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# Effective phase description of noise-perturbed and noise-induced oscillations

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**Abstract.** An effective dynamical description of a general class of stochastic phase oscillators is presented. For this, the effective phase velocity is defined either by the stochastic phase oscillators invariant probability density or its first passage times. Using the first approach the effective phase exhibits the correct frequency and invariant distribution density, whereas the second approach models the proper phase resetting curve. The discrepancy of the effective models is most pronounced for noise-induced oscillations and is related to non-monotonicity of the stochastic phase variable due to fluctuations.

## 1 Introduction

The phase is a theoretical concept lying at the heart of the description of oscillatory dynamics. In a physical context, a periodic phase variable provides a simplified description of self-sustained oscillations shown by natural, synthetic, or mathematical systems in their state space [1,2]. A phase description contains many characterizing physical properties associated to the oscillatory motion as, for example, its mean frequency and regularity of oscillations. Most importantly, a smooth or impulsive coupling of interacting oscillators may be described in terms of their phase dynamics [1,3,4].

Irregular features, interpreted as the effects of noise, may be present in oscillations. In many situations, the oscillatory dynamics is only noise-perturbed. In this case, one can start with a phase description of noiseless oscillations, from which noise can be taken into account perturbatively [5,6]. However, noise may be substantial in the sense that it induces oscillations in the system which equilibrates otherwise [7]. Such noise-induced oscillations may be quite coherent so as to disguise the excitable nature of the underlying system [8]. Therefore, it may not be possible to distinguish between the two cases in an experimental setup unless the noise amplitude is controllable.

Despite of the evident similarities between noise-perturbed and noise-induced oscillations, a phase description of noise-induced oscillations cannot be obtained through standard perturbative procedures, because in the noise-free situation there is no dynamics. Recently, this problem was addressed by means of an effective description, where noise-induced oscillations driven by an external periodic force could be characterized as self-sustained ones [9]. It was shown that the effective phase model relies on a notion of speed given by the current velocity [10].

The goal of this paper is to generalize the method described in Ref. [9]. For this, we show that an alternative effective phase model is possible and compare the two approaches. The alternative model is shown to be related to a certain concept of stochastic phase resetting.

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This paper is organized as follows. After a summary of well-known statistical properties of stochastic phase oscillators in the next section, we outline the "current model"[1] of effective phase theory by repeating the results of Ref. [9] (Sec. 3). Hereon, an alternative effective phase model based on first passage times is presented in Sec. 4, which models other features of the stochastic oscillations related to their phase resetting curve. In Sec. 5, mathematical aspects of the theory in its limiting cases of weak and strong noise are discussed highlighting differences and similarities of the effective phase models. Special emphasis is put on the description of noise-induced oscillations in these limits. In Sec. 6, the phase resetting curve of stochastic oscillations and its relation to the first passage model of effective phase theory is described in detail.

## 2 Stochastic phase oscillators and their effective description

In this section, we present some well-known results about stochastic phase oscillators in the context of noise-induced and noise-perturbed oscillations. The section is concluded with a discussion on the notion of velocity.

Our basic model of stochastic oscillations is a one-dimensional stochastic phase oscillator that, for example, may have been obtained by the phase reduction method applied to a high dimensional oscillating system [1,5]. The basic model is described by a $2\pi$-periodic random variable $\theta(t)$, called stochastic *protophase*, which is defined on the circle. Let us consider the *unwound* protophase that is not taken modulo $2\pi$: An *oscillation* is defined as the growth[2] of $\theta(t)$ by $2\pi$.

The protophase obeys the Langevin dynamics (in the Stratonovich interpretation [11])

$$\dot{\theta}(t) = h(\theta(t)) + g(\theta(t))\xi(t), \tag{1}$$

where $\xi(t)$ is $\delta$-correlated Gaussian noise $\langle \xi(t)\xi(t') \rangle = 2\delta(t-t')$. If Eq. (1) is obtained by phase reduction, the *deterministic part* $h(\theta)$ and the *noise amplitude* $g(\theta)$ reflect to a certain degree a specific choice of parameterization of oscillations and are thus not *invariant* under coordinate transformations of the high dimensional oscillating system. For a strictly positive deterministic part $h(\theta)$, the oscillator (1) can show *noise-perturbed* oscillations that persist with finite frequency if the noise amplitude vanishes. However, if $h(\theta)$ has at least two zero crossings, oscillations cease as the noise amplitude vanishes: They are *noise-induced*.

A well-known example showing the most prominent features of stochastic oscillations is the stochastic *Adler equation* [12] (generalized theta model [13])

$$\dot{\theta}(t) = a + \cos\theta + \sigma\xi(t). \tag{2}$$

For $|a| \leq 1$ it shows noise-induced oscillations, whereas for $|a| > 1$ oscillations are noise-perturbed. Furthermore, the Adler equation belongs to the subclass of stochastic phase models with *additive* noise, for which the noise amplitude $g(\theta)$ is a constant $\sigma$.

