Complex dynamics of an oscillator ensemble with uniformly distributed natural frequencies and global nonlinear coupling

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We consider large populations of phase oscillators with global nonlinear coupling. For identical oscillators such populations are known to demonstrate a transition from completely synchronized state to the state of self-organized quasiperiodicity. In this state phases of all units differ, yet the population is not completely incoherent but produces a nonzero mean field; the frequency of the latter differs from the frequency of individual units. Here we analyze the dynamics of such populations in case of uniformly distributed natural frequencies. We demonstrate numerically and describe theoretically (i) states of complete synchrony, (ii) regimes with coexistence of a synchronous cluster and a drifting subpopulation, and (iii) self-organized quasiperiodic states with nonzero mean field and all oscillators drifting with respect to it. We analyze transitions between different states with the increase of the coupling strength; in particular we show that the mean field arises via a discontinuous transition. For a further illustration we compare the results for the nonlinear model with those for the Kuramoto-Sakaguchi model.

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I. INTRODUCTION

Collective dynamics of large ensembles of interacting oscillatory units received a lot of attention within last decades [1,2]. However, this subject remains in the focus of attention of many researchers, with an emphasis on such aspects as clustering [3], effects of internal delays [4], effects of external forcing [5] or feedback [6], interaction of several populations [7], inhomogeneity in coupling and appearance of complex dynamical regimes like the so-called chimera states [8,9], search of exact solutions [9–12], etc.

A topic of particular interest is the effect of different frequency distributions [10,12–14]. So, D. Pazó [13] demonstrated that the standard Kuramoto model of phase oscillators [1] with a uniform frequency distribution exhibits a discontinuous synchronization transition with increase of the coupling strength; this transition can be viewed at as a first-order phase transition. Contrary to the typical case of a unimodal distribution, e.g., the Lorentzian one, where the transition is smooth and the synchronous cluster smoothly grows with the increase of the supercritical coupling, in the case, studied by Pazó, in the critical point all oscillators synchronize at once and therefore the nonzero mean field appears by jump.

In this paper we address dynamics of ensembles with a uniform frequency distribution and global coupling that is *nonlinear* in the sense that parameters of the coupling function depend on the amplitude of the mean field [15,16]. Namely, we consider phase oscillators with the sinusoidal coupling and a phase shift which depends on the mean field amplitude [16].

In case of identical oscillators this model exhibits the transition from a fully synchronous state to a self-organized quasiperiodic (SOQ) solution. In the latter state [16], the frequencies of oscillators differ from the frequency of the mean field; the oscillators are not entrained by the field and therefore demonstrate a quasiperiodic dynamics. Although the oscillators are identical, their phases are not locked, but also not completely incoherent. This regime emerges when the system settles at the border of stability of the completely synchronous solution. In case of Lorentzian frequency distribution, the nonlinearity in the coupling results in a nonmonotonic dependence of the mean field amplitude on the coupling strength and in the appearance of multistability [17]. Here we analyze an interesting complex dynamics of the nonlinearly coupled ensemble with a uniform frequency distribution. We demonstrate numerically and describe analytically transitions between states of complete synchrony, states with coexisting synchronous and drifting oscillators, and self-organized quasiperiodic states. As a particular case of our theory we discuss the dynamics of the linear Kuramoto-Sakaguchi model [18].

Our theoretical analysis is based on the Watanabe-Strogatz theory [19] and the Ott-Antonsen ansatz [10,11]. With the help of these analytical tools we derive closed equations for the amplitude and frequency of the mean field and find their stationary solutions.

The paper is organized as follows. Section II introduces the model and presents a numerical demonstration of the main effects. In Sec. III we derive the equations for the complex order parameter and determine the critical parameter values, corresponding to transitions between different states. In Sec. IV we present more numerical results for the nonlinear and the Kuramoto-Sakaguchi model, compare theory with numerics, and discuss our results. The technical details of the derivation of the main equations are given in Appendices.

II. MODEL AND THE MAIN EFFECTS

We consider a system of N globally coupled phase oscillators. Here N is a large number; in fact, we are interested in the dynamics of large oscillator populations and therefore exploit the thermodynamic limit $N \to \infty$ in the forthcoming theoretical analysis. The equations of the model are

$$\dot{\phi}_k = \omega_k + \varepsilon r \sin[\Theta - \phi_k + \beta(\varepsilon, r)], \quad (1)$$

where $k=1,\ldots,N$, ϕ_k and ω_k are phase and natural frequency of the kth oscillator, respectively, ε is the coupling strength, r and Θ are the amplitude and phase of the complex Kuramoto mean field

$$Y = re^{i\Theta} = N^{-1}\sum_{j=1}^{N} e^{i\phi_j},$$

and β is an additional phase shift.

The case β=const yields the well-known Kuramoto-Sakaguchi model [18]. For $|\beta| < \pi/2$ and sufficiently large coupling factor ε, this model exhibits a stable solution with the nonzero mean field Y. The appearance of the nonzero mean field is due to the fact that, in dependence on the distribution of frequencies ω_k, some part of the ensemble, or even all oscillators, adjust their frequencies and form a synchronous cluster.

The case $\beta=\beta(\varepsilon,r)$ corresponds to the recently introduced model of nonlinear global coupling [16]. If β monotonically depends on r, ε and approaches $\pi/2$, i.e., the border of stability of the synchronous solution, with variation of ε, the system exhibits complicated collective dynamics. In the examples treated below we follow [16] and choose for definiteness the phase shift function in the form

$$\beta = \beta_0 + \beta_1 \varepsilon^2 r^2. \quad (2)$$

Moreover, we take $0 < \beta_0 < \pi/2$ and $\beta_1 > 0$; the case of negative β_0 and β_1 can be treated in a similar way.

In the following the frequencies of oscillators ω_k are taken to be uniformly distributed in an interval $[-\delta, \delta]$. Notice that without loss of generality we can take the central frequency of the distribution to be zero; it corresponds to the transformation to a rotating coordinate frame. Next, we note that generally it is not possible to reduce the number of parameters of Eq. (1) by an appropriate rescaling of time because of the function $\beta(\varepsilon,r)$. Although it is possible to perform this reduction for the particular choice of β [see Eq. (2)] by simultaneous rescaling of β_1, in our simulations we fixed $\beta_1=1$ and changed δ. The latter become then a crucial quantity. As we see below, for the linear Kuramoto-Sakaguchi model with β=const the dynamics is independent of δ.

