interaction between populations can be described by the dependencies of the parameters ω_0,Δ,a,b on the amplitudes of the mean fields ρ_l, where index l counts the subpopulations. Moreover, one can see from (3) and (4) that the equation for the amplitude is independent on the phase; therefore, we can restrict our attention to the amplitude dynamics (3). This means that the frequencies of the mean field are only slightly influenced by interactions, and thus a synchronization of mean fields with each other (in the sense that frequencies of different subpopulations become locked) is impossible. Furthermore, we assume the coupling to be weak, so only the leading order corrections $\sim\rho^2$ are included. All this leads to the following general model for interacting populations:

$$\dot\rho_l = \left(-\Delta_l - \Gamma_{lm}\rho_m^2\right)\rho_l + \left(a_l + A_{lm}\rho_m^2\right)\left(1-\rho_l^2\right)\rho_l,\qquad l,m=1,\ldots,L, \tag{5}$$

with coupling constants Γ_{lm},A_{lm}. Here index l denotes a subpopulation of oscillators and Γ_{lm},A_{lm} describe the action of subpopulation m on subpopulation l. As is demonstrated in Appendix A [cf. Eq. (A7)], the microscopic equations corresponding to this model read

$$\dot\phi_k^{(l)} = \omega_k^{(l)} + \eta_k\Gamma_{lm}\rho_m^2 + \mathrm{Im}\left[2\left(a_l + A_{lm}\rho_m^2\right)Z^{(l)}\exp\left(-i\phi_k^{(l)}\right)\right], \tag{6}$$

where $\omega_k^{(l)}$ are Cauchy distributed with mean $\omega^{(l)}$ and width Δ_l, and η_k are Cauchy distributed with zero mean and unit width. In general, different kinds of configurations can be treated on this level; for example, coupling may be symmetric or directional (l influences m but m does not influence l), subpopulations may form a "ring" with coupling $1\to2\to3\to\cdots\to L\to 1$, and so forth. Below we give a rather general analysis for two subpopulations describing different possible regimes in dependence on the coupling configurations, but for $L=3,4$ we do not study all possible configurations—instead we exemplify interesting regimes of heteroclinic synchrony cycles and chaos with properly chosen examples. Note that because the widths of the frequencies distribution cannot be negative, coefficients Γ_{lm} must satisfy $\Delta_l+\Gamma_{lm}\geqslant 0$. Below we assume that there is no nonlinearity inside ensembles $\Gamma_{ll}=A_{ll}=0$.

In this paper, we do not investigate model (5) in its full generality, as it would require a rather tedious analysis. Instead, we consider two simpler models, which describe particular types of interaction but nevertheless allow us to demonstrate interesting dynamical patterns. In model A, we assume that only frequencies are influenced by the coupling, that is, $A_{lm} = 0$. This leads to a system

$$\dot{\rho}_l = \left(a_l - \Delta_l - \Gamma_{lm}\rho_m^2 - a_l\rho_l^2\right)\rho_l. \tag{7}$$

Model B takes into account the interaction via coupling constants only (i.e., $\Gamma_{lm} = 0$); additionally we assume here that the distributions of frequencies in all interacting ensembles are narrow $\Delta_l \to 0$. In the limit of identical oscillators, we obtain from (5)

$$\dot{\rho}_l = \left(a_l + A_{lm}\rho_m^2\right)\left(1 - \rho_l^2\right)\rho_l. \tag{8}$$

Here we note that the Ott-Antonsen equations for the ensemble of identical oscillators describe not a general case but a particular solution, while a general description is delivered by the Watanabe-Strogatz theory [41,42]. Thus the dynamics of model B should be considered as a special singular limit $\Delta \to 0$.

In Secs. III and IV, we describe the dynamics of these two models, for the cases of two (Sec. III) and three and more interacting ensembles (Sec. IV).

III. TWO INTERACTING ENSEMBLES

Let us first rewrite models (7) and (8) for the simplest case of only two interacting ensembles. Additionally, for model A we assume $a_l = 1$ (equivalently, one could renormalize the amplitudes of order parameters $\rho_{1,2}$ to get rid of these coefficients). Thus the model A reads

$$\begin{aligned}\dot{\rho}_1 &= \rho_1\left(\delta_1 - d_{12}\rho_2^2 - \rho_1^2\right),\\ \dot{\rho}_2 &= \rho_2\left(\delta_2 - d_{21}\rho_1^2 - \rho_2^2\right).\end{aligned} \tag{9}$$

For model B, a normalization of amplitudes is not possible, and it reads

$$\begin{aligned}\dot{\rho}_1 &= \varepsilon_1\rho_1\left(1 - D_{12}\rho_2^2\right)\left(1 - \rho_1^2\right),\\ \dot{\rho}_2 &= \varepsilon_2\rho_1\left(1 - D_{21}\rho_1^2\right)\left(1 - \rho_2^2\right).\end{aligned} \tag{10}$$

Generally, parameters $\delta_{1,2} = 1 - \Delta_{1,2}$, $d_{ik} = \Gamma_{ik}$, $\varepsilon_{1,2} = a_{1,2}$, and $D_{ik} = -A_{ik}/a_i$ can have different signs.

As the first property of both models, we mention that the dynamics is restricted to the domain $0 \leqslant \rho_{1,2} \leqslant 1$. Formally, this follows directly from (5), and physically this corresponds to the admissible range of values of the order parameter. Furthermore, for model A (9) we can apply the Bendixon-Dulac criterion

$$\frac{\partial}{\partial\rho_1}\left(\frac{1}{\rho_1\rho_2}\dot{\rho}_1\right) + \frac{\partial}{\partial\rho_2}\left(\frac{1}{\rho_1\rho_2}\dot{\rho}_2\right) = -2\frac{\rho_1^2 + \rho_2^2}{\rho_1\rho_2} < 0,$$

from which it follows that it cannot possess periodic orbits.

Remarkably, model B (10) can be written as a Hamiltonian one. With an ansatz

$$\exp y_{1,2} = \rho_{1,2}^2\left(1 - \rho_{1,2}^2\right)^{-1}, \tag{11}$$

it can be represented in a Hamiltonian form

$$\dot{y}_1 = \frac{\partial H(y_1,y_2)}{\partial y_2}, \quad \dot{y}_2 = -\frac{\partial H(y_1,y_2)}{\partial y_1},$$
$$H = 2\varepsilon_1 y_2 - 2\varepsilon_2 y_1 - 2\varepsilon_1 D_{12}\ln(1 + e^{y_2}) + 2\varepsilon_2 D_{21}\ln(1 + e^{y_1}). \tag{12}$$

Thus model B may demonstrate a family of periodic orbits if the levels of the Hamiltonian are closed curves. We stress that the Hamiltonian structure of the model does not exclude existence of stable equilibria at $\rho = 0,1$ because the transformation (11) is singular at these states; in the Hamiltonian formulation (12) these stable equilibria correspond to trajectories moving toward $\mp\infty$.

The dynamics of both models is mainly determined by the existence and stability of equilibria. For model A (9), possible equilibria are the trivial one, $S_1(0,0)$; two states where one of the order parameters vanish, $S_2(\delta_1^{1/2},0)$ and $S_3(0,\delta_2^{1/2})$; and a state where both order parameters are nonzero, $S_4[(\delta_1 - d_{12}\delta_1)(1 - d_{12}d_{21})^{-1}, (\delta_2 - d_{21}\delta_1)(1 - d_{12}d_{21})^{-1}]$. Similarly, model B (10) always has equilibria $M_1(0,0)$, $M_2(1,0)$, $M_3(0,1)$, and $M_4(1,1)$, and additionally a nontrivial state $M_5(D_{21}^{-1/2}, D_{12}^{-1/2})$ exists if $D_{12}, D_{21} > 1$.

We illustrate possible types of dynamics (up to symmetry $1 \leftrightarrow 2$) in models A and B in Figs. 1 and 2. Here it is worth mentioning that model (9) is structurally of the same type as typical models of interacting populations in mathematical ecology [43]. Model B (10) resembles them as well but has a distinctive property that fully synchronized cluster state $\rho = 1$ is invariant. Referring for the details to Appendix B, we describe briefly possible regimes in these models.

