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Hyperbolic chaos in a system of resonantly coupled weakly nonlinear oscillators

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ABSTRACT

We show that a hyperbolic chaos can be observed in resonantly coupled oscillators near a Hopf bifurcation, described by normal-form-type equations for complex amplitudes. The simplest example consists of four oscillators, comprising two alternatively activated, due to an external periodic modulation, pairs. In terms of the stroboscopic Poincaré map, the phase differences change according to an expanding Bernoulli map that depends on the coupling type. Several examples of hyperbolic chaos for different types of coupling are illustrated numerically.

1. Introduction

Investigation of coupled oscillators is one of the central problems of nonlinear dynamics, with applications to a great variety of natural and technological applications. Quite often an interaction leads to more order, in particular, oscillators can synchronize [1]. There are also many examples of an opposite effect, when coupling leads to a chaotization of the dynamics (see, e.g., [2]). Quite recently it was demonstrated that coupled oscillators can demonstrate a hyperbolic chaos. The notion of hyperbolicity is a central one in the mathematical theory of dynamical systems [3], and hyperbolic chaotic systems are the “cleanest” examples of pure chaos. Nevertheless, until recently, only abstract models of hyperbolic chaos have been known, mainly as topologically defined maps; most prominent examples being the Smale–Williams [4] and Plykin [5] attractors. While it was suggested that such attractors play an important role at the transition to turbulence [6], until recently no realistic model with such a dynamics has been developed. In a seminal paper [7] S. Kuznetsov has presented a first example of a realistic model with a Smale–Williams attractor. This model has been thoroughly studied numerically (in particular, mathematical criteria for hyperbolicity have been checked in Ref. [8]) and experimentally [9]. Other examples followed in Refs. [10,11]. In all these works hyperbolic strange attractors have been observed in coupled oscillator systems, described, e.g. by non-autonomous Van der Pol equations.

The goal of this Letter is to present an analysis of hyperbolicity in coupled oscillator systems based on the “normal form”-like description of their dynamics. Indeed, in the theory of nonlinear oscillators it is known that a universal description of the dynamics close to a Hopf bifurcation point can be formulated in terms of equations for the slowly varying complex amplitudes (in different contexts one speaks about Van der Pol method, asymptotic method, method of averaging, method of normal forms, etc.). In this approach particular properties of an underlying dynamical system become irrelevant, as the oscillators are described by a few parameters like growth rate and complex coefficient at nonlinear term. Most importantly, the frequency of the oscillations does not play any role, so that the description is equally valid for very slow (like biological ones) and very fast (like lasers) oscillators.

Before proceeding to a description of our basic model, we outline the main ingredients that are necessary for a hyperbolic chaos to occur. A hyperbolic chaos is achieved by implementing a transformation of the phase of the oscillations, which is close to the Bernoulli doubling map $\phi \to 2\phi$ or to its generalization. To implement such a doubling of the phase, in [7] it was suggested to modulate an oscillator in such a way that it slowly passes through a Hopf bifurcation, and to apply simultaneously a small force. In such a process the oscillator acquires the phase of the forcing, and if the forcing has the doubled phase of the oscillations, the desired transformation can be realized. According to this general scheme, an oscillator should pass repeatedly through a Hopf bifurcation – this is achieved by modulating its parameter responsible for the bifurcation (growth rate). Additionally, the coupling between the oscillators should be organized in such a way, that the forcing terms contain multiples of the phase. In the next section we follow these ideas in constructing our basic model.

2. Basic model of coupled oscillators

In this section we construct a model of coupled oscillators that demonstrates a hyperbolic chaos. Our aim is to arrive at the sim-

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plest possible situation, therefore in course of consideration we will make simplifying assumptions. Consider a set of oscillators marked by index k, and described by general dynamical equations

$$\frac{d\vec{x}_k}{dt} = \vec{F}_k(\vec{x}_k, t) + \varepsilon \vec{f}_k(\vec{x}_j). \tag{1}$$

Here terms $\vec{F}_k$ describe individual oscillating systems and terms $\vec{f}$ their coupling. Let us first neglect the coupling by setting $\varepsilon = 0$. We suppose that each oscillator is close to a supercritical Hopf bifurcation from a stable steady state to periodic self-sustained oscillations. In this regime the dynamics can be reduced to that on a two-dimensional center manifold (see, e.g. [12,13]), where a slowly varying complex amplitude a_k can be introduced so that $\vec{x}_k(t) \sim \mathrm{Re}[a_k(t)\vec{X}_k e^{i\omega_k t}]$. Furthermore, we assume that the parameter Γ responsible for the growth rate of the oscillations is a slow (compared to the frequency) function of time. Then the equation for the amplitude a_k reads

$$\frac{da_k}{dt} = a_k\big(\Gamma_k(t) - |a_k|^2\big). \tag{2}$$

In this equation we have neglected a possible the nonlinear frequency shift, which would correspond to a complex coefficient at nonlinear term.