For oscillator (1), the most relevant quantities are accessible analytically. The probability density $P(\theta)$ is governed by the Fokker-Planck equation associated to Eq. (1), which is given by

$$\partial_t P = -\partial_\theta [hP] + \partial_\theta [g\partial_\theta [gP]] = -\partial_\theta J. \tag{3}$$

For the stationary probability density, the *probability flux* $J$ is constant and we obtain the simpler equation

$$J = hP - g\partial_\theta [gP]. \tag{4}$$

It has the well-known solution [11]

$$P(\theta) = C \int_\theta^{2\pi+\theta} \frac{d\psi}{g(\theta)g(\psi)} \, \mathrm{e}^{-\int_\theta^\psi \frac{h(\varphi)}{g^2(\varphi)}\, d\varphi}, \tag{5}$$

[1] The naming "current model/velocity" is derived from the probability current of the Fokker-Planck equation and is well-established in the field of stochastic mechanics [10].

[2] A phase can always be defined such that it has a non-negative growth on average.

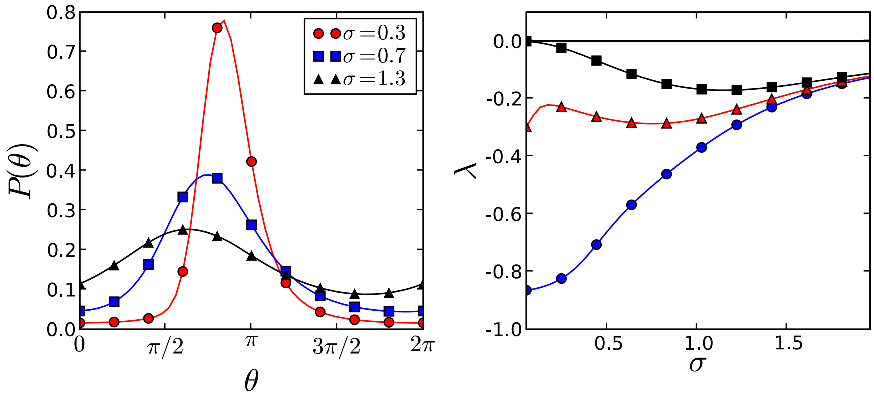


**Fig. 1.** *Left*: the probability density (5) of the Adler equation becomes singular as the noise amplitude vanishes if the oscillator shows noise-induced oscillations (here: $a = 0.9$ and $\sigma$ as indicated). *Right*: dependent on $a$, the Lyapunov exponent (9) of the Adler equation as a function of $\sigma$ shows three characteristic dependencies, here shown for noise-induced oscillations at $a = 0.5$ (blue circles), and $a = 0.95$ (red triangles), and for noise-perturbed oscillations at $a = 1.5$ (black squares).

where $C$ is a *normalization constant* ensuring $\int_0^{2\pi} P(\theta)\, d\theta = 1$. If oscillations are noise-induced, the probability density becomes singular as the noise amplitude vanishes (left plot of Fig. 1).

Among the traditional quantities of stochastic phase oscillators are the mean frequency and the diffusion coefficient

$$\omega = \lim_{t\to\infty} \frac{\langle \theta(t) \rangle}{t}, \text{ and } D = \lim_{t\to\infty} \frac{\langle [\theta(t) - \omega t]^2 \rangle}{2t}, \tag{6}$$

which are defined for the unwound protophase. They are expressed in terms of $h(\theta)$ and $g(\theta)$ through the well-known formulas [11,14–16]

$$\omega = 2\pi J = 2\pi C \left[1 - \mathrm{e}^{-\int_0^{2\pi} \frac{h(\varphi)}{g^2(\varphi)}\, d\varphi}\right], \quad \text{and} \tag{7}$$

$$D = \frac{\frac{1}{2\pi}\int_0^{2\pi} \frac{d\psi}{g(\psi)} \left[\int_{\psi-2\pi}^{\psi} \frac{d\varphi}{g(\varphi)} \rho(\varphi,\psi)\right]^2 \int_{\psi}^{\psi+2\pi} \frac{d\varphi}{g(\varphi)} \rho(\psi,\varphi)}{\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{d\psi}{g(\psi)} \int_{\psi-2\pi}^{\psi} \frac{2d\varphi}{g(\varphi)} \rho(\varphi,\psi)\right]^3}, \tag{8}$$

where $\rho(\theta,\varphi) = \exp[-\int_\theta^\varphi \frac{h(\eta)}{g^2(\eta)}\, d\eta]$. For noise-induced oscillations, the mean frequency converges to zero in the limit of vanishing noise amplitude (left plot of Fig. 2). For additive noise, the mean frequency converges to a finite value in the limit $\sigma \to \infty$ (cf. Sec. 5). The quotient of the diffusion coefficient and the mean frequency is a measure for decoherence of oscillations and inversely proportional to the Péclet number [14,16]. For noise-induced oscillations, the well-known effect of coherence resonance is observable where decoherence decreases for increasing noise amplitude [8] (right plot of Fig. 2). The effect is not observable for noise-perturbed oscillations. For a detailed discussion of coherence resonance the reader is referred to Ref. [7].