(1) Stability of only of the trivial equilibrium point $S_1(0,0)$, $M_1(0,0)$ [Figs. 1(a), 2(a), and 2(b)]. A fully asynchronous state is stable in both ensembles.

(2) Stability of a nontrivial state off coordinate axes S_4 and M_4 [Figs. 1(b), 1(c), 2(c), and 2(d)]. Here both ensembles are synchronized (in model A not completely because of a distribution of frequencies, in model B completely because we assume identical oscillators in ensembles).

(3) Competition between ensembles [Figs. 1(d) and 2(e)]. Only one ensemble synchronizes while the other one desynchronizes. Which ensemble is synchronous depends on initial conditions.

(4) Suppression. One ensemble always "wins" and is synchronous while the other one desynchronizes (steady states S_2,M_2 are global attractors; of course, stability of "symmetric" states S_3,M_3 is also possible) [Figs. 1(e), 1(f), and 2(f)–2(h)].

(5) Bistability of the trivial and the fully synchronous states of both ensembles [Fig. 2(i)]. This is possible in model B only.

(6) Periodic behavior [Fig. 2(j)]. This is possible only in ensemble B; it corresponds to an interaction of populations of "predator-prey" type. Because the system is Hamiltonian, the oscillations are conservative like in the Lotka-Volterra system.

While in our analysis we studied models (9) and (10) describing dynamics of the order parameters in the Ott-Antonsen ansatz, all the regimes described above can be observed when one simulates original equations of the ensembles of phase oscillators (1) at sufficiently large number of units N. In Fig. 3, we illustrate two nontrivial regimes of two subpopulations of phase oscillators at $N = 10^3$. Figure 3(a) shows the

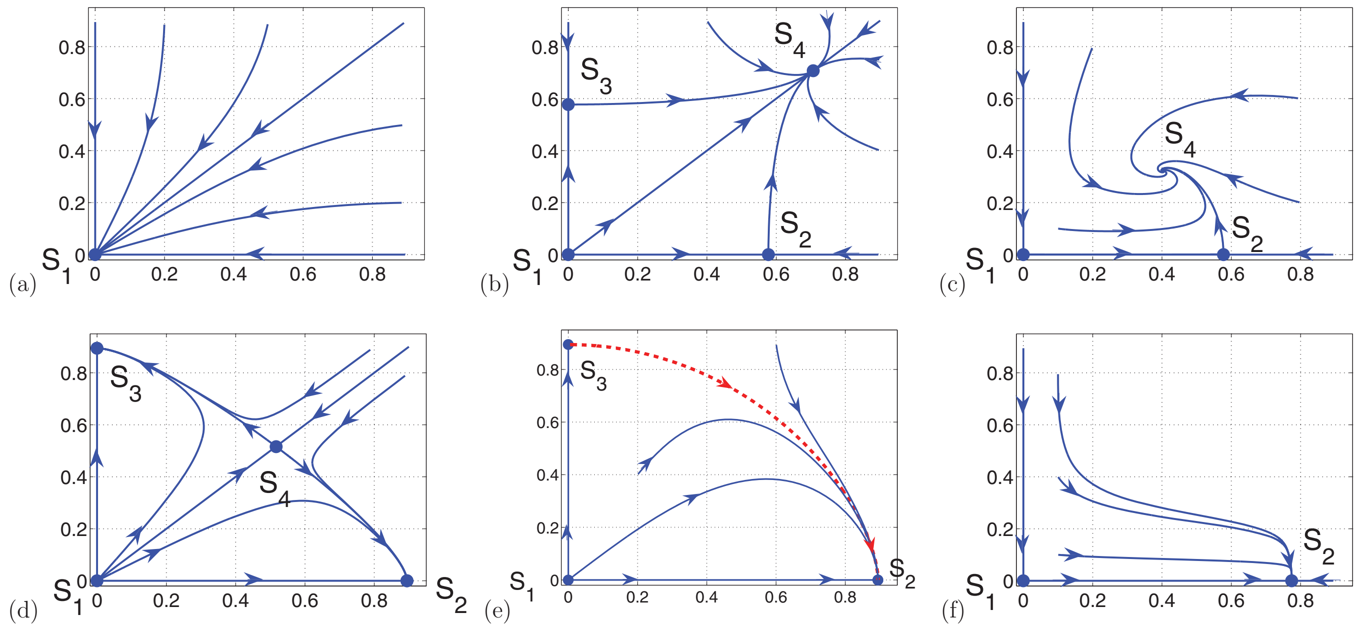

FIG. 1. (Color online) Six different patterns of the dynamics of system (9). (a) Global stability of a trivial state (for $\delta_{1,2} < 0$). (b), (c) Stability of S_4 when both populations are partially synchronous [conditions for this are (B2) for (b) or (B3) for (c)]. (d) Competition between clusters if the coupling is strongly suppressive (B6); here we have bistablity of states $S_{2,3}$ describing synchronous activity of one cluster and asynchronous of another one. (e) Asymmetric interaction between clusters arises under condition (B9); here always a heteroclinic trajectory from saddle point S_3 to stable node S_2 exists (red dashed line). (f) Stability of S_2 under condition (B8).

dynamics of mean fields in the case of a competition between two subpopulations that interact via frequency mismatch modulation; see Fig. 1(d). Figure 3(b) illustrates a periodic behavior of two subpopulations like in Fig. 2(j).

IV. THREE AND MORE INTERACTING ENSEMBLES

In this section, we generalize the results of Sec. III to many interacting ensembles. We do not aim here at the full generality but rather present interesting regimes based on the elementary dynamics depicted in Figs. 1 and 2. According to the consideration above, we restrict our attention to two basic models: A (7) and B (8). Generally, model B cannot be rewritten in a Hamiltonian form, but by applying transformation (11) one can easily see that this system has a Liouvillian property—the phase volume is conserved.

A. Symmetric case: Cosynchrony and competition

Here we describe mostly simple regimes that are observed in a symmetric case of all-to-all coupling, where parameters of all ensembles and their interaction are equal. This corresponds to equal values $a_l = a, \Delta_l = \Delta$, and $\Gamma_{lm} = \Gamma$ in (7) and $a_l = a$ and $A_{lm} = A$ in (8). In model A, the only nontrivial regimes are those where asynchronous states are unstable, $\Delta < a$. Then one observes either a coexistence of synchrony like in Fig. 1(b) (for $\Gamma < a$) or a competition like in Fig. 1(d) (for $\Gamma > a$). In the latter case, only one ensemble is synchronous, while others desynchronize. Similar regimes can be observed in model B for $a > 0$, $A < -a$. Additionally, in model B a coexistence of full synchrony in all ensembles and a full asynchrony, like in Fig. 2(i), can be observed for $a < 0$, $A > -\frac{a}{L-1}$. We illustrate the regimes of competition in Fig. 4 for the case of three interacting populations.

B. Heteroclinic synchrony cycle

Here we discuss a multidimensional generalization of the interactions where one group in a pair of ensembles always synchronizes while another one is asynchronous [see Figs. 1(e) and 2(f)]. In the examples presented in these graphs, both ensembles would self-synchronize separately, but due to interaction, synchrony in ensemble 2 disappears while ensemble 1 remains synchronous. One can say that in synchrony competition between the first and the second ensembles, the first ensemble wins. Suppose now that a third self-synchronizing ensemble is added, which wins in the competition with the first one but loses in the competition to the second one. Then a cycle $2 \to 1 \to 3 \to 2 \to 1 \to 3 \ldots$ will be observed. Moreover, because in the dynamics [Figs. 1(e) and 2(f)] the transition $2 \to 1$ follows the heteroclinic orbit connecting steady states S_3 and S_2, the cycle in the system of three ensembles will be a heteroclinic one, with asymptotically infinite period. Such a cycle has been studied in different contexts [44–46]. For a review of robust heteroclinic cycles, see [47,48] (sometimes one uses the term *winnerless competition* to describe such a dynamics [49,50]).