We start by a numerical illustration of the dynamics of the nonlinear model (1) and (2). We simulated the system for $N=1000$, $\beta_0=0.15\pi$, and $\beta_1=1$, for different values of the coupling strength ε, and computed the amplitude r and frequency Ω of the mean field. Next, we computed the frequencies of the slowest (with the natural frequency $\omega=-\delta$) and the fastest (with the natural frequency $\omega=\delta$) oscillators in the ensemble. Due to coupling, their frequencies change; we denote these observed frequencies by ν_{min} and ν_{max}, respectively. The dependencies of r, Ω, ν_{min}, and ν_{max} on ε for $\delta=0.1$ are shown in Fig. 1.

We see that the system undergoes transitions between five different states. For small coupling $\varepsilon \lesssim 0.109$ the system is asynchronous: the mean field is zero (up to finite size fluctuations) and the frequencies of oscillators remain unchanged. When the coupling achieves a critical value

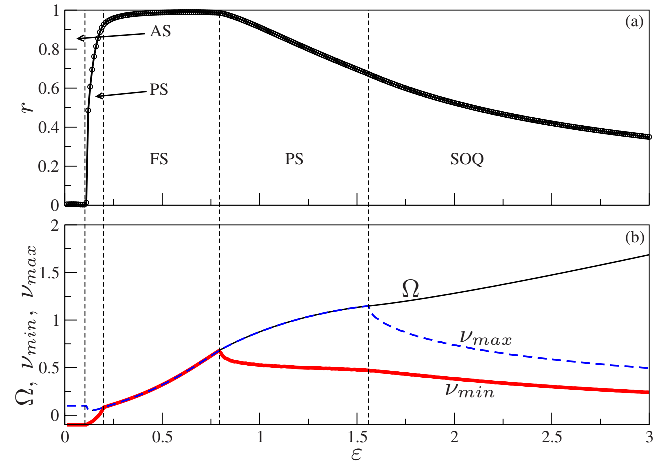

FIG. 1. (Color online) Dynamics of the ensemble with the nonlinear coupling, see Eqs. (1) and (2), for varying coupling strength ε. (a) Mean field amplitude r vs ε. (b) Mean field frequency Ω (black solid line), frequencies of the slowest and fastest oscillators in the population, ν_{min} (red bold line) and ν_{max} (blue dashed line), as functions of ε. Notice that in a large range of ε two or three curves coincide exactly (so that only one is seen); this equality of frequencies ($\Omega=\nu_{max}$ or $\Omega=\nu_{max}=\nu_{min}$) means existence of partial and full synchrony, respectively. The vertical dotted lines separate different dynamical states: asynchrony (AS), partial synchrony (PS), full synchrony (FS), and self-organized quasiperiodicity (SOQ). Width of the frequency distribution δ=0.1, for other parameters see text.

$\varepsilon_{p_1} \approx 0.109$, synchronization, reflected in the amplitude r of the mean field, appears by jump. From the frequency plot Fig. 1(b) we see that the frequency of the fastest oscillator coincides with the frequency of the mean field, whereas the slowest oscillator has a different frequency. Thus, in this regime the subpopulation of the fast oscillators forms a frequency-locked cluster and the slow oscillators are drifting. We denote this regime as the state of *partial synchrony* [20]. This is also illustrated in Fig. 2(a). When the coupling achieves the critical value $\varepsilon_s \approx 0.198$, the regime of *full synchrony* (FS) sets in. From Fig. 1(b) we see that the frequencies of the slowest and of the fastest oscillators now coincide and are equal to the frequency of the mean field. Obviously, now all oscillators form a synchronous cluster. However, though oscillators in this regime have identical observed frequencies, their phases differ due to difference in the natural frequencies, see Fig. 2(b); as a result, the amplitude of the mean field is $r<1$. Full synchrony is preserved until the coupling achieves another critical value $\varepsilon_{p_2} \approx 0.793$. At this coupling strength the slowest oscillator leaves the synchronous cluster, whereas the frequency of the fastest one still equals the mean field frequency Ω. It means that the system is again in the state of partial synchrony [Fig. 2(c)]. With the further increase of ε, more and more oscillators fall out of synchrony what is reflected in the decrease of the mean field amplitude r. The PS regime holds for $\varepsilon_{p_2} < \varepsilon < \varepsilon_q \approx 1.558$. At the critical value ε_q this regime is destroyed: even the fastest oscillator now lags behind the mean field and the whole ensemble is in the state of the *self-organized quasiperiodicity* (SOQ). Although all oscillators in this state have different

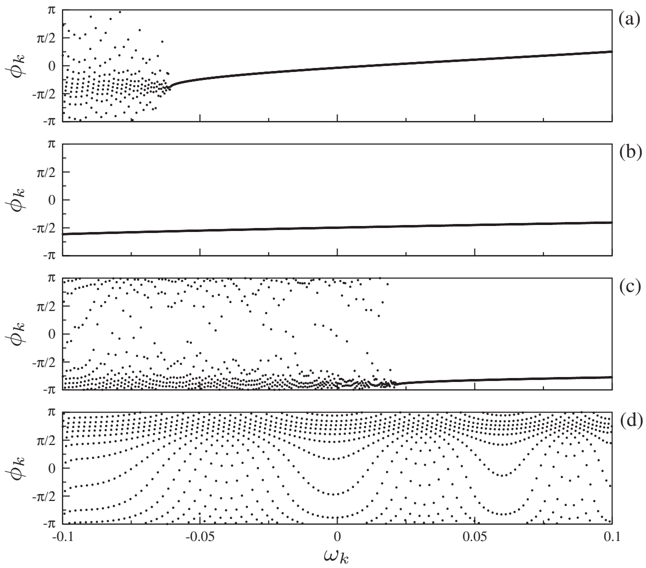

FIG. 2. Snapshots of phases of the ensemble (1) and (2) demonstrate that with the increase of coupling strength the system exhibits transitions between several states. (a) State of partial synchrony, ε=0.15. (b) Fully synchronous state, ε=0.4. (c) Second state of partial synchrony, ε=1.2. (d) Regime of self-organized quiasiperiodicity (SOQ), ε=2. The parameters are same as in Fig. 1.

frequencies, the distribution of their phases is not uniform [Fig. 2(d)], what results in the nonzero mean field.