Another interesting quantity that characterizes stochastic oscillations is the Lyapunov exponent $\lambda$ associated to noise [17–20]. It quantifies whether two copies of oscillator (1) under the influence of the same noise representation $\xi(t)$ will synchronize by stochasticity. For oscillator (1), $\lambda$ is computed by [18,21]

$$\lambda = \langle h'(\theta) + g''(\theta) g(\theta) \rangle. \tag{9}$$

If noise is additive, $\lambda$ vanishes in the limit of large noise amplitude $\sigma$. In the limit $\sigma \to 0$, there are three cases (right plot of Fig. 1): For noise-perturbed oscillations, the Lyapunov exponent goes to zero as $\lambda \propto -\sigma^2$ [21]. For noise-induced oscillations, it converges to a finite negative

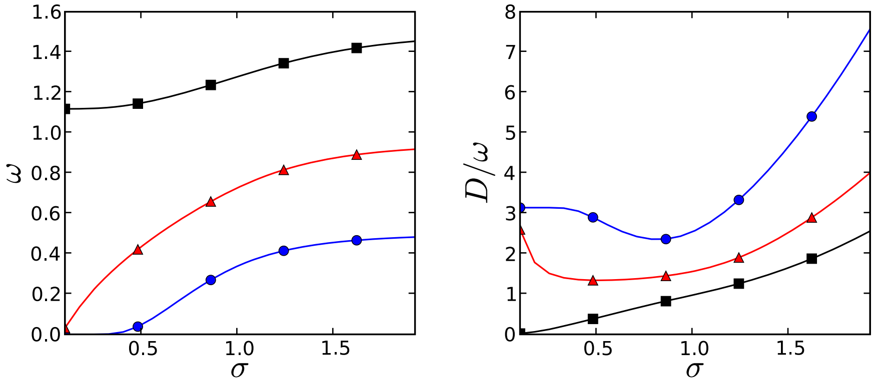


**Fig. 2.** *Left plot*: the mean frequency (7) goes to zero for noise-induced oscillations (triangles and squares) in the Adler equation in the limit of weak noise. *Right plot*: the quotient $D/\omega$ as a function of $\sigma$ is a measure of decoherence which may have negative slope for noise-induced oscillations (blue circles: $a = 0.5$, red triangles: $a = 0.95$) in the Adler equation, but not for noise-perturbed ones (black squares: $a = 1.5$).

value dependent on the quantitative stability of the oscillators fixed point at $\sigma = 0$. Here, $\lambda$ may as well show a local minimum as a function of $\sigma$ (red triangles).

We have seen that the dynamics of noise-induced oscillations can be quite close to that of noise-perturbed ones what suggests a theoretical description by a purely deterministic phase equation

$$\dot{\theta} = F(\theta) \tag{10}$$

to be possible. Generally, one cannot use the approximation $F(\theta) = h(\theta)$ by considering only the deterministic part $h(\theta)$ of oscillator (1) because $h(\theta)$ may have zero-crossings, and thus $\theta$ would be non-oscillating. Instead, we have to construct an *effective phase model* using some criteria to determine $F(\theta)$. Generally, we demand that the effective phase model of type (10) represents as many characteristic properties of stochastic phase oscillators as possible. In Ref. [9], one approach was proposed that allowed for a construction of an effective phase model with the same mean frequency (i) (equivalently, the same mean period) and the same invariant distribution density (ii) as the stochastic phase oscillator. Without violating these conditions, the diffusion coefficient (iii) could be modeled, too, by adding noise to Eq. (10) in a certain way. To draw a more general framework, we will present the effective phase model from Ref. [9] in the next section, and thereafter introduce another effective phase model based on the first passage times, in a coherent way. Roughly, the difference of these models lies in the definition of the average speed of oscillator (1). Given an interval of length $\Delta\theta$, the speed can be measured as the quotient of $\Delta\theta$ and the mean time $\Delta t$ that oscillator (1) spends in this interval. This leads exactly to the effective phase description presented in Ref. [9]. Alternatively, the speed can be measured as the quotient of $\Delta\theta$ and the mean time taken to reach the opposite boundary of the interval, which leads to the *first passage velocity.* For deterministic phase oscillators the two definitions of speed coincide, whereas for oscillations showing irregular features, there is a difference, which is especially pronounced for noise-induced ones.

## 3 Current model of effective phase dynamics

In this section, the effective phase theory presented in Ref. [9] is repeated in detail in order to make this work more self-contained. A deterministic model of the stochastic phase oscillator (1) is constructed, that shows the same mean frequency and invariant probability density. Furthermore, it allows for an invariant description of stochastic oscillations.

We construct the *current model* of oscillator (1) that obeys the deterministic phase equation

$$\dot{\theta} = H(\theta). \tag{11}$$

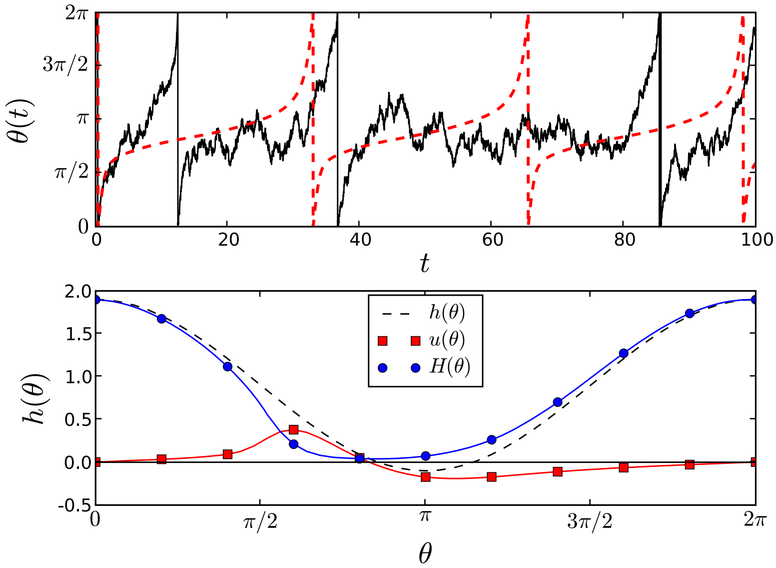


**Fig. 3.** The current model shows (top plot) a trajectory (dashed line) strongly resembling that of the Adler equation (2) (solid line) even if its oscillations are noise-induced (here at $a = 0.9$ and $\sigma = 0.3$). For this, the osmotic contribution $u(\theta)$ to the current velocity (13) is non-negligible as it accounts for the strict positivity of $H(\theta)$ (bottom plot).