We demonstrate the heteroclinic synchrony cycle for three interacting ensembles in Fig. 5. One can see that synchronous states of ensembles appear for longer and longer time intervals. It is interesting to note that heteroclinic cycles have been observed in ensembles of *identical* coupled oscillators [51–56]. There the nontrivial dynamics is in the switchings of full synchrony between different clusters. In this respect, the heteroclinic cycle in model B resembles such a regime. On the other hand, the heteroclinic cycle in model A is different: Here the natural frequencies of oscillators are

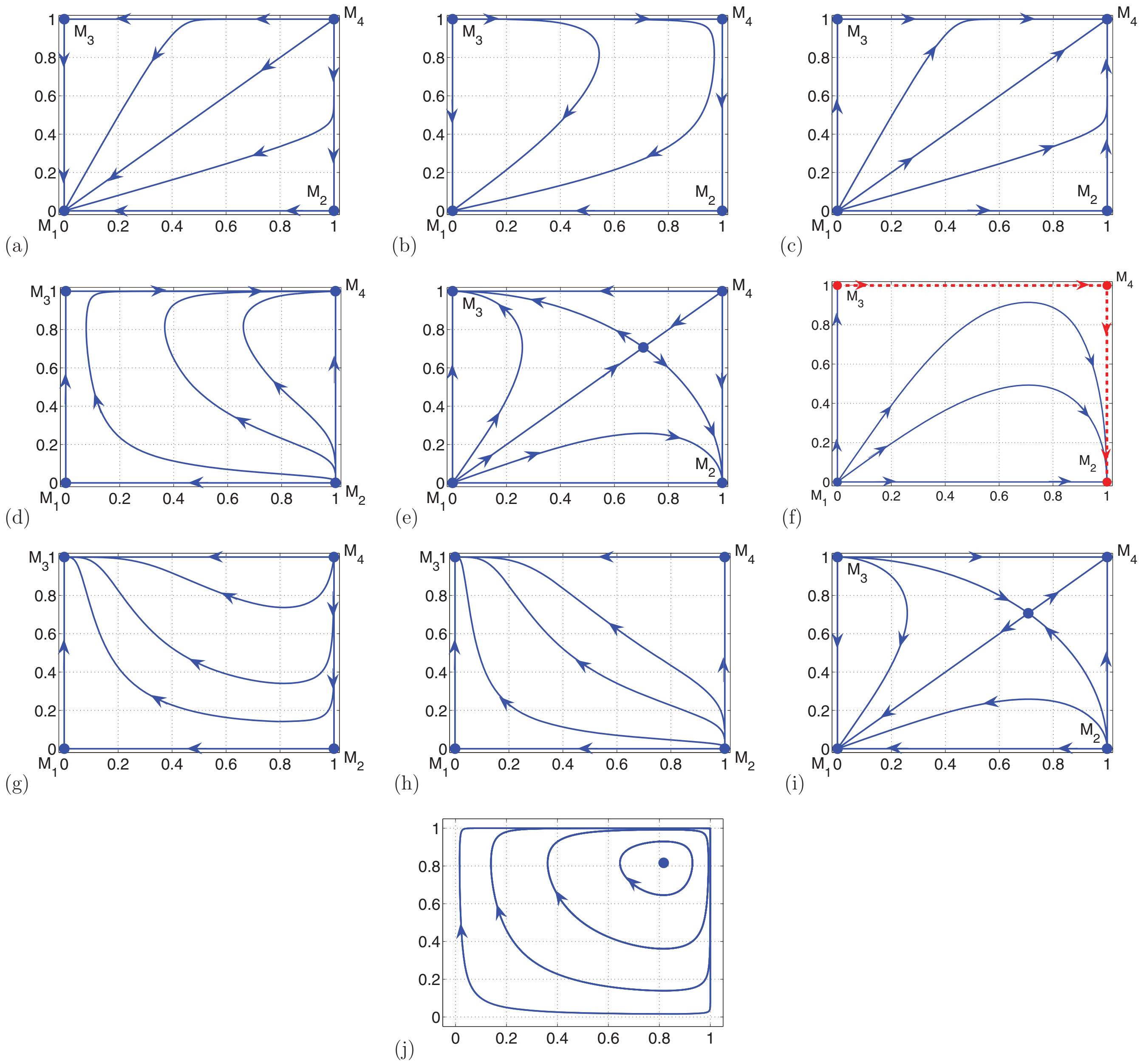

FIG. 2. (Color online) Ten different dynamical regimes in system (10). (a), (b) Stability of the trivial state arises at conditions (B1). Case (a): $D_{12} < 1$; case (b): $D_{12} > 1$. (c), (d) Stability of $M_4(1,1)$ when both clusters are in the synchronized state, under condition of a weak suppressive coupling, (B4) for (c) or (B5) for (d). (e) Competition between clusters arises at strong suppressive coupling (B7). Here we have bistability of steady states $M_{2,3}$; each of these points corresponds to synchronous activity of one cluster and asynchronous activity of another one. (f) Asymmetric interaction between clusters at asymmetric coupling (B11). Here, a sequence of heteroclinic trajectories $M_3 \to M_4 \to M_2$ (red dashed lines) is always present. (g), (h) The situation of stability of fixed point M_3 while conditions (B10) are satisfied [case (g): $D_{21} > 1$, case (h): $D_{21} < 1$]. (i) Bistability of fully asynchronous and fully synchronous states arises if (B12) is valid. In this case, stable manifolds of the saddle point M_5 divide basins of attraction of stable points M_1, M_4. (j) The case of periodic behavior arises at conditions (B13).

different and the states of synchrony are not complete, so the identical clusters never appear.

Finite size effects are nontrivial for the heteroclinic cycles described. Indeed, it is known that while in the thermodynamic limit deterministic equations for the order parameters can be used, finite size effects can be modeled via noisy terms that scale roughly as $\sim N^{-1/2}$ [57–59]. On the other hand, noisy terms destroy perfect heteroclinic orbit, making the transition times between the states finite and irregular. Exactly this is observed at modeling the interacting finite size ensembles (Fig. 6). While for small N the heteroclinic cycle is completely destroyed, for large N it looks like a noisy limit cycle.

C. Chaotic oscillations

Here we discuss possible "predator-prey"-type regimes [cf. Fig. 2(j)] for many ensembles. An elementary "oscillator," depicted in Fig. 2(j), can be represented as a Hamiltonian

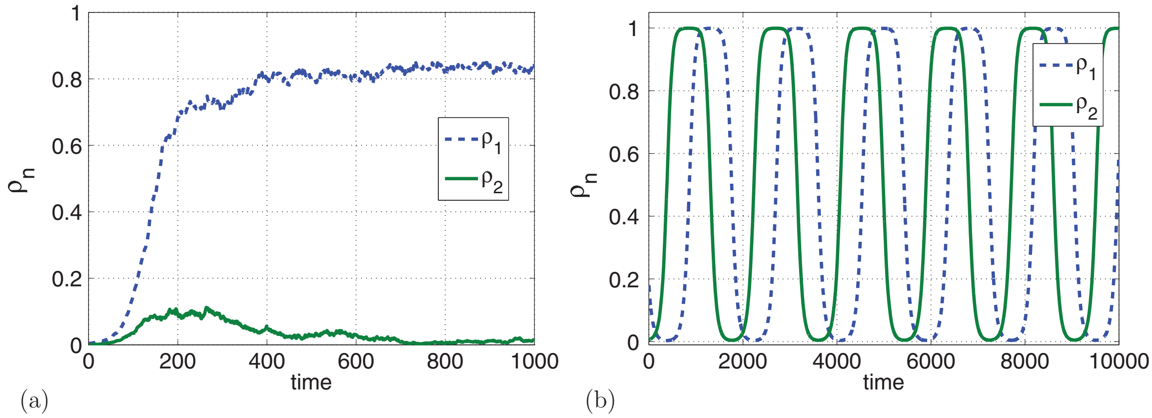

FIG. 3. (Color online) Modeling of ensemble consisting of two subpopulations of $N = 10^3$ phase oscillators. (a) Subpopulations interact via modulation of effective frequency mismatch (9). Case of competition between subpopulations for parameter values $\Delta_{1,2} = 1$, $a_{1,2} = 5$, and $\Gamma_{12} = \Gamma_{21} = 12$. (b) Subpopulations interact via coupling modulations (10). A periodic regime is presented at parameter values $a_1 = -1$, $a_2 = 1$, and $A_{12} = A_{21} = -2$. To avoid a spurious clustering and to ensure validity of the Ott-Antonsen description, a small mismatch was added: ω_n were randomly distributed in the range $[-0.025, +0.025]$.