The picture of the dynamics is essentially different for the semiwidth of the frequency distribution δ=0.5, see Fig. 3. Now we observe a transition from the asynchronous state of the system (r=0) directly to the state of PS and then to SOQ; the regime of FS is absent here. In the following section we develop the theoretical description of the observed behavior and derive the expressions for the critical values of the coupling.

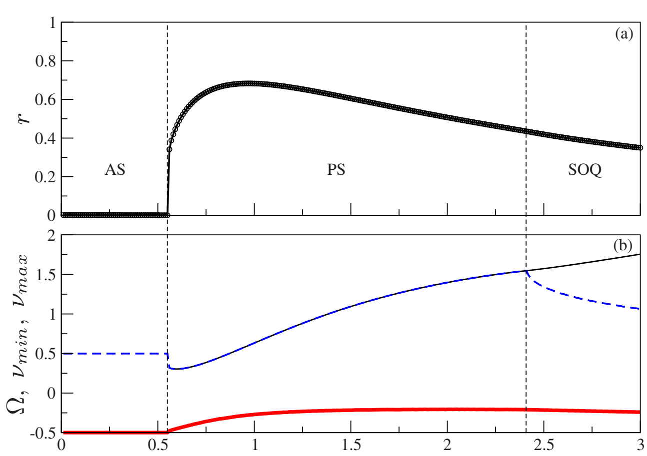

FIG. 3. (Color online) Dynamics of the ensemble with the nonlinear coupling, see Eqs. (1) and (2), for varying coupling strength ε. Width of the frequency distribution δ=0.5, other parameters are same as in Fig. 1. (a) Mean field amplitude r vs ε. (b) Mean field frequency Ω (black solid line), frequencies of the slowest and fastest oscillators in the population, ν_{min} (red bold line) and ν_{max} (blue dashed line), as functions of ε. and ν_{max} (red bold line), as functions of ε. The vertical dotted lines separate different dynamical states: asynchrony (AS), partial synchrony (PS), and self-organized quasiperiodicity (SOQ).

III. THEORY

In this Section we use the Watanabe-Strogatz (WS) theory [19,21], its extension to the case of nonidentical oscillators [17,22]), and the Ott-Antonsen (OA) ansatz [10,11] to derive the equations for the amplitude and phase of the mean field. Next we analyze different regimes and transitions between them.

A. Watanabe-Strogatz equations

For the following it is convenient to rewrite the model (1) as

$$\dot{\phi}_k = \omega_k + \mathrm{Im}(He^{-i\phi_k}), \tag{3}$$

where

$$H = \varepsilon e^{i\beta(\varepsilon,r)}Y \tag{4}$$

is the effective forcing, common for all oscillators. Since we are interested in the dynamics of large ensembles, we exploit the thermodynamical limit $N\to\infty$ for the theoretical treatment. The frequencies of oscillators are now described by the distribution $g(\omega)$ that is $g(\omega)=(2\delta)^{-1}$ for $-\delta\le\omega\le\delta$ and $g(\omega)=0$, otherwise. The distribution of the oscillator phases is described by a density function $w(\omega,\phi,t)$. Now we can introduce the complex *local order parameter*

$$Z(\omega,t) = \int_0^{2\pi} w(\omega,\phi,t)e^{i\phi}d\phi \tag{5}$$

with an obvious relation to the mean field,

$$Y = re^{i\Theta} = \int_{-\delta}^{\delta} g(\omega)Z(\omega,t)d\omega = (2\delta)^{-1}\int_{-\delta}^{\delta} Z(\omega,t)d\omega. \tag{6}$$

The dynamics of the ensemble [Eq. (3)] with a general function H can be efficiently and *exactly* described by means of the WS equations. For an ensemble of *nonidentical* oscillators in the continuous limit $N\to\infty$ these equations read (see [17]),

$$\frac{\partial\rho(\omega,t)}{\partial t} = \frac{1-\rho^2}{2}\mathrm{Re}(He^{-i\Phi}), \tag{7}$$

$$\frac{\partial\Phi(\omega,t)}{\partial t} = \omega + \frac{1+\rho^2}{2\rho}\mathrm{Im}(He^{-i\Phi}), \tag{8}$$

$$\frac{\partial\Psi(\omega,t)}{\partial t} = \frac{1-\rho^2}{2\rho}\mathrm{Im}(He^{-i\Phi}). \tag{9}$$

Here $\rho(\omega,t)$, $\Phi(\omega,t)$, and $\Psi(\omega,t)$ are the modified WS variables. Their relation to the original WS variables and their physical meaning is discussed in details in [17,22]; the meaning will also become clear later on. For an unambiguous description of the ensemble one has to compliment the

WS variables by integrals of motion ψ, having constant in time distribution $\sigma(\omega,\psi)$. The original variables ϕ are related to the new ones ψ, ρ, Φ, Ψ by an invertible transformation [19], see also [17,22],

$$\tan\left(\frac{\phi-\Phi}{2}\right)=\frac{1-\rho}{1+\rho}\tan\left(\frac{\psi-\Psi}{2}\right). \quad (10)$$

We briefly give an idea of how Eqs. (7)–(9) can be derived. The density function $w(\omega,\phi,t)$ obeys the continuity equation which expresses the conservation of the number of oscillators:

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial \phi}(wv)=0, \quad (11)$$

where the velocity $v=\dot{\phi}=\omega+\mathrm{Im}(He^{-i\phi})$ is determined by the microscopic equation of motion. Following Watanabe and Strogatz, we performed a variable substitution

$$t,\phi,\omega \to \tau=t,\psi=\psi(\omega,\phi,t;\rho,\Phi,\Psi)$$

in the continuity equation and demonstrated that the density $w(\omega,\phi,t)$ becomes a stationary distribution $\sigma(\omega,\psi)$, i.e., ψ become constants of motion, provided the macroscopic WS variables obey Eqs. (7)–(9), see [17] for details.