Here, the mean time that oscillator (1) spends in an interval $[\theta, \theta + d\theta]$ leads us to a notion of speed. According to the invariant probability density, the mean time is defined as $dt = Pd\theta/J$. It follows, that the *current velocity* $H(\theta)$ is given by

$$H(\theta) = \frac{Jd\theta}{P(\theta)d\theta} = \frac{\omega}{2\pi P(\theta)}. \tag{12}$$

This gives the same effective phase model introduced in Ref. [9]. Model (11) obeys conditions (i)[3] and (ii)[4] because it fulfills the stationary Liouville equation for $P(\theta)$ and it shows the correct period $T$ as seen by

$$-\partial_\theta \left[H(\theta)P(\theta)\right] = 0, \text{ and } \int_0^{2\pi} \frac{d\theta}{H(\theta)} = T.$$

Using Eqs. (5) and (7), the current model of the Adler equation may be constructed and integrated numerically. A comparison of respective realizations $\theta(t)$ of Eqs. (11) and (2) shows that even though the stochastic components are missing in the current model, the observed dynamics is comparable (top plot of Fig. 3).

The current velocity $H(\theta)$ can be expressed in terms of $h(\theta)$ and $g(\theta)$. For this, Eq. (4) is divided by $P(\theta)$, and the result is compared to Eq. (12) yielding

$$H(\theta) = h(\theta) - \tfrac{1}{2}\left[g^2(\theta)\right]' - g^2(\theta)\left[\ln P(\theta)\right]' = h(\theta) - u(\theta). \tag{13}$$

It consists of the deterministic contribution $h(\theta)$ and an osmotic contribution $u(\theta)$ that is especially pronounced for noise-induced oscillations (cf. bottom plot of Fig. 3). The current velocity corresponds to the point-wise average of central differences [22]. Therefore, it may be constructed from an observed (e.g., experimentally) time series $\theta_n = \theta(n\Delta t)$ by the simple

[3] The frequency of the effective phase model is equal to the mean frequency of stochastic oscillations.

[4] The invariant distribution density of the effective phase model is equal to the probability density of stochastic oscillations.

averaging procedure

$$H(\theta) \approx \left.\frac{\langle \theta_{n+1} - \theta_{n-1} \rangle}{2\Delta t}\right|_{\theta_n = \theta}. \tag{14}$$

Having constructed the current model, we can transform its protophase $\theta(t)$ to a uniformly rotating phase variable $\varphi(t)$. It has simple properties:

$$\dot{\varphi} = \omega; \qquad P(\varphi) = \frac{1}{2\pi}. \tag{15}$$

As it can be easily checked, the nonlinear transformation $\theta \to \varphi$ is given by

$$\varphi = S(\theta) = 2\pi \int_0^{\theta} P(\theta')\ d\theta'. \tag{16}$$

Herewith the coordinate-dependent differences in the protophase $\Delta\theta = \theta_2 - \theta_1$ can be transformed to the invariant differences in the phase

$$\Delta\varphi = S(\theta_2) - S(\theta_1) = 2\pi \int_{\theta_1}^{\theta_2} P(\theta)\ d\theta. \tag{17}$$

Given a time series $\theta_n = \theta(n\Delta t)$ containing $N$ data points, the transformation can be obtained numerically [3]. If one is not interested in the transformation $S(\theta)$ but in the transformed data $\varphi_n = S(\theta_n)$ only, one may alternatively evaluate

$$\varphi_n = \frac{2\pi}{N} \sum_{l=0}^{N-1} \Theta(\theta_n - \theta_l) \tag{18}$$

which is implemented quickly by sorting. ($\Theta(x)$ is the Heaviside function.)

Phase diffusion, mean frequency and probability density may be modeled simultaneously by the addition of $\delta$-correlated[5] additive noise $\sqrt{D}\eta(t)$ to the invariant phase dynamics (15):

$$\dot{\varphi} = \omega + \sqrt{D}\eta(t). \tag{19}$$

Now, $\varphi$ shows a diffusion constant $D$ while preserving uniform density and mean frequency. Therefore, application of the inverse transformation $\theta = S^{-1}(\varphi)$ gives us the stochastic current model

$$\dot{\theta} = H(\theta) + \frac{\sqrt{D}}{\omega} H(\theta)\eta(t) \tag{20}$$

that fulfills conditions (i) and (ii) for any value of $D$. It may be chosen freely, and we chose it uniquely from condition (iii): The diffusion coefficients of stochastic current model (20) and oscillator (1) should be equal. This condition is fulfilled if the diffusion coefficient (8) is used for $D$.

## 4 First passage model of effective phase dynamics

In this section, an alternative effective phase model for the stochastic phase oscillator (1) is introduced. It is based on first passage time statistics. In Sec. 6, it is shown that this alternative model exhibits the correct phase resetting curve of stochastic oscillations.