Next, we discuss the coupling terms. We assume that all the oscillators have the same natural frequency $\omega_1 = \omega_2 = \cdots = \omega$. Then a general forcing term acting on oscillator k in terms of complex amplitudes a is obtained via averaging $\langle \vec{f}_k e^{-i\omega t}\rangle$ over the oscillation period. Generally, such a term contains products of powers (because for a small nonlinearity a power-series expansion of all terms is possible) of the complex amplitudes, i.e. terms like $a_{k'}^{s'_k}|a_{k'}|^{\sigma'_k} a_{k''}^{s''_k}|a_{k''}|^{\sigma''_k}\ldots$. From the resonance condition it follows that

$$s'_k + s''_k + \cdots = 1, \tag{3}$$

all other terms vanish due to the averaging. Here the powers s_i are integers and we use a convention that negative powers mean complex conjugation $a^{-1} = a^*$. In general, there can be several terms with different combinations $s'_k, \sigma'_k, s''_k, \sigma''_k, \ldots,$ but in order to have a well-defined phase of the forcing, we assume that it consists of just one such term. Furthermore, the components $|a_{k'}|^{\sigma'_k}, |a_{k''}|^{\sigma''_k}, \ldots$ do not have the phases, therefore we can set $\sigma_k = 0$. Summarizing, the governing equations for the resonantly coupled oscillators can be written as

$$\frac{da_k}{dt} = a_k\big(\Gamma_k(t) - |a_k|^2\big) + \varepsilon a_{k'}^{s'_k} a_{k''}^{s''_k}\ldots. \tag{4}$$

As outlined above, due to a periodic modulation of the growth rates $\Gamma_k(t)$, the oscillators pass through a Hopf bifurcation. In the simplest situation, there are two groups of oscillators that are excited alternatively: for the first group $\Gamma(t) = \Gamma_+(t) = \gamma_0 + \gamma_1\cos(\Omega t)$ while for the second group $\Gamma(t) = \Gamma_-(t) = \gamma_0 - \gamma_1\cos(\Omega t)$. We denote the complex amplitudes of the oscillators belonging to the first and the second group by b_n and c_m, correspondingly. Furthermore, in the coupling terms it is essential to have a large forcing acting on the non-excited oscillators as they pass through a Hopf bifurcation. Thus, oscillators b_n should be forced by oscillators c_m and vice versa. The model then reads

$$\frac{db_n}{dt} = b_n\big(\Gamma_+(t) - |b_n|^2\big) + \varepsilon c_{m'}^{s'_m} c_{m''}^{s''_m}\ldots, \tag{5}$$

$$\frac{dc_m}{dt} = c_m\big(\Gamma_-(t) - |c_m|^2\big) + \varepsilon b_{n'}^{s'_n} b_{n''}^{s''_n}\ldots. \tag{6}$$

We now discuss, what is the simplest system of type (5)–(6) that demonstrates a nontrivial dynamics of the phases. One can easily check that system (5)–(6) (and Eq. (4) as well) is invariant under the transformation $b_n \to b_n e^{i\phi}$, $c_m \to c_m e^{i\phi}$, i.e. invariant with respect to a common phase shift of all complex amplitudes (formally this follows from the resonance condition (3); physically this is due to the fact that all oscillators have the same frequency which becomes irrelevant in the complex amplitude formulation). This means that the transformation of the phases over the period of modulation $T = 2\pi/\Omega$ has a multiplier equal to one, corresponding to this invariance. In order to have a nontrivial multiplier, the transformation of each group of oscillators must be at least two-dimensional. Thus, the simplest nontrivial case is that of two oscillators in each group b_n and c_m. Therefore, the minimal model reads

$$\frac{db_1}{dt} = b_1\big(\gamma_0 + \gamma_1\cos(\Omega t) - |b_1|^2\big) + \varepsilon c_1^{\alpha} c_2^{1-\alpha}, \tag{7}$$

$$\frac{db_2}{dt} = b_2\big(\gamma_0 + \gamma_1\cos(\Omega t) - |b_2|^2\big) + \varepsilon c_1^{\beta} c_2^{1-\beta}, \tag{8}$$

$$\frac{dc_1}{dt} = c_1\big(\gamma_0 - \gamma_1\cos(\Omega t) - |c_1|^2\big) + \varepsilon b_1^{\kappa} b_2^{1-\kappa}, \tag{9}$$

$$\frac{dc_2}{dt} = c_2\big(\gamma_0 - \gamma_1\cos(\Omega t) - |c_2|^2\big) + \varepsilon b_1^{\delta} b_2^{1-\delta}. \tag{10}$$

Here α, β, κ, δ are integers describing the coupling.