The crucial issue is that the system Eqs. (7)–(9) can be further simplified, if we are interested only in the asymptotic behavior for $t\to\infty$. As argued by Ott and Antonsen, a system of oscillators with global sinusoidal coupling and with a continuous frequency distribution settles for $t\to\infty$ at the so-called reduced manifold [10,11]. We discuss now the reduction to this manifold in terms of the WS theory. First, as shown in [22], if for each frequency ω the integrals of motion ψ are distributed uniformly, then the WS variables ρ and Φ simply coincide with the amplitude and phase of the local mean field

$$\rho(\omega)e^{i\Phi(\omega)}=Z(\omega),$$

and Eqs. (7) and (8) become equivalent to the Ott-Antonsen Eq. [10]:

$$\frac{\partial Z}{\partial t}=i\omega Z+\frac{1}{2}H-\frac{Z^2}{2}H^*.$$

Generally, the time-independent distributions of ψ, unambiguously determined by initial conditions, can be arbitrary. However, as argued in [17], the asymptotic dynamics of the global mean field Y is the same as if the distributions were uniform, provided the frequency distribution is *smooth*. The reason is that computation of Y requires averaging over the frequency distribution [cf. Eq. (5)], and as a result of this averaging the contribution of the inhomogeneities in the distributions of ψ is eliminated, see [17] for a detailed discussion.

Thus, for the asymptotic description of the mean field we make use of the reduction to the Ott-Antonsen manifold and obtain closed system of Eqs. (7), (8), (4), and (6), whereas Eq. (9) decouples. Thus, we are left with the equations

$$\frac{\partial \rho(\omega,t)}{\partial t}=\frac{1-\rho^2}{2}\varepsilon r\cos(\Theta-\Phi+\beta), \quad (12)$$

$$\frac{\partial \Phi(\omega,t)}{\partial t}=\omega+\frac{1+\rho^2}{2\rho}\varepsilon r\sin(\Theta-\Phi+\beta). \quad (13)$$

Since in the following we are looking for the harmonic solution for the mean field with $\Theta=\Omega t$, it is convenient to introduce the frequency dependent phase shift between the phase Φ of the local mean field and the phase Θ of the global one,

$$\alpha(\omega)=\Phi(\omega)-\Theta.$$

As a result, we obtain the closed equation system,

$$\frac{\partial \rho(\omega,t)}{\partial t}=\frac{1-\rho^2}{2}\varepsilon r\cos(\beta-\alpha), \quad (14)$$

$$\frac{\partial \alpha(\omega,t)}{\partial t}=\omega-\Omega+\frac{1+\rho^2}{2\rho}\varepsilon r\sin(\beta-\alpha), \quad (15)$$

$$2\delta r=\int_{-\delta}^{\delta}\rho(\omega)e^{i\alpha(\omega)}d\omega. \quad (16)$$

Below we present an analysis of these equations.

B. Full synchronization

We start by analyzing the regime of full synchrony. In this state we have $\rho(\omega)=1$ and $\dot{\Phi}(\omega)=\Omega$. Hence, $\dot{\alpha}(\omega)=0$ and the system [Eqs. (14)–(16)] yields

$$\Omega=\omega+\varepsilon r\sin(\beta-\alpha), \quad (17)$$

$$2\delta r=\int_{-\delta}^{\delta}e^{i\alpha(\omega)}d\omega. \quad (18)$$

From the first equation we obtain

$$\alpha(\omega)=\beta+\arcsin\left(\frac{\omega-\Omega}{\varepsilon r}\right);$$

this solution exists if

$$|\omega-\Omega|\le\varepsilon r. \quad (19)$$

Substituting α into the second equation, separating real and imaginary parts and solving the corresponding integrals, we obtain two equations for yet unknown r and Ω,

$$\Omega=\varepsilon r^2\sin\beta, \quad (20)$$

$$\frac{4\delta}{\varepsilon}\cos\beta=\arcsin\frac{\Omega+\delta}{\varepsilon r}-\arcsin\frac{\Omega-\delta}{\varepsilon r}+\frac{\Omega+\delta}{\varepsilon r}\sqrt{1-\left(\frac{\Omega+\delta}{\varepsilon r}\right)^2}-\frac{\Omega-\delta}{\varepsilon r}\sqrt{1-\left(\frac{\Omega-\delta}{\varepsilon r}\right)^2}. \quad (21)$$

For given nonlinearity $\beta=\beta(\varepsilon,r)$ these equations shall be solved numerically. We analyze them for the particularly chosen nonlinear function Eq. (2), which includes as a particular case the Kuramoto-Sakaguchi model, and find the critical values of ε_s, r_s, and Ω_s, where the subscript s stands for full synchrony.

Notice that for the chosen nonlinearity and for the chosen signs of β_0 and β_1 we have $\Omega > 0$ [23] and the condition (19) becomes $\Omega + \delta \le \varepsilon r$. Thus, the critical condition for the onset of the full synchrony is

$$\Omega_s + \delta = \varepsilon_s r_s. \tag{22}$$

1. Kuramoto model

Before proceeding with the analysis of the nonlinear model, we check the obtained result for the simplest case of the Kuramoto model where $\beta=0$. For this case Eq. (20) yields $\Omega=0$ and Eq. (21) becomes an equation for r (for given ε and δ),

$$\arcsin\frac{\delta}{\varepsilon r} + \frac{\delta}{\varepsilon r}\sqrt{1-\frac{\delta^2}{\varepsilon^2 r^2}} = \frac{2\delta}{\varepsilon}. \tag{23}$$

This equation has been derived by Pazó [13]. Its solution exists if $\varepsilon r \ge \delta$. Thus, the critical value of the coupling ε_s and the corresponding mean field amplitude r_s are determined from the condition $\delta = \varepsilon_s r_s$ [cf. Eq. (22) with $\Omega_s = 0$]. Substituting this into Eq. (23) we obtain the result of Pazó [13], i.e., $r_s = \pi/4$.