As an alternative to the current model outlined in the last section, a velocity based on first passage time statistics of oscillator (1) shall lead us to the *first passage model*

$$\dot{\theta} = N(\theta). \tag{21}$$

[5] $\langle \eta(t)\eta(t') \rangle = 2\delta(t - t')$.

To determine $\dot{\theta} \approx \frac{\Delta\theta}{\Delta t}$ we interpret $\Delta t$ as the mean first passage time of passing an interval of length $\Delta\theta$. More precisely, the *first passage velocity* is constructed using the mean first passage time $T(\alpha, \beta)$ which it takes for the stochastic phase of oscillator (1) to reach a boundary $\beta > \alpha$ starting at $\alpha$. It is given by [23]

$$\frac{1}{N(\theta)} = \frac{dt}{d\theta} = \lim_{\varepsilon \to 0} \frac{T(\theta, \theta + \varepsilon) - T(\theta, \theta)}{\varepsilon} = \partial_\beta T|_{\alpha=\beta=\theta} \,. \tag{22}$$

The mean frequency of Eq. (21) is equal to the mean frequency of the stochastic oscillations, because the mean period $T$ is nothing else but $T(\theta, 2\pi + \theta)$ and therefore $\int_0^{2\pi} d\theta/N(\theta) = T$. However, the condition (ii)[6] is generally not fulfilled as will be shown next.

The distribution density of model (21), which we call *mean first passage density*, is given by

$$R(\theta) = \frac{\omega}{2\pi N(\theta)}. \tag{23}$$

It is also known as the *speed density* [23]. In order to obtain an equation for $R(\theta)$, the well-known equation for the mean first passage time of oscillator (1) has to be derived which we repeat following Ref. [24]. For an easier description of boundaries, the unwound stochastic protophase $\theta(t)$ is considered in the following. Consider the Fokker-Planck equation (3) with the sharp initial condition $P(\theta, 0) = \delta(\theta - \alpha)$. In this case Eq. (3) describes the conditional probability density $P(\theta, t|\alpha, 0)$. The boundary conditions

$$P(-\infty, t|\alpha, 0) = P(\beta, t|\alpha, 0) = 0 \tag{24}$$

are introduced, which correspond to the fact that states starting at $\alpha$ should only be considered as long as they do not reach boundary $\beta$. Now, $P$ has to be reinterpreted since the normalization condition does not hold anymore. The *no-passage probability* $G(\alpha, t)$ is defined as the probability that at time $t$ boundary $\beta$ is not reached when starting at $\alpha$. It is given by

$$G(\alpha, t) = \int_{-\infty}^{\beta} P(\theta', t|\alpha, 0) d\theta'. \tag{25}$$

By a Kolmogorov backward expansion of $P(\theta', t|\alpha, 0)$, one finds that $G$ obeys

$$\partial_t G = h(\alpha)\partial_\alpha G + g(\alpha)\partial_\alpha \left[g(\alpha)\partial_\alpha G\right]. \tag{26}$$

For $t \in [0, \infty)$ and $\alpha < \beta$, the probability density of first passage times is given by $g(\alpha, t) = -\partial_t G$. With respect to it, the mean first passage time is given by

$$T(\alpha, \beta) = \langle t \rangle = \int_0^\infty G(\alpha, t')\, dt'. \tag{27}$$

Integrating Eq. (26) over positive times, one obtains the equation for the mean first passage time:

$$-1 = h\partial_\alpha T + g\partial_\alpha \left[g\partial_\alpha T\right]. \tag{28}$$

Because $\partial_\beta T|_{\alpha=\beta=\theta} = -\, \partial_\alpha T|_{\alpha=\beta=\theta} = 1/N(\theta)$, Eq. (28) may be rewritten for the mean first passage density $R(\theta)$ as

$$J = hR + g\partial_\theta \left[gR\right]. \tag{29}$$

This equation has the well-known solution

$$R(\theta) = C \int_{\theta-2\pi}^{\theta} \frac{d\psi}{g(\theta)g(\psi)} \mathrm{e}^{-\int_\psi^\theta \frac{h(\varphi)}{g^2(\varphi)} d\varphi} \tag{30}$$

[6] The invariant distribution density of the effective phase model is equal to the probability density of stochastic oscillations.

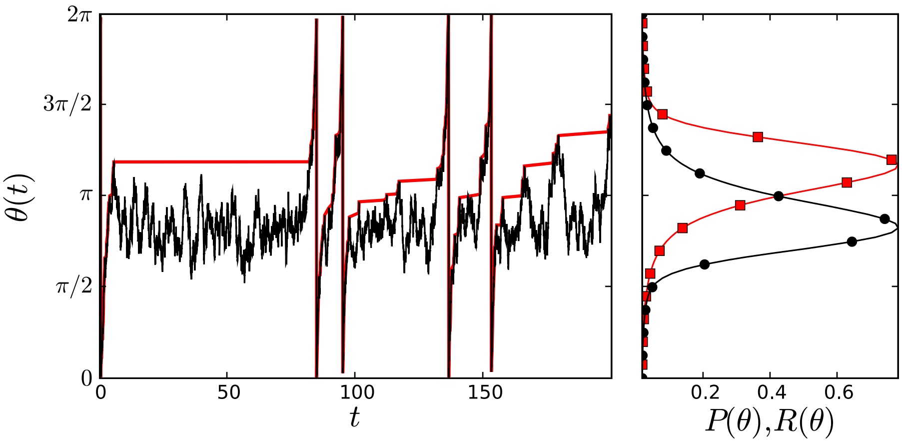


**Fig. 4.** The protophase $\theta(t)$ (black [solid] line) of the Adler equation (2) and its envelope (red [thick] line) have similar long term dynamics (left plot), but the probability densities $P(\theta)$ (black circles) and mean first passage density $R(\theta)$ (red squares) are different (right plot), especially for noise-induced oscillations shown here at $a = 0.9$ and $\sigma = 0.3$.