3. Analysis of the phase transformation

In this section we perform a qualitative analysis of our basic model of four coupled oscillators (7)–(10), a numerical study will be reported in the next section. Within the period of modulation T, the oscillations are alternatively excited and suppressed. When passing through a Hopf bifurcation from suppression to excitation, the oscillator assumes the phase of the forcing. Let us denote the phases of oscillators b_1, b_2, c_1, c_2 as φ_1, φ_2, ψ_1, ψ_2 respectively. Then, the phase of the forcing term for the oscillator b_1 is $\alpha\psi_1 + (1-\alpha)\psi_2$, and correspondingly for all other oscillators.

Following the methodology of the work [7], the functioning of the system (7)–(10) can be described qualitatively as follows (we stress that arguments below are very rough qualitative, to be confirmed by numerical analysis in the next section). Suppose that the oscillators enter a period of modulation with phases φ_1, φ_2, ψ_1, ψ_2. For definiteness, we define the period as starting from the state where oscillators b_1, b_2 are excited and oscillators c_1, c_2 are not excited. Then at the beginning of the period the phases φ_1, φ_2 are well defined. When c_1, c_2 become excited, they get the phases

$$\overline{\psi}_1 = \kappa\varphi_1 + (1-\kappa)\varphi_2, \tag{11}$$

$$\overline{\psi}_2 = \delta\varphi_1 + (1-\delta)\varphi_2. \tag{12}$$

On the next stage the oscillators c_1, c_2 are excited while the oscillators b_1, b_2 are not excited. As the latter ones pass through a Hopf bifurcation, they get the phases

$$\overline{\varphi}_1 = \alpha\overline{\psi}_1 + (1-\alpha)\overline{\psi}_2, \tag{13}$$

$$\overline{\varphi}_2 = \beta\overline{\psi}_1 + (1-\beta)\overline{\psi}_2. \tag{14}$$

The overall transformation of the phases φ_1, φ_2 over the period is

$$\begin{pmatrix}\overline{\varphi}_1\\ \overline{\varphi}_2\end{pmatrix} = M\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \qquad M = \begin{pmatrix}\alpha & 1-\alpha\\ \beta & 1-\beta\end{pmatrix}\begin{pmatrix}\kappa & 1-\kappa\\ \delta & 1-\delta\end{pmatrix}. \tag{15}$$

One eigenvalue of matrix M is one, in accordance with the mentioned above invariance to an overall shift of the phases, and the nontrivial eigenvalue is

$$\mu = (\alpha-\beta)(\kappa-\delta). \tag{16}$$

One can easily see that the eigenvector corresponding to this eigenvalue is the difference of the phases, which during the period of modulation is transformed as

$$\overline{\varphi}_1 - \overline{\varphi}_2 = \mu(\varphi_1 - \varphi_2). \tag{17}$$

The transformation of the phases will be chaotic, if $|\mu| > 1$. According to (16), one then easily finds the values of factors $\alpha, \beta, \kappa, \delta$ leading to a chaotic transformation of the phases. Below we consider two examples: (A) the case $\alpha = -1$, $\beta = 1$, $\kappa = 0$ and $\delta = 1$ yielding $\mu = 2$ and (B) the case $\alpha = 2$, $\beta = -1$, $\kappa = 2$ and $\delta = -1$ yielding $\mu = 9$.

4. Examples of coupled oscillators with hyperbolic chaos

We start with model A, governed by equations

$$\frac{db_1}{dt} = b_1\big(\gamma_0 + \gamma_1 \cos(\Omega t) - |b_1|^2\big) + \varepsilon c_1^* c_2^2, \tag{18}$$

$$\frac{db_2}{dt} = b_2\big(\gamma_0 + \gamma_1 \cos(\Omega t) - |b_2|^2\big) + \varepsilon c_1, \tag{19}$$

$$\frac{dc_1}{dt} = c_1\big(\gamma_0 - \gamma_1 \cos(\Omega t) - |c_1|^2\big) + \varepsilon b_2, \tag{20}$$

$$\frac{dc_2}{dt} = c_2\big(\gamma_0 - \gamma_1 \cos(\Omega t) - |c_2|^2\big) + \varepsilon b_1. \tag{21}$$

