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Scaling properties of weak chaos in nonlinear disordered lattices

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We study the discrete nonlinear Schrödinger equation with a random potential in one dimension. It is characterized by the length, the strength of the random potential, and the field density that determines the effect of nonlinearity. Following the time evolution of the field and calculating the largest Lyapunov exponent, the probability of the system to be regular is established numerically and found to be a scaling function of the parameters. This property is used to calculate the asymptotic properties of the system in regimes beyond our computational power.

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The nonlinear Schrödinger equation (NLSE), in absence of a potential, in the continuum is an integrable problem [1]. It is relevant for the description of Bose-Einstein condensates [2] as well as for classical nonlinear optics and plasmas [3]. For the linear Schrödinger equation with a random potential in one dimension all the states are localized [4], as a manifestation of Anderson localization [5]. It is natural to ask what is the asymptotic behavior for the NLSE with a random potential, namely, what is the outcome of the competition between Anderson localization and nonlinearity. This question is directly related to several experimental situations. For BEC the nonlinear term in the NLSE models the interaction between the atoms while the random potential may be generated by lasers as in recent experiments [6]. In nonlinear optics the NLSE adequately describes light propagation in a nonlinear medium, where the randomness can be written with light that passes trough a diffuser [7] or by manufacturing [8].

Here the exploration will be performed in the framework of a discrete one-dimensional model [defined by (1)]. The specific elementary question in this field is the following: Will an initially localized wave packet spread to infinity in the long time limit? Extensive numerical simulations [9] (where although the full control of errors is impossible, one argues that the average results are statistically meaningful) exhibit subdiffusion, with the width of the wave packet growing in time as t^{σ} with $\sigma \approx 1/6$. On the other hand, recently it was argued that eventually the spreading should stop and the dynamics is eventually almost-periodic on a kind of Kolmogorov-Arnol'd-Moser torus [10]. Rigorous studies [11] lead to the conjecture that in the strong disorder limit the spreading is at most logarithmic in time, excluding subdiffusion as the asymptotic behavior. Nonrigorous results based on perturbation theory extend this conjecture beyond the regime of strong disorder with the help of a bound on the remainder term of the perturbation series [12], but the times available here at orders calculated so far turned out to be short compared with numerical calculations where subdiffusion was found. A major difficulty in the exploration of this problem is that we do not know how far in space and time one should go so that the result can be considered *asymptotic*. One reason this problem is complicated is the fact that during the spreading the effective number of degrees of freedom increases enhancing chaos, but their amplitude decreases suppressing chaos. This motivated the present work, which is designed to decide which of these competing effects wins. To address this issue we develop here a scaling theory of weak chaos in disordered nonlinear lattices, expecting it will be useful for extending results beyond our computational ability. Scaling approaches proved to be extremely powerful in equilibrium and nonequilibrium statistical physics [13] and have been also very successful in understanding Anderson localization [14]. In this letter a scaling theory for the probability distribution that chaos or regularity occurs, based on the computation of the largest Lyapunov exponent, is developed and tested for relatively small systems which are within our numerical power. Such a theory is expected to have predictive power when extended to infinite size.

We study a nonlinear disordered medium described by the discrete Anderson nonlinear Schrödinger equation (DANSE) model for a complex field $\psi_n(t)$:

$$i\frac{d\psi_n}{dt} = \epsilon_n\psi_n + J(\psi_{n+1} + \psi_{n-1}) + |\,\psi_n\,|^2\psi_n. \qquad (1)$$

This equation describes, in particular, an experimental setup in [8] where effects of nonlinearity on light localization in a disordered waveguide array have been studied. Without any limitation of generality, by rescaling time we can set the hopping to be $J = (1+W)^{-1}$ while ϵ_n are independent identically uniformly distributed in $(-JW, JW)$. With this rescaling the eigenvalues E of the linear part satisfy $|E| \leqslant 1 + (1+W)^{-1}$. Hence, for strong disorder (large W, that is the focus of our present work), the energies of the corresponding linear equation are practically in the interval $(-1,1)$. Measuring the length scale of the eigenstates of the linear problem μ by the inverse participation number $\mu^{-1} = \sum_k |\Psi_k|^4$, we find that $\mu \approx 1 + W^{-1}$ for large W. By scaling the amplitude of the field we set the coefficient of the nonlinear term in (1) to one and adopt periodic boundary conditions on a ring of length L. While DANSE model has two integrals of motion, the norm of the field $N = \sum_n |\psi_n|^2$ and the total energy, only the norm is important here; and in our treatment we do not control the energy which is always chosen to be close to zero.