2. Nonlinear model

Now we consider $\beta = \beta_0 + \beta_1 \varepsilon^2 r^2 > 0$. Substituting Eq. (22) into Eq. (21) we obtain

$$\frac{\pi}{2} + \arcsin\left(2\frac{\delta}{\varepsilon_s r_s} - 1\right) + 2\left(2\frac{\delta}{\varepsilon_s r_s} - 1\right)\sqrt{\frac{\delta}{\varepsilon_s r_s} - \frac{\delta^2}{\varepsilon_s^2 r_s^2}} = 4\frac{\delta}{\varepsilon_s}\cos\beta_s, \tag{24}$$

where $\beta_s = \beta_0 + \beta_1 \varepsilon_s^2 r_s^2$. It is convenient to introduce $x = \delta/\varepsilon_s r_s$; obviously, $0 \le x \le 1$. (Taking into account the stability condition $\beta \le \pi/2$, we obtain $x^2 \ge \beta_1 \delta^2/(\pi/2 - \beta_0)$.) Using Eqs. (20) and (22) to write the critical condition as $\Omega_s + \delta = \varepsilon_s r_s^2 \sin\beta_s + \delta = \varepsilon_s r_s$, we obtain

$$r_s = \frac{\varepsilon_s r_s - \delta}{\varepsilon_s r_s \sin\beta_s} = \frac{1-x}{\sin(\beta_0 + \beta_1 \delta^2/x^2)} \tag{25}$$

and

$$\Omega_s = \varepsilon_s r_s - \delta = \frac{1-x}{x}\delta. \tag{26}$$

Equation (24) becomes now a closed equation for x

$$\pi + 2\arcsin(2x-1) + 4(2x-1)\sqrt{x-x^2} = 8x(1-x)\cot(\beta_0 + \beta_1 \delta^2/x^2). \tag{27}$$

Solving this equation numerically for given β_0, β_1 we obtain r_s and Ω_s via Eqs. (25) and (26) and the critical value of the coupling as

$$\varepsilon_s = \frac{\delta}{x r_s} = \frac{\delta \sin(\beta_0 + \beta_1 \delta^2/x^2)}{x(1-x)}. \tag{28}$$

C. SOQ state

From the theoretical analysis of identical nonlinearly coupled oscillators [16] we can expect to observe a state when all oscillators are not locked by the mean field, i.e., the SOQ state. This expectation is also confirmed by the numerical simulation presented above. For the chosen nonlinearity, the mean field frequency in this regime is larger than frequencies of all oscillators. Hence, fully unlocked state appears when the fastest oscillator falls out of the synchrony. It means that Eq. (15) does not any more possesses a solution $\alpha = \text{const}$ for $\omega = \delta$ and $\rho = 1$. It is easy to see that this solution is lost if $\Omega - \delta \ge \varepsilon r$, i.e., the critical condition is

$$\Omega_q - \delta = \varepsilon_q r_q. \tag{29}$$

Now we look for the stationary solution $\dot{\rho}=0$, with $\rho < 1$. Equation (14) yields

$$\beta(\varepsilon, r) - \alpha = \pm\frac{\pi}{2}. \tag{30}$$

We see that in this case α is also constant in time. Hence, we obtain from Eq. (15)

$$\Omega - \omega = \pm\frac{1+\rho^2}{2\rho}\varepsilon r. \tag{31}$$

Since we consider for definiteness the case $\Omega > \omega$, in the following we choose the plus sign in Eqs. (30) and (31). Solving Eq. (31) for ρ, we obtain

$$\rho = \frac{\Omega - \omega - \sqrt{(\Omega-\omega)^2 - \varepsilon^2 r^2}}{\varepsilon r}. \tag{32}$$

(We ignore the second root because it yields $\rho \ge 1$.)

The self-consistency condition for the mean field [Eq. (16)] takes the form:

$$2\delta r = \frac{e^{i(\beta - \pi/2)}}{\varepsilon r}\int_{-\delta}^{\delta} \Omega - \omega - \sqrt{(\Omega-\omega)^2 - \varepsilon^2 r^2}\, d\omega, \tag{33}$$

or

$$2\delta\varepsilon r^2 e^{-i\beta} = -i(2\delta\Omega - I), \tag{34}$$

where

$$I = \int_{-\delta}^{\delta} \sqrt{(\Omega-\omega)^2 - \varepsilon^2 r^2}\, d\omega; \tag{35}$$

the exact expression for the integral is given in Appendix A [see Eq. (A1)].

Separating the real and the imaginary parts we obtain

$$\cos\beta(\varepsilon, r) = 0, \tag{36}$$

$$2\delta\varepsilon r^2 \sin\beta(\varepsilon, r) = 2\delta\Omega - I. \tag{37}$$

For positive β the first equation simplifies to

$$\beta(\varepsilon,r)=\frac{\pi}{2}, \tag{38}$$

it yields the mean field amplitude for given coupling strength. Remarkably, this equation does not contain δ and coincides with the corresponding equation, derived for the ensemble of *identical* nonlinearly coupled oscillators in the SOQ state [16]. As shown below, the width of the frequency distribution determines only the critical value of the coupling, when this regime sets in. With account of Eq. (36), Eq. (37) takes the form

$$2\delta(\Omega-\varepsilon r^2)=I, \tag{39}$$

see Eq. (A2) for an explicit expression. For given ε, δ, Eqs. (38) and (39) can be solved numerically to yield the mean field amplitude r and frequency Ω. The critical values, corresponding to the onset of the fully unlocked state, can be found with the help of the condition Eq. (29).

For our particular choice of nonlinearity $\beta=\beta_0+\varepsilon^2 r^2$, Eq. (38) simplifies to

$$\varepsilon r=\sqrt{\frac{\pi}{2}-\beta_0} \tag{40}$$

and the corresponding particular case of Eq. (39) is given in Appendix A [see Eq. (A2)]. The critical condition Eq. (29) immediately yields the critical value of the frequency:

$$\Omega_q=\delta+\sqrt{\pi/2-\beta_0}. \tag{41}$$

The critical value of the coupling is given by Eq. (A4).