A remarkable property of this system is that the coupling in the last three of Eqs. (19)–(21) is linear, and only one coupling term in Eq. (18) is nonlinear. Furthermore, we can significantly simplify the system by observing that oscillators b_2 and c_1 simply exchange their phases, as $\overline{\psi}_1 = \varphi_2$ and $\overline{\varphi}_2 = \overline{\psi}_1 = \varphi_2$. Thus, the phase of oscillator c_1 remains constant – this oscillator simply serves as a periodic signal allowing a resonance excitation of the oscillator b_1. Therefore we can construct a simplified version of system (18)–(21) by assuming $c_1 = C = \text{const}$:

$$\frac{db_1}{dt} = b_1\big(\gamma_0 + \gamma_1 \cos(\Omega t) - |b_1|^2\big) + \varepsilon C^* c_2^2, \tag{22}$$

$$\frac{dc_2}{dt} = c_2\big(\gamma_0 - \gamma_1 \cos(\Omega t) - |c_2|^2\big) + \varepsilon b_1. \tag{23}$$

Note that this reduced model does not possess the invariance to phase shifts, this is due a fixed phase of the "external force" C. Existence of different systems of coupled oscillators that demonstrate the same phase dynamics (17) follows from the fact that this dynamics is determined only by the nontrivial eigenvalue of matrix M, which depends on the particular parameters of coupling via relation (16). While the particular choice of these parameters in (18)–(21) allows to exclude some variables because the phase dynamics in (18)–(21) is equivalent to that in (22)–(23), such a reduction is not possible for model (24)–(27) below.

In numerical examples below we have chosen $\gamma_0 = 0.2$, $\gamma_1 = 2$, $\Omega = 1$ and $\varepsilon = 0.05$. We illustrate the dynamics of four coupled oscillators according to (18)–(21) in Fig. 1. One can clearly see that whereas the amplitudes vary regularly and alternatively, following the modulation of the growth rate, the phases demonstrate an irregular dynamics. As the chaos is in the differences of the phases, it can be transformed in an intensity observable by considering the sums of the complex amplitudes. Indeed, defining $b = b_1 + b_2$ and assuming $|b_1| \approx |b_2|$ (which is well confirmed by numerics), we obtain $|b|^2 \approx 2|b_1|^2(1 + \cos(\varphi_1 - \varphi_2))$. Thus, this intensity varies chaotically in large range, as shown in Fig. 1. Notice also the constance of the phases φ_2, ψ_1 according to the discussion above. In Fig. 2 we show the stroboscopic map at times $t = 0, T, 2T, \ldots$. The transformation of the phases clearly demonstrates the doubling and looks as Bernoulli map. The view of the strange attractor projected on the plane $(\text{Re}(b_1), \text{Im}(b_1))$ corresponds to an image of the Smale–Williams solenoid.

In Fig. 3 we illustrate the Lyapunov exponents λ_k as function of the parameter γ_1. The correspondence of the transformation of the phases with Bernoulli map presumes that the largest Lyapunov exponent of the system (18)–(21) should be equal to $\frac{\ln 2}{T}$. To compute

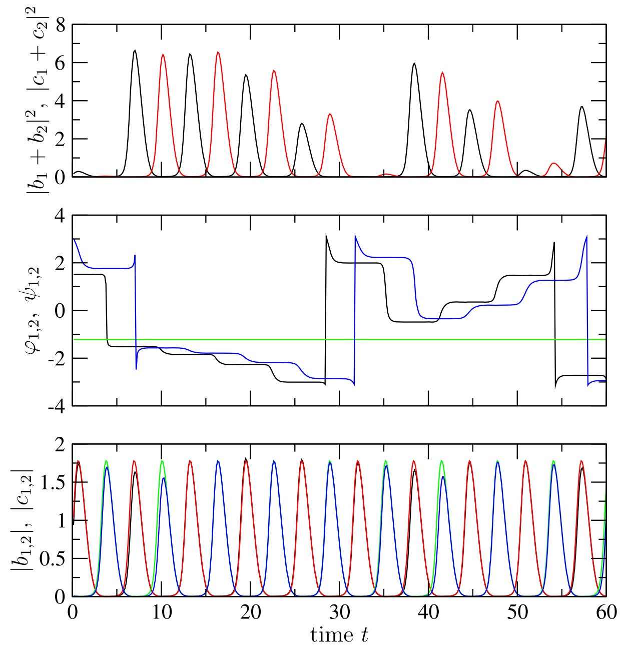

Fig. 1. Numerical analysis of model A ((18)–(21)). Middle and bottom panels: evolutions of the phases and the amplitudes of oscillators b_1 (black), b_2 (red), c_1 (green), and c_2 (blue). Top panel: Intensities of the sums $b_1 + b_2$ (black) and $c_1 + c_2$ (red). Note, that the phases φ_2 and ψ_1 of the oscillators b_2 and c_1 are constant. Thus the red and green lines overlap in the middle panel. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