with the same normalization constant $C$ as in Eq. (5), and $J = \omega/2\pi$. Remarkably, Eq. (29) is similar to Eq. (4) except for a factor minus one. The equation provides an easily manageable analytic formula for the first passage velocity $N(\theta)$.

The mean first passage density has a direct meaning for a realization $\theta(t)$ of the stochastic protophase: Let $t_n$ be the times of first passage, for which $\theta(t < t_n) < \theta(t_n)$ holds ($\theta(t)$ is unwound). Connecting adjacent values of the point process $\theta_n = \theta(t_n)$ gives the *envelope* of the realization $\theta(t)$ (left plot of Fig. 4). Each $\theta_n$ gives a starting point for a measurement of the passage time ending when $\theta(t)$ reaches $\theta_{n+1} = \theta_n + d\theta$. Although the trajectory of the corresponding time segment $\theta(t_n < t < t_{n+1})$ lies in an arbitrary region below $\theta_n$, it is attributed to the interval $\theta_n < \theta < \theta_{n+1}$ for the mean first passage density $R(\theta)$. In fact, $R(\theta)$ is the probability density of the envelope of $\theta(t)$. The probability density $P(\theta)$ and the mean first passage density $R(\theta)$ can be quite different (right plot of Fig. 4). Therefore, the current model and the first passage model are distinctly different from each other for a general stochastic phase oscillator. The fact that $R(\theta)$ is the probability density of the envelope of $\theta(t)$ can be used for its numerical construction from data, which is shown at the end of this section.

We would like to mention that going from the stochastic protophase $\theta(t)$ to its envelope, we achieve a monotonically growing protophase. Indeed, phase is often understood as a strictly monotonic variable, in some sense a "replacement" for a time variable. But for a stochastic oscillator one often observes "reverse" variations. Thus, taking the envelope is a natural way to restore a monotonic function of time.

The first passage model (21) provides an effective phase dynamics of stochastic oscillations, which is alternative to the current model. It fulfills condition (i) in that it shows the same mean frequency as oscillator (1), but instead of modeling its probability density (ii), it preserves its mean first passage density (iib) which for deterministic oscillators, is equal to the distribution density. Even for small noise amplitudes, the current model and the first passage model can be quite different if they describe noise-induced oscillations (cf. top plot of Fig. 5). For noise-perturbed oscillations the two velocities $H(\theta)$ and $N(\theta)$ converge to $h(\theta)$ as the noise amplitude vanishes (bottom plot of Fig. 5).

In the numerical example presented in Fig. 5, the current and the first passage velocities are mapped to each other by mirror symmetry. This is due to the fact that both $h(\theta)$ and $g(\theta)$ are symmetric in the Adler equation. For a stochastic phase oscillator with symmetric functions $h(\theta)$ and $g(\theta)$, the transformation $\theta \to -\theta$ transforms Eq. (29) in (4). Their solutions are mapped to each other, too. Therefore, symmetry of $h$ and $g$ implies $N(\theta) = H(-\theta)$.

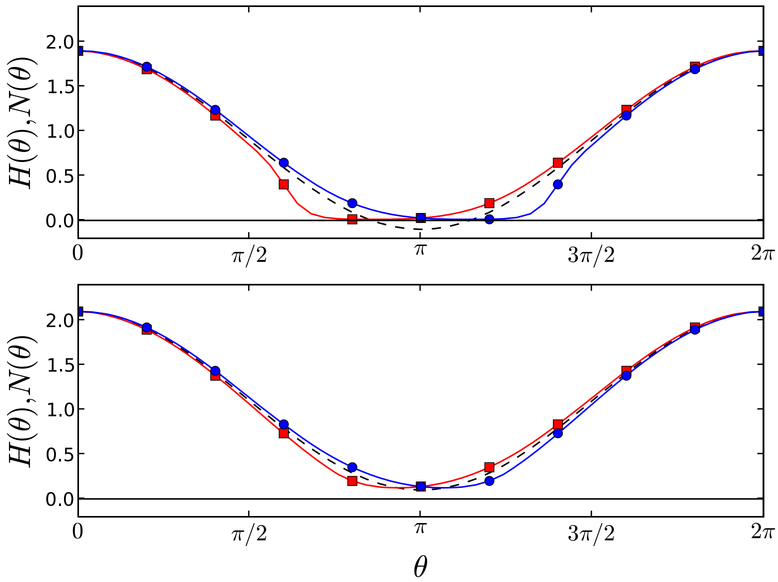


**Fig. 5.** The current velocity (12) (red squares) and the first passage velocity (22) (blue circles) for the Adler equation (2) differ essentially for both noise-induced (at $a = 0.9$, top plot) and noise-perturbed (at $a = 1.1$, bottom plot) oscillations. For both cases the noise amplitude is chosen to be $\sigma = 0.2$. The difference in effective velocities is especially pronounced for noise-induced oscillations and does not disappear for small $\sigma$, whereas both effective velocities closely approximate $h(\theta)$ (dashed line) for noise-perturbed oscillations.