D. Partially locked state

Now we are ready to describe the most complicated regime when some part of the population is locked to the mean field whereas the other part is not. Since we consider the case when $\beta>0$ and therefore $\Omega>0$, we conclude that unlocked are the oscillators with the natural frequencies

$$-\delta\leq\omega<\delta_b. \tag{42}$$

Here δ_b is yet unknown parameter which determines the border between synchronized and asynchronous subpopulations. Correspondingly, the natural frequencies of the synchronous group satisfy

$$\delta_b\leq\omega<\delta. \tag{43}$$

Parameter δ_b is determined from the condition

$$\Omega-\delta_b=\varepsilon r. \tag{44}$$

The self-consistency Eq. (16) can be written with the help of the corresponding equations for the fully locked and fully unlocked states, obtained in the previous Sections. It now becomes:

$$2\delta r=\frac{e^{i(\beta-\pi/2)}}{\varepsilon r}\int_{-\delta}^{\delta_b}\Omega-\omega-\sqrt{(\Omega-\omega)^2-\varepsilon^2 r^2}d\omega + e^{i\beta}\int_{\delta_b}^{\delta}e^{i\arcsin\omega-\Omega/\varepsilon r}d\omega. \tag{45}$$

Here the first (second) integral describes the contribution of the unlocked (locked) subpopulation. In Appendix B we present the reduction of this equation to a system of two transcendental equations for the amplitude r and frequency Ω of the mean field [see Eqs. (B3) and (B4)].

In order to find critical values of parameters we note that the borders of this regime correspond to $\delta_b=\mp\delta$, what yields $\Omega_p-\varepsilon_p r_p=\mp\delta$. This equation, together with Eqs. (B3) and (B4), constitutes an equation system for computation of ε_p, r_p, and Ω_p. Introducing variables $x=(\delta-\Omega)/\varepsilon r$ and $y=(\delta+\Omega)/\varepsilon r$, we obtain from Eqs. (B3) and (B4),

$$\frac{4\delta}{\varepsilon}\cos\beta=x\sqrt{1-x^2}+\arcsin x+\frac{\pi}{2}, \tag{46}$$

$$x^2+\frac{4\delta}{\varepsilon}\sin\beta=y^2-y\sqrt{y^2-1}-\ln(y-\sqrt{y^2-1}), \tag{47}$$

with obvious transformations

$$\varepsilon r=\frac{2\delta}{y+x}, \tag{48}$$

$$\Omega=\delta\cdot\frac{y-x}{y+x}, \tag{49}$$

$$\beta=\beta_0+\left(\frac{2\delta}{y+x}\right)^2. \tag{50}$$

Obviously, the solution of these equations exist for $x^2\leq 1$ and $y^2\geq 1$. Notice also that x, y are positive. Hence we obtain two critical conditions: $x=1$ and $y=1$. The latter one determines the point where the partial synchrony becomes the full synchrony [cf. condition (22)] and yields the already known solution, given by Eqs. (25)–(28). The former condition $x=1$ or, equivalently, $\Omega_p=\delta-\varepsilon_p r_p$, corresponds to the point where partially synchronous state emerges from the fully asynchronous state. Using this condition, we obtain from Eq. (46),

$$\varepsilon_p=\frac{4\delta\cos\beta(y_p)}{\pi}, \tag{51}$$

where y_p is the solution of the transcendental Eq. (47),

$$\pi\tan\beta(y_p)=y_p^2-1-y_p\sqrt{y_p^2-1}-\ln(y_p-\sqrt{y_p^2-1}). \tag{52}$$

Solving this equation and finding the value of y_p, we can calculate the remaining critical values r_p and Ω_p from Eqs. (48) and (49).

IV. RESULTS AND DISCUSSION

In this Section we present more numerical results and compare them with the developed theory. We start by a relatively simple particular case $\beta_1=0$, $\beta_0\neq 0$, when the model (1) and (2) reduces to the Kuramoto-Sakaguchi model with β=const.

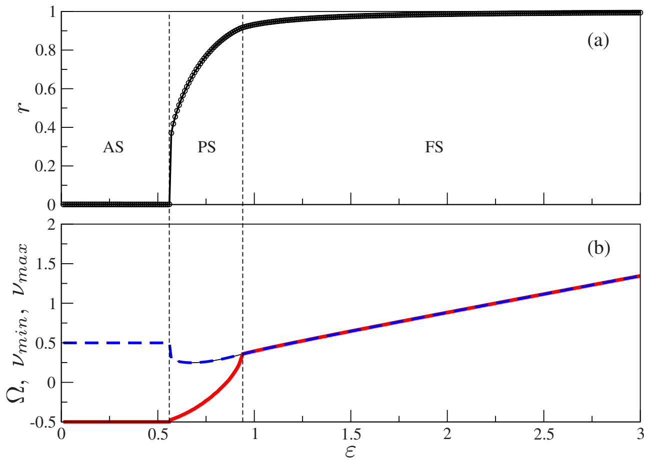

FIG. 4. (Color online) Dynamics of the Kuramoto-Sakaguchi model for varying coupling strength ε; width of the frequency distribution δ=0.5, β_0=0.15π. (a): Mean field amplitude r vs ε. (b): Mean field frequency Ω (bold black line), frequencies of the slowest and fastest oscillators in the population, ν_{min} (red bold line) and ν_{max} (blue dashed line), as functions of ε and ν_{max} (solid red line), as functions of ε. (Here Ω and ν_{max} overlap.)

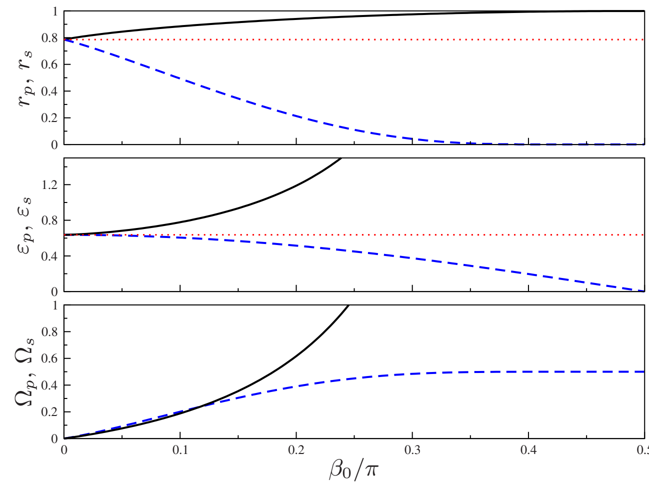

FIG. 5. (Color online) Critical parameters for the Kuramoto-Sakaguchi model in dependence on β_0, for the width of the frequency distribution δ=0.5. In all panels black solid lines correspond to the critical values for full synchrony, blue dashed lines correspond to the critical values for partial synchrony. Horizontal dotted lines in two upper panels show the corresponding values for the Kuramoto model: $r_s=\pi/4$, $\varepsilon_s=4\delta/\pi$.