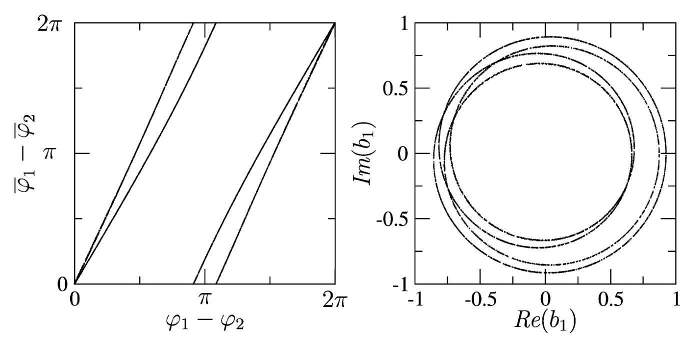

Fig. 2. Numerical analysis of model A ((18)–(21)). Left panel: the stroboscopic mapping for the phase difference. What looks like two lines is in fact a set of points. This picture proofs validity of transformation of the phases (17) with $\mu = 2$. Right panel: the stroboscopic view of the attractor in projection on plane $(\text{Re}(b_1), \text{Im}(b_1))$. In this projection the cantor transversal structure is seen better than in the left panel, where only two main branches can be distinguished.

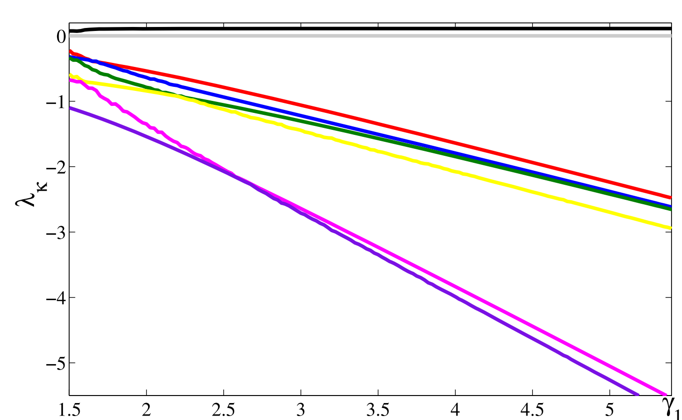

Fig. 3. Numerical analysis of model A ((18)–(21)). Dependence of the Lyapunov exponents λ_k on the parameter γ_1.

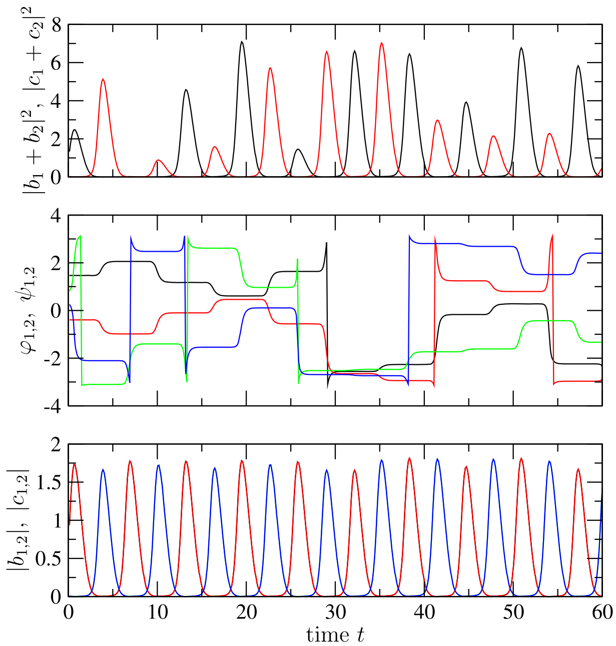

Fig. 4. Numerical analysis of model B ((24)–(27)). Middle and bottom panels: evolutions of the phases and the amplitudes of oscillators b_1 (black), b_2 (red), c_1 (green), and c_2 (blue). Top panel: intensities of the sums b_1+b_2 (black) and c_1+c_2 (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