system with one degree of freedom. Several of such elementary conservative "oscillators," being coupled, can yield quasiperiodic and chaotic regimes. In the case of two interacting conservative "oscillators" (i.e., of four interacting ensembles), system (8) can be rewritten as follows:

$$\begin{aligned}\dot{\rho}_{1,2} &= \varepsilon_{1,2}\rho_{1,2}\left(1 - D_0\rho_{2,1}^2 - D_1\upsilon_{1,2}^2\right)\left(1 - \rho_{1,2}^2\right),\\ \dot{\upsilon}_{1,2} &= \varepsilon_{1,2}\upsilon_{1,2}\left(1 - D_0\upsilon_{2,1}^2 - D_1\rho_{1,2}^2\right)\left(1 - \upsilon_{1,2}^2\right).\end{aligned} \tag{13}$$

Here the parameters of the system were chosen in such a way that each pair of subpopulations (ρ_1,ρ_2) and (υ_1,υ_2) exhibits periodic oscillation when decoupled from another pair (at $D_1 = 0$), that is, $\varepsilon_1\varepsilon_2 < 0$ and $D_0 > 1$. When the coupling between the two pairs is introduced (i.e., $D_1 \neq 0$), then in dependence on this coupling and initial conditions, the dynamics can be qusiperiodic or chaotic. Like in general Hamiltonian systems with two degrees of freedom, it is convenient to represent the dynamics as a two-dimensional Poincaré map. As a Poincaré section [Fig. 7(a)], we have taken the plane (υ_1,υ_2) at moments of time at which the variable $\rho_1(t)$ has a maximum. At small values of the coupling between the "oscillators" D_1, the dynamics is typically quasiperiodic. While increasing D_1, one can observe a transition to dominance of chaotic regimes in the system (13) [see Figs. 7(a) and 7(b) and calculation of Lyapunov exponents in Fig. 7(c)]. Furthermore,

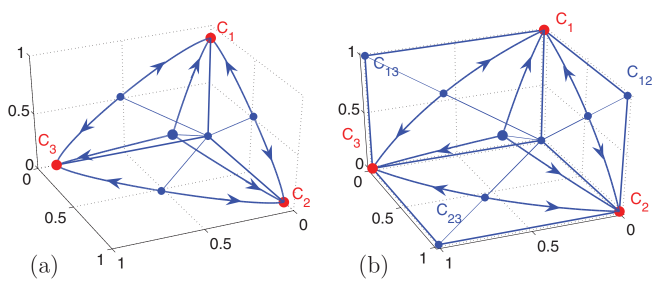

FIG. 4. (Color online) Multistability of steady states C_n corresponding to synchronous state of only one cluster for (a) system (7) ($\Delta < a$, $\Gamma > a$) and (b) system (8) ($a > 0$, $A < -a$).

we have confirmed the existence of chaotic oscillations by direct numerical simulation of four subpopulations satisfying (13), consisting of $N = 10^3$ elements each [Fig. 7(d)].

V. CONCLUSION

In this paper, we have introduced and studied a model of nonresonantly coupled ensembles of oscillators. It is assumed that oscillators form several groups; in each group the natural frequencies are close to each other, but the frequencies of different groups are rather different. This means that only oscillators within each group interact resonantly (i.e., the coupling terms depend on their phases), while interactions between the groups can be only nonresonant (i.e., depending on slow nonoscillating variables only). As a particular realization of such a setup, we considered phase oscillators, which resonantly interact according to the Kuramoto-Sakaguchi model, and the nonresonant terms appear as dependencies of the parameters of the Kuramoto-Sakaguchi model on the amplitudes of the mean fields (Kuramoto order parameters) of other groups.

We employed the Ott-Antonsen theory, allowing us to write a closed system of equations for the amplitudes of the order parameters. Analysis of this system constitutes the main part of the paper. The system resembles the Lotka-Volterra-type equations used in mathematical ecology for the dynamics of populations but has nevertheless some peculiarities. For two coupled ensembles, we demonstrated a variety of possible regimes: coexistence and bistability of synchronous states as well as periodic oscillations. For a larger number of interacting groups, more complex states appear: a stable heteroclinic cycle and a chaotic regime. Heteroclinic cycle means a sequence of synchronous epochs that become longer and longer. In a chaotic regime, the order parameters demonstrate low-dimensional chaos. While the main analysis is performed for the Ott-Antonsen equations that are valid in the thermodynamic limit of a infinite number of oscillators in ensembles, we have checked finite size effects in several regimes by modeling finite ensembles. Finiteness of ensembles