A. Kuramoto-Sakaguchi model

For this model only two nontrivial states are possible: (i) the regime of full synchrony, when all oscillators are locked to the same frequency but have different phases due to difference of natural frequencies and (ii) regime of partial synchrony, when a subpopulation forms a synchronous cluster and the rest of oscillators are asynchronous, see Fig. 4.

Now we find the critical parameters, starting with those for the state of full synchrony. Expressing from Eq. (25) $x=1-r_s \sin\beta_0$ and substituting it into Eq. (27), we obtain a closed equation for r_s

$$\pi + 2 \arcsin(1 - 2r_s \sin \beta_0) + 4(1 - 2r_s \sin \beta_0)\sqrt{(1 - r_s \sin \beta_0)r_s \sin \beta_0} = 8(1 - r_s \sin \beta_0)r_s \cos \beta_0, \tag{53}$$

which now does not contain δ. Equations (26) and (28) simplify to

$$\varepsilon_s = \frac{\delta}{r_s(1 - r_s \sin \beta_0)}, \quad \Omega_s = \varepsilon_s r_s - \delta = \frac{\delta r_s \sin \beta_0}{1 - r_s \sin \beta_0}. \tag{54}$$

For determination of the critical parameters for the partial synchrony we use Eqs. (46) and (47). Using the condition $x=1$ and $\beta=\beta_0$=const we have

$$\varepsilon_p = \frac{4\delta \cos \beta_0}{\pi}. \tag{55}$$

Note that in this case, Eq. (52) does not depend on δ. Solving the derived equations numerically, we obtain the critical parameters for given β_0, the corresponding dependencies are shown in Fig. 5. In particular, for the example, shown in Fig. 4, the theory yields the critical values ε_p=0.5672, r_p=0.3445, Ω_p=0.3046, and ε_s=0.9346, r_s=0.9155, Ω_s=0.3556, in a good correspondence with the direct numerical simulation. We note, that except for the case of the Kuramoto model, β_0=0, (i) the mean field emerges discontinuously and (ii) transition to FS occurs via PS. With $\beta_0 \to \pi/2$, ε_p and r_p also tend to zero, but the synchronization transition remains a first-order transition. Naturally, $\varepsilon_s \to \infty$ when β_0 approaches the border of stability of the synchronous solution.

B. Nonlinear model

We first discuss the effect of the width of the frequency distribution. Numerics demonstrate, that for β_0=0.15π and $\delta \lesssim 0.1858$ Eq. (27) has two roots, whereas for large δ it has no roots. We remind that these roots correspond to the borders of the fully synchronous state. This agrees with the simulation results presented in Figs. 1 and 3: if the frequency distribution is relatively small, then full synchrony is observed, and the borders of this regime are given by the solution of Eq. (27), otherwise only PS and SOQ states are possible.

The equations, derived in the previous Section, nicely describe the numerical results. To illustrate this, we present the theoretically obtained critical values for β_0=0.15π and δ=0.1 and δ=0.5 in Table I. Finally, we mention that, for β_0=0 and δ=0 the first domain of PS is absent: like in case of the Kuramoto model, the system immediately transits to FS. With the further increase of the coupling strength, FS gets destroyed due to nonlinearity.

C. Conclusions

We have numerically and theoretically analyzed the dynamics of a large oscillator population with nonlinear coupling and uniform distribution of frequencies. We have

TABLE I. Critical values for the nonlinear model with $\beta_0=0.15\pi$: comparison of theoretical and numerical results.

Regimes		$\delta=0.1$ Theory	$\delta=0.1$ Simulation	$\delta=0.5$ Theory	$\delta=0.5$ Simulation
	ε_{p1}	0.114	0.113	0.558	0.558
PS1	r_{p1}	0.343	0.341	0.317	0.319
	Ω_{p1}	0.061	0.061	0.323	0.322
	ε_s	0.195	0.199		
FS	r_s	0.921	0.926		
	Ω_s	0.080	0.083		
	ε_{p2}	0.797	0.790		
PS2	r_{p2}	0.985	0.984		
	Ω_{p2}	0.685	0.676		
	ε_q	1.560	1.560	2.416	2.410
SOQ	r_q	0.672	0.672	0.434	0.435
	Ω_q	1.149	1.149	1.548	1.547

shown, that with increase of the coupling strength the system undergoes several transitions, exhibiting states of full synchrony, partial synchrony and self-organized quasiperiodicity. In the latter state all oscillators have different frequencies which span the interval ν_{min}, ν_{max}, whereas the frequency of the mean field lies outside of this interval. For the considered positive values of β_0 and β, this state emerges when $\beta(\varepsilon,r)$ achieves $\pi/2$; in this case we have $\Omega>\nu_{max}$. Similarly, SOQ state emerges if negative values are considered and $\beta(\varepsilon,r)$ achieves $-\pi/2$. In this case $\Omega<\nu_{min}$. Interesting, the dependence of the mean field amplitude on the coupling in the SOQ state does not depend on the width of the frequency distribution and therefore coincides with that for identical oscillators [16].

Appearance of SOQ in ensemble of nonidentical oscillators is due to combination of two aspects of our model: nonlinearity and frequency distribution with a finite support. In case of infinite distribution, e.g., a Lorentzian one, the nonlinearity results in shift of the mean field frequency, but synchronous cluster always exists. Only in case of distribution with a finite support the frequency of the mean field can differ from the frequency of all oscillators. We believe that our results obtained for a uniform frequency distribution, remain quantitatively valid for other distributions with a finite support. This belief is supported by numerical simulation of the ensemble dynamics for the following frequency distribution:

$$g(\omega)=\begin{cases}\dfrac{\pi}{2}\cos\dfrac{\pi(\omega-\delta)}{2\delta}, & \omega\in[-\delta,\delta],\\ 0 & \text{otherwise}.\end{cases} \qquad (56)$$

The results of this simulation are presented in Fig. 6 which is qualitatively similar to Fig. 1.

An important manifestation of the uniform frequency distribution is the discontinuous synchronization transition. Remarkably, this property is rather robust and is observed not only for the Kuramoto model [13], but also for the nonlinear one.