the Lyapunov exponents we employ the Benettin's algorithm that requires solving simultaneously Eqs. (18)–(21) and the linearized equations for small perturbations. Advancing the solution, we perform the Gram–Schmidt orthogonalization and normalization of the perturbation vectors after each time interval T. Lyapunov exponents appear as time averaged logarithms of the norms of the perturbation vectors. In Fig. 3 one can see that the largest exponent (λ_1) is positive and constant ($\lambda_1 \simeq 0.1103\ldots$) for a wide range of parameter values; the second exponent is equal to zero, and other exponents are negative and manifest rather regular parameter dependencies. This is another evidence of a hyperbolic chaotic nature of the observed dynamics. Using the Lyapunov exponents we estimated for the attractor presented in Fig. 2 the dimension from the Kaplan–Yorke formula [14]. It is $D_l = 1.168$ and demonstrates a rather good correspondence with the correlation dimension $D_k = 1.16$ which was calculated using the Grassberger–Procaccia algorithm [15].

Next we consider model B governed by equations

$$\frac{db_1}{dt} = b_1\left(\gamma_0 + \gamma_1\cos(\Omega t) - |b_1|^2\right) + \varepsilon c_1^2 c_2^*, \tag{24}$$

$$\frac{db_2}{dt} = b_2\left(\gamma_0 + \gamma_1\cos(\Omega t) - |b_2|^2\right) + \varepsilon c_2^2 c_1^*, \tag{25}$$

$$\frac{dc_1}{dt} = c_1\left(\gamma_0 - \gamma_1\cos(\Omega t) - |c_1|^2\right) + \varepsilon b_1^2 b_2^*, \tag{26}$$

$$\frac{dc_2}{dt} = c_2\left(\gamma_0 - \gamma_1\cos(\Omega t) - |c_2|^2\right) + \varepsilon b_2^2 b_1^*. \tag{27}$$

The same analysis as for model A is illustrated in Figs. 4–6. Now the dynamics of all phases is chaotic, while the amplitudes of the oscillators vary nearly periodically. The stroboscopic map in Fig. 5 has now nine stripes. In the projection on the plane $(\mathrm{Re}(b_1), \mathrm{Im}(b_1))$ a transverse structure is smeared to the projection effect. The largest Lyapunov exponent (λ_1) (see Fig. 6) is positive and constant ($\lambda_1 \simeq 0.34496\ldots$) for a wide range of parameter values, similar to the case A. However, now its value is close to $\frac{\ln 9}{T}$, because the stroboscopic map (Fig. 5) has nine stripes.

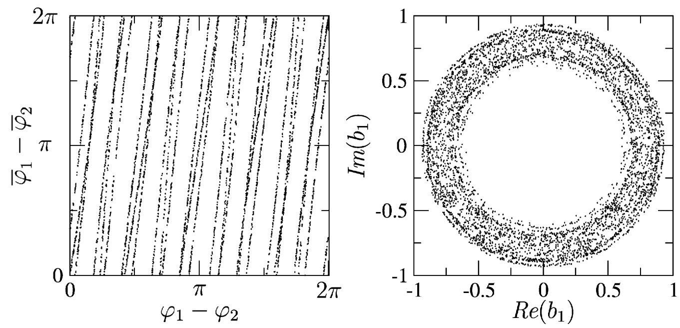

Fig. 5. Numerical analysis of model B ((24)–(27)). Left panel: the stroboscopic mapping for the phase difference. Now, in comparison to Fig. 2, the internal cantor structure of nine basic stripes is better seen in the left panel, while it is hardly distinguishable in the right one. This picture proofs validity of transformation of the phases (17) with $\mu = 9$. Right panel: the stroboscopic view of the attractor in projection on plane $(\mathrm{Re}(b_1), \mathrm{Im}(b_1))$.

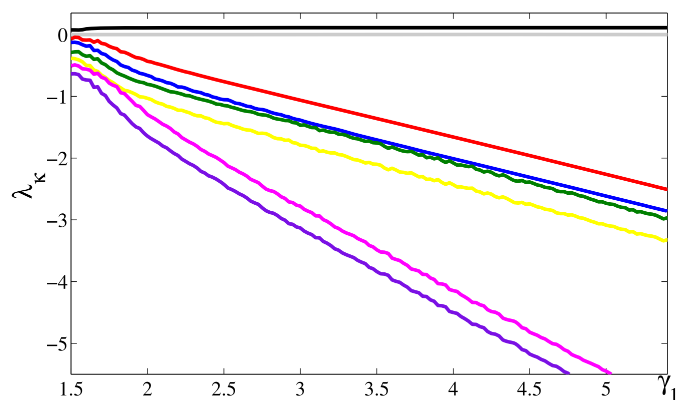

Fig. 6. Numerical analysis of model B ((24)–(27)). Dependence of the Lyapunov exponents λ_k on the parameter γ_1.