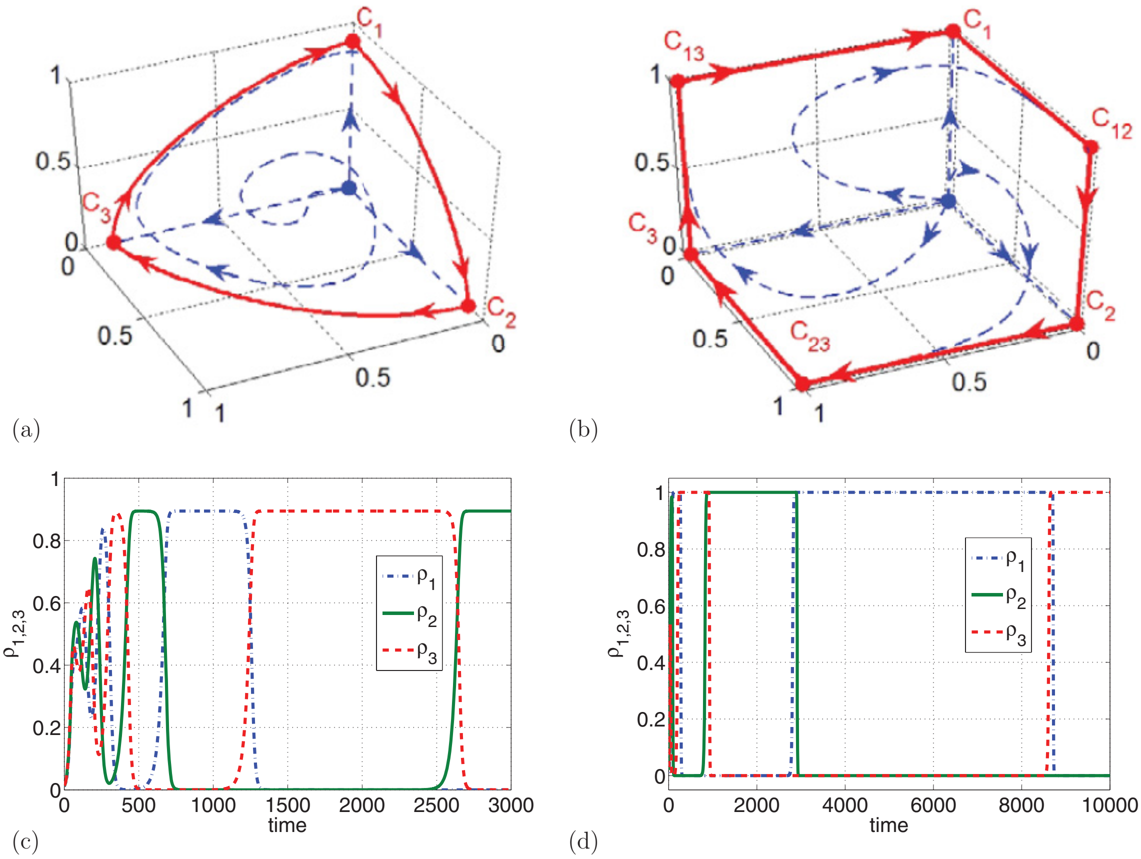

FIG. 5. (Color online) Stable heteroclinic cycles caused by asymmetric interactions between clusters in system (7) [(a) and (c)] and in system (8) [(b) and (d)]. Parameters: (a) $a_l - \Delta_l > 0$, $\Gamma_{12} > \frac{a_2(a_1-\Delta_1)}{a_2-\Delta_2}$, $\Gamma_{31} > \frac{a_1(a_3-\Delta_3)}{a_1-\Delta_1}$, $\Gamma_{23} > \frac{a_3(a_2-\Delta_2)}{a_3-\Delta_3}$, $\Gamma_{21} < \frac{a_1(s_2-\Delta_2)}{a_1-\Delta_1}$, $\Gamma_{13} < \frac{a_3(a_1-\Delta_1)}{a_3-\Delta_3}$, and $\Gamma_{32} < \frac{a_2(a_3-\Delta_3)}{a_2-\Delta_2}$; (b) $a_l > 0$, $A_{12} < -a_1$, $A_{31} < -a_3$, $A_{23} < -a_2$, $A_{21} > -a_2$, $A_{13} > -a_1$, and $A_{32} > -a_3$. Panels (a) and (b) show the phase-space portraits while time series are presented in panels (c) and (d). Parameters: (c) $a_{1,2,3} = 1$, $\Delta_{1,2,3} = 0.2$, $\Gamma_{21,32,13} = 2$, $\Gamma_{12,23,31} = 0.6$ (d) $a_{1,2,3} = 1.0$, $A_{21,32,13} = -3.0$, and $A_{12,23,31} = -0.3$.

only slightly influences the dynamics in most of the observed states, except for the heteroclinic cycle. Here a small effective noise due to finite size effects destroys the cycle, producing nearly periodic noise-induced oscillations.

One of the models we studied was that of groups of identical oscillators. Here in many cases only the states where some groups completely synchronize (i.e., all oscillators form an identical cluster) while others completely desynchronize (order parameter vanish) are possible. A heteroclinic cycle in this model also connects such states. There is, however, a nontrivial set of parameters, at which the order parameters of ensembles oscillate between zero and one, thus demonstrating time-dependent partial synchronization. Moreover, for four ensembles these oscillations are chaotic. This regime is quite interesting for a general theory of collective chaos in oscillator populations (cf. chaotic dynamics of the order parameter in

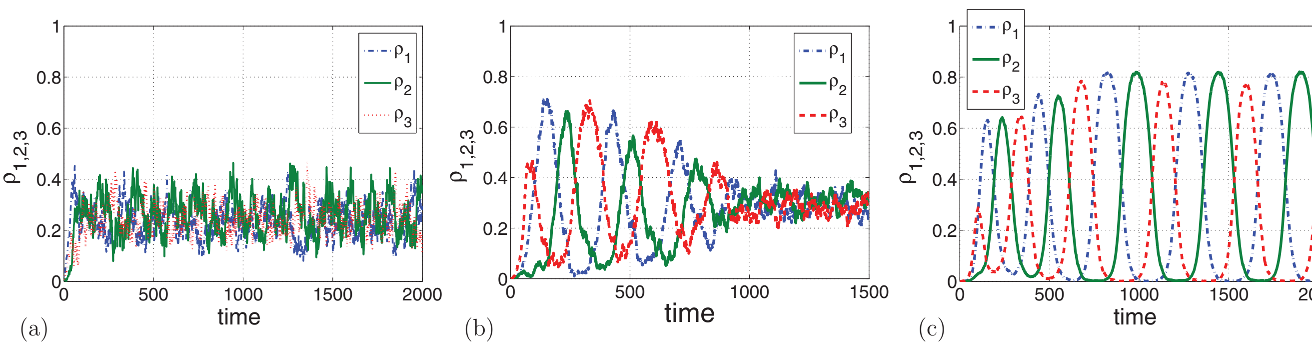

FIG. 6. (Color online) Dynamics of the order parameters of three interacting populations of oscillators for three different sizes of populations: (a) $N = 100$, (b) $N = 400$, and (c) $N = 10\,000$. Parameters: $a_{1,2,3} = 1$, $\Delta_{1,2,3} = 0.2$, $\Gamma_{21,32,13} = 3.0$, and $\Gamma_{12,23,31} = 0.6$ (such a set of parameters produces a heteroclinic cycle in the phase of the system in the thermodynamic limit).