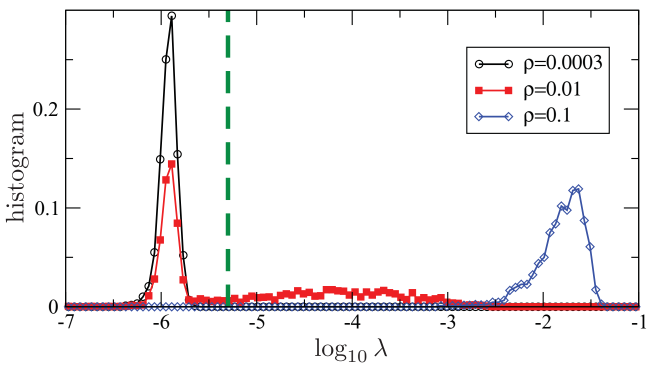

FIG. 1. (Color online) Distributions of the largest Lyapunov exponent for $L = 16$ and $W = 10$ and three values of the field density. For $\rho = 0.0003$ all the realizations are regular, for $\rho = 0.1$ all are chaotic, but for $\rho = 0.01$ part of realizations are chaotic. The vertical dashed line indicates the border $\lambda = 5 \times 10^{-6}$.

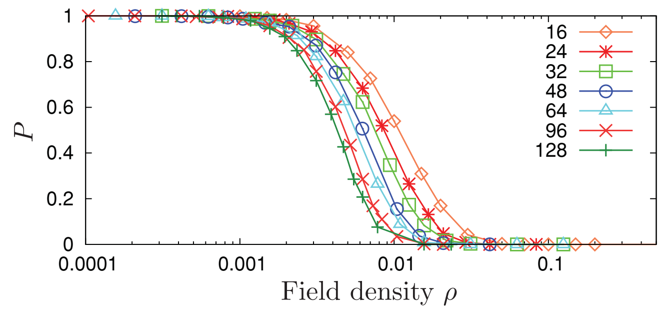

FIG. 2. (Color online) The values of $P(\rho, W, L)$ for $W = 10$ and different L, as functions of ρ. For the same density ρ the probability of regularity decreases with L.

Our goal in this letter is to study properties of the dynamics as a function of all relevant parameters: disorder W, norm N, and lattice length L. Equivalently, we introduce the density $\rho = N/L$ and consider the dependence on two intensive parameters W and ρ and on the extensive parameter L.

We characterize the dynamics by means of the largest Lyapunov exponent. For a particular realization of disorder $\{\epsilon_n\}$, we followed a dynamical solution of Eq. (1) starting with a *uniform initial field*, and for this trajectory directly calculated the largest Lyapunov exponent λ by a standard method. This calculation was repeated for a large number of realizations of disorder and the parameters above. As a result, we can construct a distribution of largest Lyapunov exponents over the realizations of disorder for given macroscopic parameters W, ρ, and L. Several examples of these distributions are presented in Fig. 1. We can see here that for very small densities ρ, where the nonlinearity in DANSE is very small, all the Lyapunov exponents are small and close to $\approx 10^{-6}$. For a regular dynamics the largest Lyapunov exponent should be exactly zero, but numerically, with a finite integrating time ($T_{\max} = 10^6$ in our numerical simulation; control runs with $T_{\max} = 10^7$ showed no significant difference) such a small value essentially indicates regular (quasiperiodic) dynamics of the field. Contrary to this, for large densities we observe Lyapunov exponents in the range 0.01–0.1, which indicates chaotic dynamics. For intermediate densities we see that for some realizations of disorder the dynamics is regular ($\lambda \approx 10^{-6}$), while for other realizations much larger exponents are observed, indicating weak chaos. As we want to perform a statistical analysis rather than going into details of particular dynamics for particular realizations of disorder, we adopt the following operational definition to distinguish between regularity and chaos: All runs with $\lambda < 5 \times 10^{-6}$ are regular ones, and all runs with $\lambda > 5 \times 10^{-6}$ are chaotic ones. We stress that the threshold value used is determined solely by the finite integration time; increasing this time would allow us to use a smaller threshold (cf. [10]). In this way we directly define the quantities of our main interest in this letter, a probability that chaos occurs P_{ch} and a probability that regular dynamics occurs $P = 1 - P_{\text{ch}}$.