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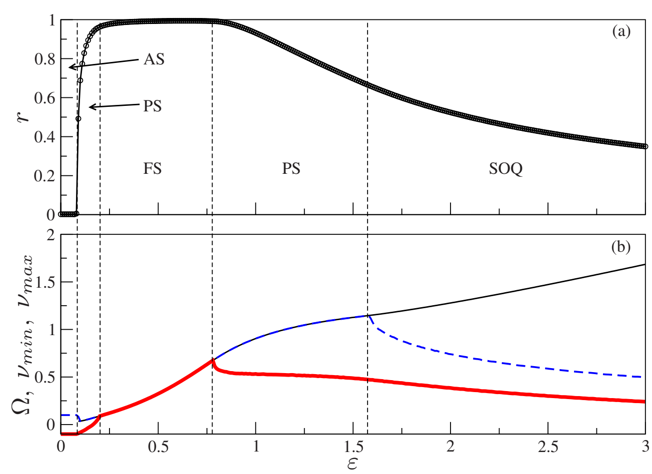

FIG. 6. (Color online) Dynamics of the system with the nonuniform frequency distribution, see Eq. (56), and $\delta=0.1$, to be compared with Fig. 1. Mean field amplitude r vs ε. (b) Mean field frequency Ω (black solid line), frequencies of the slowest and fastest oscillators in the population, ν_{min} (red bold line) and ν_{max} (blue dashed line), as functions of ε. The vertical dotted lines separate different dynamical states: asynchrony (AS), partial synchrony (PS), full synchrony (FS), and self-organized quasiperiodicity (SOQ). Note the differencies with the results of Fig. 1: (i) the synchronization transition is smooth, like for any other frequency distribution with a maximum; (ii) due to the same reason, the synchronous cluster in the first PS state is formed around the center of the distribution.

APPENDIX A: AUXILIARY EQUATIONS FOR THE FULLY UNLOCKED STATE

Computation of the integral [Eq. (35)] yields

$$I=\int_{-\delta}^{\delta}\sqrt{(\Omega-\omega)^2-\varepsilon^2r^2}d\omega=\frac{1}{2}[(\Omega+\delta)\sqrt{(\Omega+\delta)^2-\varepsilon^2r^2}-(\Omega-\delta)\sqrt{(\Omega-\delta)^2-\varepsilon^2r^2}]+\frac{\varepsilon^2r^2}{2}\ln\left[\frac{(\Omega+\delta)-\sqrt{(\Omega+\delta)^2-\varepsilon^2r^2}}{(\Omega-\delta)-\sqrt{(\Omega-\delta)^2-\varepsilon^2r^2}}\right]. \tag{A1}$$

Equation (39) becomes

$$\frac{4\delta}{\varepsilon}=\frac{4\delta\Omega}{\varepsilon^2r^2}+\frac{\Omega-\delta}{\varepsilon r}\sqrt{\left(\frac{\Omega-\delta}{\varepsilon r}\right)^2-1}-\frac{\Omega+\delta}{\varepsilon r}\sqrt{\left(\frac{\Omega+\delta}{\varepsilon r}\right)^2-1}+\ln\left(\frac{(\Omega-\delta)-\sqrt{(\Omega-\delta)^2-\varepsilon^2r^2}}{(\Omega+\delta)-\sqrt{(\Omega+\delta)^2-\varepsilon^2r^2}}\right). \tag{A2}$$

For the chosen nonlinearity $\beta=\beta_0+\varepsilon^2r^2$, Eq. (39) becomes

$$4\delta\left(\Omega-\frac{\pi/2-\beta_0}{\varepsilon}\right)=(\Omega+\delta)\sqrt{(\Omega+\delta)^2+\beta_0-\pi/2}-(\Omega-\delta)\sqrt{(\Omega-\delta)^2+\beta_0-\pi/2}+\left(\frac{\pi}{2}-\beta_0\right)\ln\left[\frac{(\Omega+\delta)-\sqrt{(\Omega+\delta)^2+\beta_0-\pi/2}}{(\Omega-\delta)-\sqrt{(\Omega-\delta)^2+\beta_0-\pi/2}}\right], \tag{A3}$$

which is an equation for Ω. Substituting here $\Omega_q=\delta+\sqrt{\gamma}$, $\gamma=\pi/2-\beta_0$ we find the critical value for the coupling

$$\varepsilon_q^{-1}=\frac{\delta+\sqrt{\gamma}}{\gamma}-\frac{2\delta+\sqrt{\gamma}}{2\delta\gamma}\sqrt{\delta^2+\delta\sqrt{\gamma}}+\frac{1}{8\delta}[\ln\gamma-4\ln(\sqrt{\delta+\sqrt{\gamma}}-\sqrt{\delta})]. \tag{A4}$$

APPENDIX B: AUXILIARY EQUATIONS FOR THE PARTIALLY LOCKED STATE

Separating real and imaginary parts of Eq. (45), we obtain

$$2\delta r\cos\beta=\int_{\delta_b}^{\delta}\sqrt{1-\left(\frac{\omega-\Omega}{\varepsilon r}\right)^2}d\omega, \tag{B1}$$

$$2\delta\varepsilon r^2\sin\beta=-\int_{-\delta}^{\delta_b}\sqrt{(\Omega-\omega)^2-\varepsilon^2r^2}d\omega-\int_{-\delta}^{\delta}(\omega-\Omega)d\omega. \tag{B2}$$

Computing the integrals and using $\delta_b=\Omega-\varepsilon r$, we obtain after straightforward manipulations,

$$\frac{4\delta}{\varepsilon}\cos\beta=\frac{\delta-\Omega}{\varepsilon r}\sqrt{1-\left(\frac{\delta-\Omega}{\varepsilon r}\right)^2}+\arcsin\frac{\delta-\Omega}{\varepsilon r}+\frac{\pi}{2}, \tag{B3}$$

$$2\delta\varepsilon r^2\sin\beta=2\delta\Omega-\frac{\varepsilon^2r^2}{2}\left[\frac{\Omega+\delta}{\varepsilon r}\sqrt{\left(\frac{\Omega+\delta}{\varepsilon r}\right)^2-1}+\ln\left(\frac{\Omega+\delta}{\varepsilon r}-\sqrt{\left(\frac{\Omega+\delta}{\varepsilon r}\right)^2-1}\right)\right]. \tag{B4}$$

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