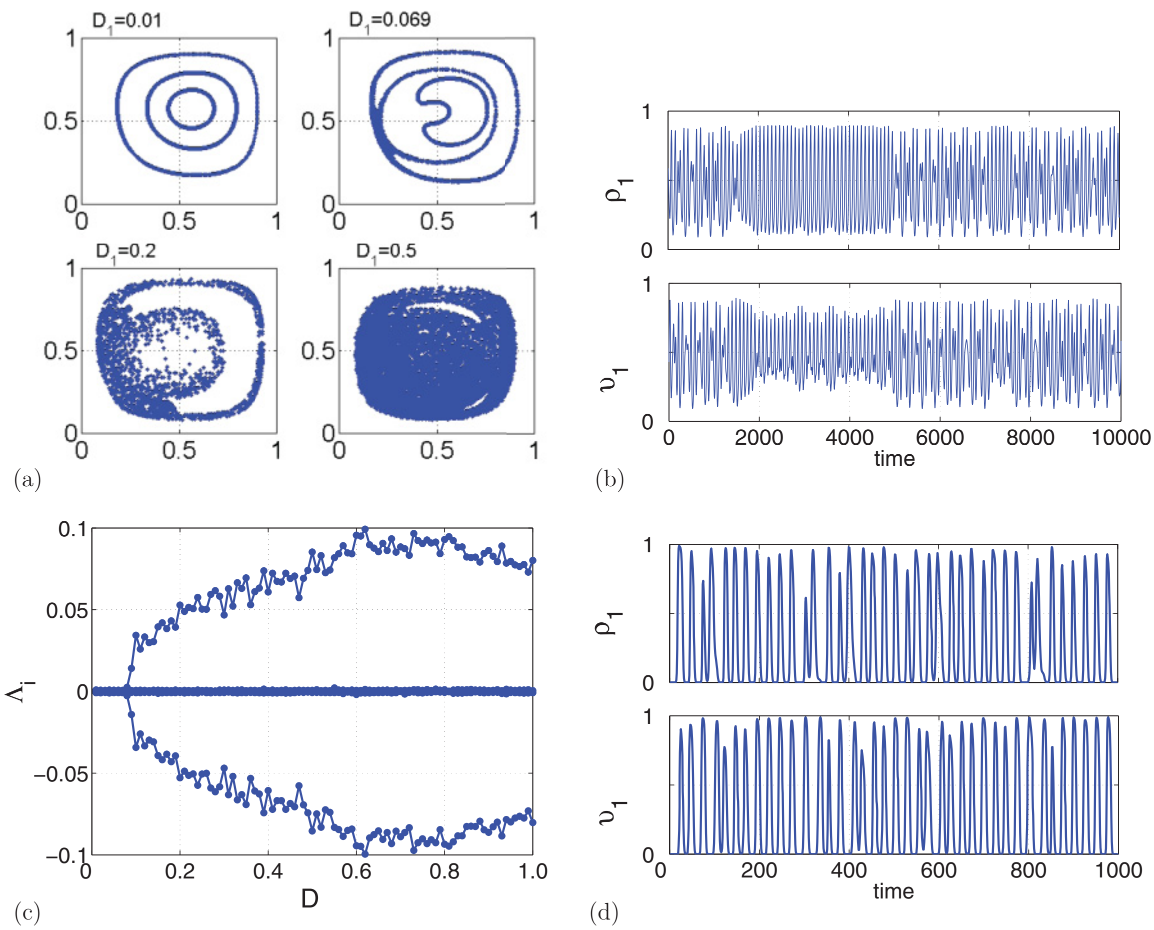

FIG. 7. (Color online) (a) Poincaré sections on the plane (υ_1,υ_2) demonstrating regular and chaotic dynamics at different values of D_1 in the system (13). (b) Time series of a chaotic regime of system (13), for parameter values $D_1 = 0.5$, $D_0 = 2.0$, $\varepsilon_1 = -1.0$, and $\varepsilon_2 = 1.0$. (c) Lyapunov exponents calculated at different values of D_1, for some particular values of the Hamiltonian. From four Lyapunov exponents, two always vanish, while other two vanish for small D_1 (quasiperiodicity) and are nonzero for larger coupling (chaos). (d) Chaotic time series of order parameters of four subpopulations of oscillators consisting of $N = 10^3$ elements each [the coupling configuration and the parameters are like in panel (b)].

an ensemble of Josephson junctions reported in [41]) and certainly deserves further investigation.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF EQUATIONS FOR NONRESONANT COUPLING

Consider two types of oscillators x and y that are governed by equations

$$\dot{x} = \mathcal{X}(x), \quad \dot{y} = \mathcal{Y}(y),$$

and have frequencies ω and Ω respectively. We consider two coupled ensembles of such oscillators of sizes N and M respectively, with a global coupling. For simplicity of presentation, we write equations for the ensemble x only. We assume that in general the coupling terms include coupling of 2, 3, 4, etc., oscillators:

$$\begin{aligned}\dot{x}_k = {} & \mathcal{X}(x_k) + \frac{1}{N}\sum_{l=1}^{N}\mathcal{A}(x_l) + \frac{1}{N^2}\sum_{l,m=1}^{N}\mathcal{B}(x_l,x_m) + \frac{1}{M}\sum_{l=1}^{M}\mathcal{C}(y_l)\\ & + \frac{1}{M^2}\sum_{l,m=1}^{M}\mathcal{D}(y_l,y_m) + \frac{1}{MN}\sum_{l=1}^{M}\sum_{m=1}^{N}\mathcal{F}(y_l,x_m)\\ & + \frac{1}{M^2N}\sum_{l,m=1}^{M}\sum_{j=1}^{N}\mathcal{G}(y_l,y_m,x_j)\\ & + \frac{1}{N^2M}\sum_{l,m=1}^{N}\sum_{j=1}^{M}\mathcal{H}(x_l,x_m,y_j)\\ & + \frac{1}{N^2M^2}\sum_{l,m=1}^{N}\sum_{j,n=1}^{M}\mathcal{U}(x_l,x_m,y_j,y_n) + \cdots .\end{aligned} \tag{A1}$$

Such coupling terms appear, for example, if one considers two groups of electronic oscillators (or Josephson junctions) with a common nonlinear load. By applying the phase reduction as described in [9], we characterize the oscillations by the phases φ_k of oscillators x_k and ψ_k of oscillators y_k, and obtain for their dynamics

$$\begin{aligned}\dot\varphi_k &= \omega_k + \frac{1}{N}\sum_{l=1}^{N} A(\varphi_l,\varphi_k) + \frac{1}{N^2}\sum_{l,m=1}^{N} B(\varphi_l,\varphi_m,\varphi_k)\\ &\quad + \frac{1}{M}\sum_{l=1}^{M} C(\psi_l,\varphi_k) + \frac{1}{M^2}\sum_{l,m=1}^{M} D(\psi_l,\psi_m,\varphi_k)\\ &\quad + \frac{1}{MN}\sum_{l=1}^{M}\sum_{m=1}^{N} F(\psi_l,\varphi_m,\varphi_k)\\ &\quad + \frac{1}{M^2N}\sum_{l,m=1}^{M}\sum_{j=1}^{N} G(\psi_l,\psi_m,\varphi_j,\varphi_k)\\ &\quad + \frac{1}{N^2M}\sum_{l,m=1}^{N}\sum_{j=1}^{M} H(\varphi_l,\varphi_m,\psi_j,\varphi_k)\\ &\quad + \frac{1}{N^2M^2}\sum_{l,m=1}^{N}\sum_{j,n=1}^{M} U(\varphi_l,\varphi_m,\psi_j,\psi_n,\varphi_k) + \cdots .\end{aligned}$$

We stress here that even if some coupling terms (e.g., four-oscillator couplings) are absent in the full model (A1), they appear generally in the phase description in the second order, third order, and so on of the perturbative reduction procedure.

All the functions $A,B,\dots$ are 2π periodic in all arguments, and thus they can be represented as multiple Fourier series:

$$A(\varphi_l,\varphi_k) = \sum_{p,q} a_{pq} e^{ip\varphi_l + iq\varphi_k},\quad C(\psi_l,\varphi_k) = \sum_{p,q} c_{pq} e^{ip\psi_l + iq\varphi_k},$$
$$\dots$$
$$U(\varphi_l,\varphi_m,\psi_j,\psi_n,\varphi_k) = \sum_{p,q,r,s,t} u_{pqrst} e^{ip\varphi_l + iq\varphi_m + ir\psi_j + is\psi_n + it\varphi_k}.$$

Substituting these expressions in coupling terms allows one to express them via the pth-degree Daido order parameters

$$X_p = \frac{1}{N}\sum_{l=1}^{N} e^{ip\varphi_l},\quad Y_p = \frac{1}{M}\sum_{l=1}^{M} e^{ip\psi_l},$$

as follows:

$$\frac{1}{N}\sum_{l=1}^{N} A(\varphi_l,\varphi_k) = \frac{1}{N}\sum_{l=1}^{N}\sum_{p,q} a_{pq} e^{ip\varphi_l + iq\varphi_k} = \sum_{p,q} a_{pq} e^{iq\varphi_k} X_p,$$
$$\frac{1}{N^2}\sum_{l,m=1}^{N} B(\varphi_l,\varphi_m,\varphi_k) = \sum_{p,q,r} b_{pqr} X_p X_q e^{ir\varphi_k},$$
$$\dots$$
$$\frac{1}{N^2M^2}\sum_{l,m=1}^{N}\sum_{j,n=1}^{M} U(\varphi_l,\varphi_m,\psi_j,\psi_n,\varphi_k) = \sum_{p,q,r,s,t} u_{pqrst} X_p X_q Y_r Y_s e^{it\varphi_k}.$$

With these order parameters, we write the phase equations as

$$\begin{aligned}\dot\varphi_k &= \omega_k + \sum_{p,q} a_{pq} X_p e^{iq\varphi_k} + \sum_{p,q,r} b_{pqr} X_p X_q e^{ir\varphi_k}\\ &\quad + \sum_{p,q} c_{pq} Y_p e^{iq\varphi_k} + \sum_{p,q,r} d_{pqr} Y_p Y_q e^{ir\varphi_k}\\ &\quad + \sum_{p,q,r} f_{pqr} Y_p X_q e^{ir\varphi_k} + \sum_{p,q,r,s} g_{pqrs} Y_p Y_q X_r e^{is\varphi_k}\\ &\quad + \sum_{p,q,r,s} h_{pqrs} X_p X_q Y_r e^{is\varphi_k}\\ &\quad + \sum_{p,q,r,s,t} u_{pqrst} X_p X_q Y_r Y_s e^{it\varphi_k} + \cdots .\end{aligned}$$