As explained above, we have determined the probability of regular dynamics $P(\rho, W, L)$ as a function of parameters of the model. Typical profiles of P for fixed disorder W and different values of ρ and L are depicted in Fig. 2. One can see a typical sigmoidal function with limits $P \to 1$ for $\rho \to 0$ and $P \to 0$ for $\rho \to \infty$.

We first concentrate on the dependence on the extensive parameter L. According to Fig. 2, for the fixed W and ρ, the probability that regularity occurs decreases with the length L. To understand this, let us consider a lattice of large length L as composed of (still large) subsystems of lengths L_0 and relate the probability that regularity occurs on the large lattice $P(\rho, W, L)$ to the corresponding probabilities for smaller lattices $P(\rho, W, L_0)$. It is reasonable to assume that to observe regularity in the whole lattice we need to have all the subsystems regular, because any one chaotic subsystem will destroy regularity. This immediately leads us to the relation

$$P(\rho, W, L) = [P(\rho, W, L_0)]^{L/L_0}. \tag{2}$$

Equation (2) implicitly assumes that chaos appears not due to an interaction between the subsystems, but in each subsystem separately. This appears reasonable if the interaction between the subsystems is small, that is, if their lengths are large compared to the length scale associated with localization in the linear problem: $L_0 \gg \mu$. On these scales the various subsystems are statistically independent. This is the content of (2). It motivates the definition of the L-independent quantity:

$$R(\rho, W) = [P(\rho, W, L)]^{1/L}. \tag{3}$$

We check the scaling relations (2) and (3) in Figs. 3 and see that the data for lattices of sizes $16 < L < 128$ collapse, so that R is independent of L. Remarkably, a short lattice with $L = 8$ obeys the scaling for large disorders but deviates significantly for small disorders $W \lesssim 10$; this corresponds to the expected validity condition that L should be larger than μ (the spatial size of eigenfunctions).

The scaling relation (2) describes dependence on the extensive parameter L (and will allow us to extrapolate results to long lattices beyond our numerical resources), so we can concentrate on considering dependencies on intensive parameters W and ρ. Therefore, below we fix $L = L_0 = 16$ and study the scaling properties of $P_0(\rho, W) = P(\rho, W, L_0)$.

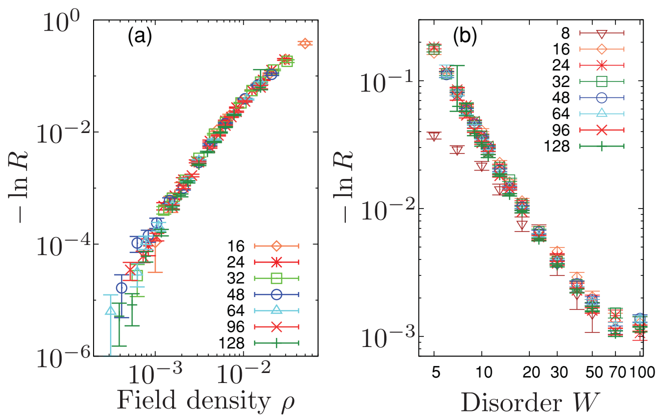

FIG. 3. (Color online) Rescaled according to (3) probabilities of regularity for different lattice sizes. (a) Fixed disorder $W = 10$, dependence on density (the same data as in Fig. 2); (b) fixed density $\rho = 0.01$, dependence on the disorder.