At the end of this section we discuss, whether the basic models (18)–(21) and (24)–(27) can be further reduced. As already discussed, model A can be exactly reduced to (22)–(23) which is five-dimensional (two complex variables and time). The nine-dimensional model (24)–(27) can be reduced to eight dimensions because of invariance to a simultaneous phase shift of all complex amplitudes. A further exact reduction appears not possible. On the other hand, one could try to construct a similar model without periodic forcing where modulations of the growth rates appear due to internal dynamics, like in [11], what however goes beyond the scope of this Letter.

5. Hyperbolic chaos in the non-reduced system of coupled oscillators

Above we have focused on general properties of hyperbolic chaos in a system of coupled equations based on the normal form (2). Here we will demonstrate that a similar regime can be found in the non-reduced system of type (1). For this example we choose a model of coupled nanoscale electromechanical oscillators studied in details in [16]. The model suggested in [16] are coupled Van der Pol equations of type

$$\ddot{x} - (\gamma - x^2)\dot{x} + \omega^2 x = f \tag{28}$$

where x is the coordinate of a nanoelectromechanical oscillator and f describes coupling terms; this model is based on comparison with experiments [17]. To obtain hyperbolic chaos as described above, we consider four oscillators of type (28) that are excited alternatively and are nonlinearly coupled:

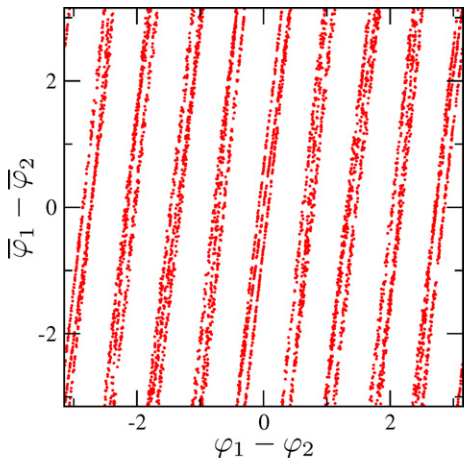

Fig. 7. Numerical analysis of non-reduced model (29) with $\gamma_1 = 3$, $\omega = 25$, $\Omega = 1$, $\epsilon = 0.2$. The stroboscopic mapping for the phase difference, where φ_i is phase of the i-th oscillator, should be compared with Fig. 5.

$$\begin{aligned}
&\ddot{x}_1 - \left(\gamma_1 \cos(\Omega t) - x_1^2\right)\dot{x}_1 + \omega^2 x_1 = \epsilon x_4 x_3 \dot{x}_3,\\
&\ddot{x}_2 - \left(\gamma_1 \cos(\Omega t) - x_2^2\right)\dot{x}_2 + \omega^2 x_2 = \epsilon x_3 x_4 \dot{x}_4,\\
&\ddot{x}_3 - \left(-\gamma_1 \cos(\Omega t) - x_3^2\right)\dot{x}_3 + \omega^2 x_3 = \epsilon x_2 x_1 \dot{x}_1,\\
&\ddot{x}_4 - \left(-\gamma_1 \cos(\Omega t) - x_4^2\right)\dot{x}_4 + \omega^2 x_4 = \epsilon x_1 x_2 \dot{x}_2. \qquad (29)
\end{aligned}$$

We stress that the switching on and off of the activity of oscillators has been experimentally demonstrated in [17]. We integrated the system (29) numerically and have obtained the phase chaos very similar to that found for the normal form (24)–(27), see Fig. 7. This proves that the hyperbolic chaos is not the property that appears in course of a reduction of the original equations, but is the intrinsic feature of the dynamics.

6. Conclusion

In this work we have demonstrated that a system of coupled oscillators can possess a hyperbolic chaos of Smale–Williams type. We have argued that a minimal model is that of four oscillators consisting of two alternatively excited pairs. All oscillators have the same frequency and only a resonant interaction is taken into account. The description is based on the normal-form-type equations for the complex amplitudes, thus it is independent of a particularities of the oscillating dynamics – only a closeness to a Hopf bifurcation is needed. We stress that the basic frequency does not enter the problem, so that the theory is equally applicable to very slow and very fast oscillators. In order to show that the conclusions based on the consideration of normal form equations are valid also for the original dynamical system, we considered a model previously suggested for a description of active nanoelectromechanical oscillators and demonstrated that the hyperbolic chaos of the same type as in normal form formulation is observed in the full system. This opens a perspective for a possible experimental realization. Another possible experimental setup is one where the active oscillators are realized as electronic devices, like in [9].