As for the current model, a uniformly rotating phase variable is constructed by the transformation

$$\psi = Z(\theta) = 2\pi \int_0^{\theta} R(\theta')\ d\theta' \tag{31}$$

with which differences in the protophase $\theta_2 - \theta_1$ are transformed to differences in the phase

$$\Delta\psi = Z(\theta_2) - Z(\theta_1) = 2\pi \int_{\theta_1}^{\theta_2} R(\theta)\ d\theta. \tag{32}$$

By adding effective noise to the dynamics of $\psi$, phase diffusion can be taken into account leading to (cf. Eqs. (19) and (20))

$$\dot{\theta} = N(\theta) + \frac{\sqrt{D}}{\omega} N(\theta)\eta(t). \tag{33}$$

This stochastic first passage model fulfills conditions (i), (iib) and (iii). However, it does not fulfill condition (ii), as its probability density is $R(\theta) \neq P(\theta)$. Therefore, an application of transformation (31) to the stochastic protophase $\theta(t)$ of oscillator (1) does not lead to an invariant phase $\psi(t)$ that is uniformly distributed as it does for transformation (16).

The fact, that $R(\theta)$ is the probability density of the envelope of a realization $\theta(t)$, can be used to obtain transformation $Z(\theta)$ from a sampled protophase $\theta(n\Delta t)$ of oscillator (1). To construct the envelope, we need to find the times of first passage $t_n$ for which the unwound protophase has a history that is strictly smaller, i. e. $\theta(j\Delta t < t_n) < \theta(t_n)$. At these first passages $\theta_n = \theta(t_n)$, transformation (31) is estimated by

$$\psi_n = Z(\theta_n) = \frac{2\pi}{T} \sum_{\theta_j < \theta_n} (t_j - t_{j-1})\,. \tag{34}$$

To find the transformation on the whole domain one can proceed by an appropriate interpolation of $Z(\theta_n)$, for example, via smoothing splines. Note that Eq. (34) gives a biased estimator. This can be fixed by inserting a central difference scheme $(t_{j+1} - t_{j-1})/2$.

## 5 Asymptotic properties

In this section, the limiting cases of weak and strong noise in a stochastic phase oscillator with additive noise are considered. For the interesting case of a vanishing noise amplitude in noise-induced oscillations, the effective velocity shows an interesting singularity. Furthermore, a formula for the estimation of mean frequency in the limit of strong noise is derived.

### 5.1 Singular perturbation for weak noise

In the Fokker-Planck equation (3), the noise amplitude $g(\theta)$ appears in front of the derivative with the highest order in $\theta$. Therefore, a perturbation expansion in $g(\theta)$ for a deterministic approximation of the dynamics (1) might be singular. For a stochastic phase oscillator with additive noise

$$\dot{\theta} = h(\theta) + \sigma\xi(t), \tag{35}$$

we show that a perturbation expansion becomes singular if oscillations are noise-induced. We discuss the case in which the deterministic phase description should model the probability density of the stochastic oscillations. The singularity that arises in the limit of weak noise $\sigma \to 0$ is discussed by a perturbation expansion starting from $\sigma = 0$, and starting from $\sigma > 0$ using the current model of effective phase theory.

For $\sigma = 0$, our considerations start with the arbitrary model $\dot{\theta} = h(\theta)$. Let us suppose that oscillator (35) shows noise-perturbed oscillations, i. e. $h(\theta)$ is strictly positive. Then, the invariant distribution density of the arbitrary model gives a proper zeroth order approximation to the probability density:

$$P(\theta) = \frac{C}{h(\theta)} + \mathcal{O}(\sigma).$$

Better approximations can be obtained analytically by a Taylor expansion of $P(\theta)$. For the alternative case of noise-induced oscillations, we impose, without loss of generality, that $h(\theta)$ has two zero crossings. Then, the arbitrary model has a stable fixed point at $\theta_-$ as well as an unstable one at $\theta_+ > \theta_-$. Its distribution density is given by

$$P(\theta) = \delta(\theta - \theta_-) + \mathcal{O}(\sigma), \tag{36}$$

corresponds to the probability density of oscillator (35). However, higher order terms in $\sigma$, that should lead to a smooth distribution density, must necessarily be singular. One can see that a perturbation theory becomes singular for noise-induced oscillations. Note that in the limit $\sigma \to 0$, the mean first passage density also has a singular limit for noise-induced oscillations, which is located at the unstable fixed point. In the above example it is given by

$$R(\theta) = \delta(\theta - \theta_+) + \mathcal{O}(\sigma). \tag{37}$$

For noise-perturbed oscillations the mean first passage density and the invariant probability density converge in this limit reflecting the fact that for deterministic self-sustained oscillators the quantities are synonymous.