As this quantity is roughly a sigmoidal function of ρ [see Fig. 4(a)], it is convenient to perform a transformation to a new quantity $Q(\rho, W)$ as $Q = \frac{P_0}{1-P_0}$. In this representation,

$$P_0 = \frac{Q(\rho, W)}{1 + Q(\rho, W)} = \frac{1}{1 + Q^{-1}(\rho, W)}, \tag{4}$$

so that the asymptotic behaviors $P_0 \to 1$ as $Q \to \infty$ and $P_0 \to 0$ as $Q \to 0$ can be easily visualized as in Fig. 4(b).

The next crucial observation is that the function $Q(\rho, W)$ is not an arbitrary function of ρ and W, but it can be written in a scaling form,

$$Q = \frac{1}{W^\alpha} q\left(\frac{\rho}{W^\beta}\right), \tag{5}$$

where $q(x)$ is as usual a singular function at its limits $q(x) \sim c_1 x^{-\zeta}$ for small x, while $q(x) \sim c_2 x^{-\eta}$ for large x.

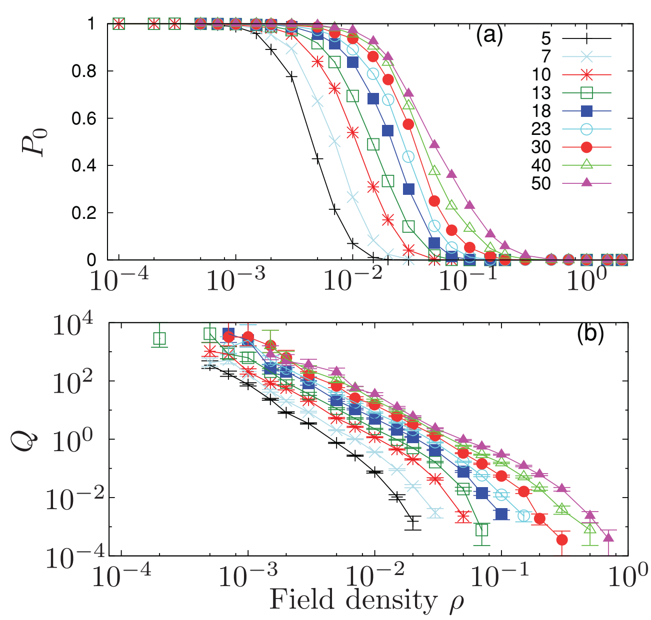

FIG. 4. (Color online) Profiles of $P_0(\rho, W)$ vs ρ for different values of W (a) and the same data in terms of $Q(\rho, W)$ (b).

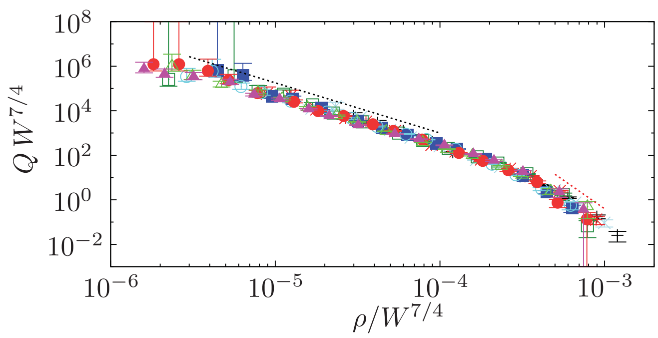

FIG. 5. (Color online) The same data as in Fig. 4 but in scaled coordinates. The black dashed and red dotted lines, showing asymptotics for small and large arguments of q, have slopes $\zeta = 9/4$ and $\eta = 5.2$.

These are found from the straight lines in Fig. 5. The data of Fig. 4 collapses to one curve, as shown in Fig. 5. This is the numerical justification for (5). It also provides the values of the exponents $\alpha = \beta = 1.75$, $\zeta \approx 9/4 = 2.25$, $\eta \approx 5.2$, $c_1 \approx 2.5 \times 10^{-7}$, and $c_2 \approx 1.8 \times 10^{-18}$.