In order to obtain the averaged (over fast periods of oscillations) equations, we introduce slowly varying phases $\tilde\varphi$, $\tilde\psi$ as $\tilde\varphi = \varphi - \omega t$, $\tilde\psi = \psi - \Omega t$ where $\omega = \langle\omega_k\rangle$, $\Omega = \langle\Omega_k\rangle$. Correspondingly, the order parameters can be written as $X_p = \tilde X_p e^{i\omega t}$, $Y_p = \tilde Y_p e^{i\Omega t}$ via the slowly varying order parameter amplitudes $\tilde X_p, \tilde Y_p$. Substituting this yields

$$\begin{aligned}\dot{\tilde\varphi}_k &= \Delta\omega_k + \sum_{p,q} a_{pq} \tilde X_p e^{iq\tilde\varphi_k} e^{i(p+q)\omega t}\\ &\quad + \sum_{p,q,r} b_{pqr} \tilde X_p \tilde X_q e^{ir\tilde\varphi_k} e^{i(p+q+r)\omega t}\\ &\quad + \sum_{p,q} c_{pq} \tilde Y_p e^{iq\tilde\varphi_k} e^{ip\Omega t + iq\omega t}\\ &\quad + \sum_{p,q,r} d_{pqr} \tilde Y_p \tilde Y_q e^{ir\tilde\varphi_k} e^{i(p+q)\Omega t + ir\omega t}\\ &\quad + \sum_{p,q,r} f_{pqr} \tilde Y_p \tilde X_q e^{ir\tilde\varphi_k} e^{ip\Omega t + i(q+r)\omega t}\\ &\quad + \sum_{p,q,r,s} g_{pqrs} \tilde Y_p \tilde Y_q \tilde X_r e^{is\tilde\varphi_k} e^{i(p+q)\Omega t + i(r+s)\omega t}\\ &\quad + \sum_{p,q,r,s} h_{pqrs} \tilde X_p \tilde X_q \tilde Y_r e^{is\tilde\varphi_k} e^{ir\Omega t + i(p+q+s)\omega t}\\ &\quad + \sum_{p,q,r,s,t} u_{pqrst} \tilde X_p \tilde X_q \tilde Y_r \tilde Y_s e^{it\tilde\varphi_k} e^{is\varphi_k} e^{i(p+q+t)\omega t + i(r+s)\Omega t}\\ &\quad + \cdots .\end{aligned}$$

Now we can average the right-hand side just by integration over the explicit time dependence, neglecting at this stage the time dependence of the slow variables. Because the frequencies ω and Ω are incommensurate, $\overline{e^{im\omega t + in\Omega t}}^{\,t} = \delta_{n,0}\delta_{m,0}$, where $\delta_{n,m}$ is the Kronecker δ. This yields

$$\begin{aligned}\dot{\tilde\varphi}_k &= \Delta\omega_k + \sum_{p} a_{p,-p} \tilde X_p e^{-ip\tilde\varphi_k}\\ &\quad + \sum_{p,r} b_{p,-p+r,r} \tilde X_p \tilde X_{-p+r} e^{-ir\tilde\varphi_k} + c_{00}\\ &\quad + \sum_{p} d_{p,-p,0} \tilde Y_p \tilde Y_{-p} + \sum_{q} f_{0,q,-q} \tilde X_q e^{-iq\tilde\varphi_k}\\ &\quad + \sum_{p,r} g_{p,-p,r,-r} \tilde Y_p \tilde Y_{-p} \tilde X_r e^{-ir\tilde\varphi_k}\end{aligned}$$

$$+\sum_{p,q} h_{p,q-p,0,-q}\tilde{X}_p\tilde{X}_{-p+q}e^{-iq\tilde{\varphi}_k}$$
$$+\sum_{p,r,t} u_{p,-p+t,r,-r,t}\tilde{X}_p\tilde{X}_{-p+t}\tilde{Y}_r\tilde{Y}_{-r}e^{-it\tilde{\varphi}_k}.$$

We now make an assumption of sine coupling, namely that only the first harmonics of the phase $\tilde{\varphi}_k$ enters the coupling terms. This means, for example, that in the sum $\sum_p a_{p,-p}\tilde{X}_p e^{-ip\tilde{\varphi}_k}$ only the terms with $p=\pm 1$ are present (these two terms are complex conjugate), and similarly for other coupling terms. Furthermore, we explore the Ott-Antonsen ansatz [38], according to which the higher order parameters can be represented through the main one: $X_p=(X_1)^p$ and $Y_p=(Y_1)^p$. To simplify the notations, we drop the index of these main order parameters X_1,Y_1 and obtain

$$\dot{\tilde{\varphi}}_k=\frac{1}{2}\left[\Delta\omega_k+c_{00}+\sum_p d_{p,-p,0}|\tilde{Y}|^{2p}\right]+a_{1,-1}\tilde{X}e^{-i\tilde{\varphi}_k}$$
$$+\sum_p b_{p,-p+1,1}|\tilde{X}|^{2p}\tilde{X}e^{-i\tilde{\varphi}_k}+f_{0,1,-1}\tilde{X}e^{-i\tilde{\varphi}_k}$$
$$+\sum_p g_{p,-p,1,-1}|\tilde{Y}|^{2p}\tilde{X}e^{-i\tilde{\varphi}_k}$$
$$+\sum_p h_{p,1-p,0,-1}|\tilde{X}|^{2p}\tilde{X}e^{-i\tilde{\varphi}_k}$$
$$+\sum_{p,r} u_{p,-p+1,r,-r,1}|\tilde{X}|^{2p}\tilde{X}|\tilde{Y}|^{2r}e^{-i\tilde{\varphi}_k}+\text{c.c.},$$

where c.c. means complex conjugate (because of adding complex conjugate terms, we multiplied the real terms by $\frac{1}{2}$). One can see that the coupling involves only the amplitudes of the mean fields but not their phases; such a situation is typical for nonresonant couplings of oscillators. Combining the terms as

$$\dot{\tilde{\varphi}}_k=\frac{1}{2}\left(\Delta\omega_k+c_{00}+\sum_p d_{p,-p,0}|\tilde{Y}|^{2p}\right) \tag{A2}$$
$$+a_{1,-1}\tilde{X}e^{-i\tilde{\varphi}_k}+f_{0,1,-1}\tilde{X}e^{-i\tilde{\varphi}_k} \tag{A3}$$
$$+\sum_p(b_{p,-p+1,1}+h_{p,-p+1,0,-1})|\tilde{X}|^{2p}\tilde{X}e^{-i\tilde{\varphi}_k} \tag{A4}$$
$$+\sum_p g_{p,-p,1,-1}|\tilde{Y}|^{2p}\tilde{X}e^{-i\tilde{\varphi}_k} \tag{A5}$$
$$\sum_{p,r} u_{p,-p+1,r,-r,1}|\tilde{X}|^{2p}|\tilde{Y}|^{2r}\tilde{X}e^{-i\tilde{\varphi}_k}+\text{c.c.}, \tag{A6}$$

we can interpret the physical effects of different coupling terms. Here (A2) describes the shift of frequencies due to coupling, (A3) are the usual terms describing the Kuramoto interaction inside populations, terms (A4) describe nonlinear effects in interaction within a population like in [42], terms (A5) describe nonresonant interaction between populations, and terms (A6) describe nonlinear cross-coupling.

Now one can transform back to the original phases and order parameters to get a system to be modelled numerically:

$$\dot{\varphi}_k=\frac{1}{2}\left[\omega_k+c_{00}+\sum_p d_{p,-p,0}|Y|^{2p}\right]+(a_{1,-1}+f_{0,1,-1})Xe^{-i\varphi_k}$$
$$+\sum_p(b_{p,-p+1,1}+h_{p,-p+1,0,-1})|X|^{2p}Xe^{-i\varphi_k}$$
$$+\sum_p g_{p,-p,1,-1}|Y|^{2p}Xe^{-i\varphi_k}$$
$$\times\sum_{p,r} u_{p,-p+1,r,-r,1}|X|^{2p}|Y|^{2r}Xe^{-i\varphi_k}+\text{c.c.} \tag{A7}$$

In our simulations of ensembles, we used systems of type (A7), where we restricted ourselves to the simplest nontrivial case $p,r=\pm 1$. In this way, one formulates the ensemble dynamics corresponding to Eqs. (5) for the order parameters. Additionally, we ensured that the distribution of frequencies of coupled oscillators remained Lorentzian by choosing the coefficients $d_{p,-p,0}$ according to such a distribution.