We have argued that there are many ways to couple the oscillators to achieve an expanding transformation of the phase difference. For demonstration we have chosen two examples. The first one, yielding the weakest chaos described by the Bernoulli doubling map for the phases, includes only one nonlinear coupling term, all other couplings are linear. In another example, where all the couplings are nonlinear (but of the lowest possible order of nonlinearity) the resulting chaos is rather strong – the transformation for the phase differences is a Bernoulli map with multiplier 9.

A natural question arises whether the regimes observed are stable with respect to variations of the parameters. The answer just comes from the stability of Bernoulli-type maps: small variations of parameters does not lead to topological changes. In particular, a small detuning from a perfect coincidence of the frequencies does not lead to a qualitative change in the dynamics. This conclusion is, however, restricted to the perturbations within the equations for the complex amplitudes that preserve the invariance to phase shifts. As this invariance is, strictly speaking, not valid for the original equations prior to the normal mode reduction, the relation to the hyperbolicity properties of the original system remains a subject for a further research.

As a interesting development of the research reported we mention a possibility to observe hyperbolic chaos in ensembles of coupled oscillators. Indeed, recently Ott and Antonsen [18] have shown that ensembles of globally coupled phase oscillators (so-called Kuramoto model) are described by equations of type (2) for the complex order parameter. Therefore, one can expect that a nonlinear coupling between such alternatively synchronized ensembles may lead to a hyperbolic chaos of the order parameters.

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References

[1] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Science, Cambridge University Press, Cambridge, England, 2001.
[2] T. Kapitaniak, W.H. Steeb, Phys. Lett. A 152 (1991) 33;
I. Pastor-Diaz, V. Perez-Garcia, F. Encinas-Sanz, J.M. Guerra, Phys. Rev. E 48 (1993) 171;
T. Ueta, H. Miyazaki, T. Kousaka, H. Kawakami, Internat. J. Bifur. Chaos 14 (2004) 1305.
[3] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995.
[4] S. Smale, Bull. Amer. Math. Soc. 13 (1967).
[5] R. Plykin, Math. USSR Sbornik 23 (1974) 233.
[6] D. Ruelle, F. Takens, Comm. Math. Phys. 20 (1971) 167;
S.E. Newhouse, D. Ruelle, F. Takens, Comm. Math. Phys. 64 (1978) 35.
[7] S.P. Kuznetsov, Phys. Rev. Lett. 95 (2005) 144101.
[8] S.P. Kuznetsov, I.R. Sataev, Phys. Lett. A 365 (2007) 97.
[9] S.P. Kuznetsov, E.P. Seleznev, J. Exp. Theor. Phys. 102 (2006) 355;
S.P. Kuznetsov, V.I. Ponomarenko, Tech. Phys. Lett. 34 (2008) 771.
[10] O.B. Isaeva, A.Y. Jalnine, S.P. Kuznetsov, Phys. Rev. E 74 (2006) 046207;
S.P. Kuznetsov, J. Exp. Theor. Phys. 106 (2008) 380;
S.P. Kuznetsov, A. Pikovsky, Europhys. Lett. 84 (2008) 10013;
S.P. Kuznetsov, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 3487;
S.P. Kuznetsov, Chaos 19 (2009).
[11] S.P. Kuznetsov, A. Pikovsky, Physica D 232 (2007) 87.
[12] N.N. Bogoliubov, Y.A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York, 1961.
[13] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, New York, 1986.
[14] J.L. Kaplan, J.A. Yorke, in: Functional Differential Equations and Approximations of Fixed Point, Lecture in Mathematics, vol. 730, Springer, Berlin, 1979, p. 204;
J.L. Kaplan, J.A. Yorke, Ann. N.Y. Acad. Sci. 316 (1979) 400.
[15] P. Grassberger, I. Procaccia, Physica D 9 (1983) 189;
P. Grassberger, Phys. Lett. A 97 (1983) 227.
[16] M.C. Cross, A. Zumdieck, R. Lifshitz, J.L. Rogers, Phys. Rev. Lett. 93 (2004) 224101;
M.C. Cross, J.L. Rogers, R. Lifshitz, A. Zumdieck, Phys. Rev. E 73 (2006) 036205.
[17] M. Zalalutdinov, A. Zehnder, A. Olkhovets, S. Turner, L. Sekaric, B. Ilic, D. Czaplewski, J.M. Parpia, H.G. Craighead, Appl. Phys. Lett. 79 (2001) 695;
B. Ilic, S. Krylov, K. Aubin, R. Reichenbach, H.G. Craighead, Appl. Phys. Lett. 86 (2005) 193114.
[18] E. Ott, Th. Antonsen, Chaos 18 (2008) 037113.