The existence of the scaling function enables us to analyze the behavior at various limits. Most interesting is the limit of small densities where Q is large (see Fig. 5) and probability of regular behavior is close to one; correspondingly, the probability P_{ch} that chaos occurs is small. Using the general relation following from (2), (4), and (5),

$$P(\rho, W, L) = \left[1 + W^\alpha q^{-1}\left(\frac{\rho}{W^\alpha}\right)\right]^{-\frac{L}{L_0}},$$

we obtain for large q and P_{ch} small

$$P_{ch} = \approx -\log(1 - P_{ch}) = -\log P = L L_0^{-1} W^\alpha q^{-1}\left(\frac{\rho}{W^\alpha}\right).$$

Using the asymptotics $q(x) \sim c_1 x^{-\zeta}$ we get [15]

$$P_{ch} \approx L L_0^{-1} W^{\alpha(1-\zeta)} \rho^\zeta c_1^{-1}. \tag{6}$$

Now let us assume that we consider the states with the same fixed norm N on lattices of different length L. Then $\rho = N/L$ and from Eq. (6) it follows

$$P_{ch} \approx \frac{L^{1-\zeta} N^\zeta W^{\alpha(1-\zeta)}}{c_1 L_0} = \frac{L^{-5/4} N^{9/4}}{c_1 L_0 W^{35/16}}. \tag{7}$$

This quantity, as expected, grows with the norm N and decreases with the disorder W. We see that because $\zeta > 1$, probability that chaos occurs in large lattices at fixed norm tends to zero. This result may have implications for the problem of spreading of an initially local wave packet in large lattices. In this setup the norm of the field is conserved, and the effective density decreases in the course of the spreading. If one assumes that the dynamics follows the scaling above (although we established it for a special setup of lattices of finite lengths with periodic boundary conditions), and if one assumes that chaos is essential for spreading, then one concludes that the spreading should eventually stop as the probability that chaos occurs eventually vanishes. To estimate, according to arguments above, at which length we can expect the spreading to stop, we have to start with large densities (where the probability to observe regular dynamics is negligible) and to estimate, at what lattice size L_{max}

chaos becomes extinct. Assume that this happens when the probability that chaos occurs reaches some small level D. Substituting $P_{\mathrm{ch}} \approx D$ in Eq. (7), we obtain for the following estimate for $L_{\max}$:

$$L_{\max} = N^{\frac{\zeta}{\zeta-1}} W^{-\alpha} (D L_0 c_1)^{\frac{1}{1-\zeta}}.$$

Substituting $DL_0 \approx 1$ and the constants found, we obtain

$$L_{\max}(N,W) \approx 2 \times 10^5 \times N^{9/5} W^{-7/4}. \quad (8)$$

This is our estimation for the maximal spreading of a wave packet of norm N in a lattice with disorder W. Taking values typical for the numerical experiments [9], namely, $N = 1$, $1 \leqslant W \leqslant 10$, we obtain from (8) $L_{\max}$ in the range from 3×10^4 to 2×10^5. This explains why in the simulations, where typically values $L \approx 100$ are achieved, no saturation of the power law spreading is observed.

In conclusion, we established a full scaling theory for the probability that chaos occurs in disordered nonlinear Schrödinger lattice: The scaling with the extensive parameter—lattice length—is given by (3), while the scaling dependence on intensive parameters (disorder and density) is given by Eq. (5). Existence of such a scaling function is natural since the limits of vanishing disorder and of vanishing nonlinearity are singular ones. The found scaling indices indicate that for long lattices with the same norm chaos extincts and regularity prevails. Furthermore, we use the system presented here as a model for the chaotic region of high density typically observed at the initial stage of evolution in all numerical experiments [9]. In this context the scaling with the length allows us to estimate the maximal length that could be reached by spreading from initially local chaotic wave packet. We stress here that the scaling properties of the intermediate time spreading itself (cf. [16]) do not follow from the scaling properties of weak chaos established above; finding the former for the DANSE model (1) remains a challenge for future studies. Remarkably, the scaling relations established include large and small constants, which explains previous observations of energy spreading over extremely large time scales. Nevertheless, by applying the scaling we were able to estimate the final stages of evolution (hardly achievable with current computational abilities) from the studies of relatively small lattices.

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