APPENDIX B: DETAILS OF ANALYSIS OF TWO INTERACTING ENSEMBLES

Here we present details of the analysis of models (9),(10), giving the conditions for different regimes presented in Figs. 1 and 2.

(1) The case when only the trivial equilibrium point $S_1(0,0)$, $M_1(0,0)$ [Figs. 1(a), 2(a), and 2(b)] is stable. For system (9), such a situation occurs in the case $\delta_{1,2}<0$. For system (10), stability of only the trivial state M_1 occurs if $\varepsilon_{1,2}$ are negative and at least one of D_{12} or D_{21} less than 1:

$$\varepsilon_{1,2}<0,\quad \min(D_{12},D_{21})<1. \tag{B1}$$

(2) The case of stability of nontrivial state off-coordinate axes S_4 and M_4 [Figs. 1(b), 1(c), 2(c), and 1(d)]. For system (9), this situation occurs in two cases. The first situation appears if $\delta_{1,2}>0$ (when isolated subpopulations tends to synchrony) and suppressive couplings are weak:

$$\lambda_1=\delta_2-\delta_1 d_{21}>0,\quad \lambda_2=\delta_1-\delta_2 d_{12}>0. \tag{B2}$$

The states $S_{2,3}$ have eigenvalues $-2\delta_1$, λ_1, and $-2\delta_2$, λ_2, respectively, and therefore are saddles. The origin is an unstable node ($\delta_{1,2}>0$) and therefore the state S_4 is an attractor [note that S_4 always exists while (B2) holds]. We call this situation "case of weak suppressive couplings" because (B2) can be written as $d_{12}<\frac{\delta_1}{\delta_2}$, $d_{21}<\frac{\delta_2}{\delta_1}$.

The second situation appears if one of the subpopulations approaches the asynchronous state (negative δ) while another group tends to synchrony and has positive influence on the first subpopulation:

$$\delta_1>0,\ \delta_2<0,\ \lambda_1>0\quad \text{or}\quad \delta_1<0,\ \delta_2>0,\ \lambda_2>0. \tag{B3}$$

Condition $\lambda_1>0$ is equivalent to $d_{21}<\frac{\delta_2}{\delta_1}$, which means that coupling d_{21} should be negative and absolute value of d_{21} should be large enough to maintain a partially synchronous state inside the second cluster (positive influence). For the system (10), the situation when only M_4 is stable can be produced by two types of conditions. The first case is that of positive $\varepsilon_{1,2}$ and weak suppressive couplings:

$$\varepsilon_{1,2} > 0, \quad D_{12} < 1, \quad D_{21} < 1. \tag{B4}$$

Another case of stability of M_4 occurs if

$$\varepsilon_1 < 0, \quad \varepsilon_2 > 0, \quad D_{12} > 1, \quad D_{21} < 1 \quad \text{or} \quad \varepsilon_1 > 0, \quad \varepsilon_2 < 0, \quad D_{12} < 1, \quad D_{21} > 1. \tag{B5}$$

The latter case differs from the previous one only by the direction of the flow on lines $\rho_{1,2} = 0$ and the type of unstable points M_1, M_2, M_3 [Fig. 2(d)].

(3) The case of competition between subpopulations [Figs. 1(d) and 2(e)]. In model (9), this type of behavior arises when

$$\delta_{1,2} > 0, \quad \lambda_1 < 0, \quad \lambda_2 < 0. \tag{B6}$$

According to (B6), the points $S_{2,3}$ are stable, while S_1 is unstable node and S_4 is a saddle. This case corresponds to the situation of strong suppressive couplings

$$d_{12} > \frac{\delta_1}{\delta_2}, \quad d_{21} > \frac{\delta_2}{\delta_1}.$$

Competitive behavior in the system (10) is produced by positive $\varepsilon_{1,2}$ and strong suppressive couplings between subpopulations

$$\varepsilon_0 > 0, \quad D_{12} > 1, \quad D_{21} > 1. \tag{B7}$$

(4) The case of stability of synchronous state of only one cluster (S_2, M_2) [Figs. 1(e), 1(f), and 2(f)–2(h). In model (9), only one group is synchronous in two cases. The first trivial situation is similar to conditions (B3) (when one group approaches to asynchronous state while another one tends to synchrony), but in this case the active group does not have sufficient positive coupling to maintain synchronization in the asynchronous subpopulation [Fig. 1(f)]:

$$\delta_1 > 0\ \delta_2 < 0\ \lambda_1 < 0 \quad \text{or} \quad \delta_1 < 0\ \delta_2 > 0\ \lambda_2 < 0. \tag{B8}$$

Under conditions (B8), only one of the fixed points S_2 or S_3 exists and S_1 is always unstable. The second case occurs at an asymmetric interaction of intrinsically active clusters (isolated clusters tend to synchronous regime):

$$\delta_{1,2} > 0, \quad \text{and} \quad \lambda_1 < 0\ \lambda_2 > 0 \quad \text{or} \quad \lambda_1 > 0\ \lambda_2 < 0. \tag{B9}$$

In other words, it appears when one coupling coefficient is strong enough to fully suppress the synchrony in the opponent, for example, $d_{21} > \frac{\delta_2}{\delta_1}$, while another one is weak or even nonsuppressing $d_{12} < \frac{\delta_1}{\delta_2}$. In this case, one can prove that S_4 does not exist, point S_2 is a stable node, and S_3 and S_1 are saddles. Thus all trajectories approach stable node S_2, which corresponds to the synchronous state of the first group and to the asynchronous state of the second one. Because of this, on the plane (ρ_1, ρ_2) a heteroclinic trajectory always exists that connects saddle point S_3 and stable equilibrium S_2 [red line in Fig. 1(e)]. The stability of point $M_2(M_3)$ of system (10) occurs in several different cases. The first case is similar to the situation in the system (9) at conditions (B8) when one group tends to synchrony ($\delta_n > 0$), another one approaches trivial state ($\delta_m < 0$), and the synchronous group does not have sufficient positive influence to maintain synchronization in the asynchronous group:

$$\varepsilon_1 < 0, \varepsilon_2 > 0, D_{12} < 1 \quad \text{or} \quad \varepsilon_1 > 0, \varepsilon_2 < 0, D_{21} < 1. \tag{B10}$$

Corresponding phase planes are presented in Figs. 2(g) and 1(h). Another case is that of positive $\varepsilon_{1,2} > 0$ and asymmetric couplings:

$$\varepsilon_{1,2} > 0, D_{12} > 1\ D_{21} < 1 \quad \text{or} \quad D_{12} < 1\ D_{21} > 1. \tag{B11}$$

Under conditions in (B10) and (B11), it is easy to show that only one stable fixed point $M_2(1,0)$ exists, so all trajectories approach M_2. In the case of (B11), a sequence of heteroclinic orbits connecting M_2 and M_3 [red lines in Fig. 2(f)] appears.

(5) The case of bistability of trivial and fully synchronous states [Fig. 2(i)]. In model (10), this happens for negative $\varepsilon_{1,2}$ and strong synchronizing couplings:

$$\varepsilon_{1,2} < 0, \quad D_{12} > 1, \quad D_{21} > 1. \tag{B12}$$

(6) Periodic behavior [Fig. 2(j)]. In model (10), periodic solutions can be observed. Conditions

$$D_{12} > 1, \quad D_{21} > 1, \quad \varepsilon_1\varepsilon_2 < 0 \tag{B13}$$

provide saddle type of points M_{1-4} and existence of equilibrium M_5 with imaginary eigenvalues $\pm i\sqrt{-\frac{\varepsilon_1\varepsilon_2(D_{12}-1)(D_{21}-1)}{4D_{12}D_{21}}}$. Because model (10) can be rewritten as a Hamiltonian, one has a family of periodic orbits.

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