Starting with the current model of oscillator (35) computed for finite $\sigma \neq 0$, another view can be gained on the limit of weak noise $\sigma \to 0$. Using Eqs. (7) and (12), the current velocity is given by

$$H(\theta) = \frac{1 - \mathrm{e}^{-\frac{r(0,2\pi)}{\sigma^2}}}{\int_\theta^{\theta+2\pi} \frac{d\psi}{\sigma^2} \mathrm{e}^{-\frac{r(\theta,\psi)}{\sigma^2}}}, \text{ with } r(\theta,\psi) = \int_\theta^\psi h(\varphi)\, d\varphi. \tag{38}$$

Let us assume without loss of generality that $r(0, 2\pi)$ is positive. In the limit of weak noise, the integral in the denominator of Eq. (38) is dominated by the minimum of $r(\theta, \psi)$ with respect to $\psi$. Using the method of stationary phase, the zeroth order expansion

$$H(\theta) = \mathcal{O}(\sigma) + \begin{cases} h(\theta) & : \text{ if } \operatorname{argmin}_\psi [r(\theta,\psi)] = \theta \\ 0 & : \text{ elsewise} \end{cases} \tag{39}$$

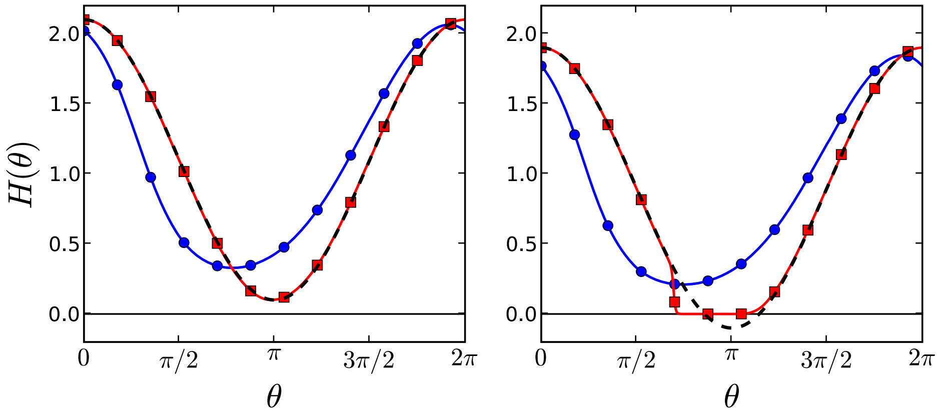


**Fig. 6.** The current velocity converges to $h(\theta)$ (black [dashed] line) at small $\sigma$ (red squares) for noise-perturbed oscillations ($a = 1.1$, left plot), but for noise-induced oscillations (at $a = 0.9$, right plot), it becomes discontinuous (cf. Eq. (39)). Furthermore, the total area under the curve is independent of $\sigma$ (cf. Eq. (40)). Red squares: $\sigma = 0.1$, Blue circles: $\sigma = 0.7$. Parameter $a$ is chosen to be 1.1 (left plot), and 0.9 (right plot).

is obtained. Surprisingly, the limit of weak noise leads to a vanishing of current velocity in a finite interval around the fixed point $\theta_-$ and not just at this point. A similar derivation is possible for the first passage velocity.

In Fig. 6, the peculiar nature of $H(\theta)$ is illustrated for the Adler equation. If $\sigma$ is small, the effective velocity becomes discontinuous for noise-induced oscillations (right plot of Fig. 6), whereas $H(\theta)$ converges to $h(\theta)$ for noise-perturbed ones (left plot of Fig. 6).

### 5.2 Estimating frequency for strong noise

For strong noise, an estimation of the mean frequency can be troublesome if one has to rely on Monte-Carlo simulations. Here, we want to provide a formula that allows for an estimation of frequency at large noise amplitudes if an effective velocity at arbitrary $\sigma$ is available.

The current velocity can be expressed as $H = h - \sigma^2 [\ln P]'$ (cf. Eq. (13)). Therefore, the integral over $H(\theta)$ from 0 to $2\pi$ does not depend on $\sigma$. To evaluate the integral we consider the limit of strong noise $\sigma \to \infty$ for which $P(\theta) \to 1/2\pi$, and we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} H(\theta)\, d\theta = \lim_{\sigma \to \infty} \omega = \omega_\infty. \tag{40}$$

The same result can be obtained for the first passage velocity. Thus for additive noise, the area under effective velocities is independent of the noise amplitude and proportional to the asymptotic mean frequency $\omega_\infty$.

## 6 Stochastic phase resetting

In this section, a theory of phase resetting of stochastic phase oscillators is developed. The first passage model represents the phase resetting curve of the stochastic phase oscillator correctly, whereas the current model does not.

The theory of phase resetting is concerned with the response of a general high-dimensional oscillating system to a brief stimulus, a kick, applied at a certain protophase $\alpha$ with a certain strength and direction $\mathbf{k}$. The freely rotating period $T$ is compared to the period $T_{\mathbf{k}}(\alpha)$ wherein the kick is applied by computing the *phase resetting curve* [25,26]

$$\mathrm{PRC}(\alpha, \mathbf{k}) = 2\pi \frac{T - T_{\mathbf{k}}(\alpha)}{T}. \tag{41}$$

It gives the shift in the uniformly rotating phase, which the oscillator experiences due to the kick. For example, Eq. (17) provides the phase resetting curve of the current model (11), whereas Eq. (32) provides it for the first passage model (21), if one sets[7] $\theta_2 = \theta_1 + k$ for either. Up to here, the quantities appearing in Eq. (41) are only well-defined for deterministic systems that show limit-cycle oscillations. We extend the applicability of Eq. (41) to stochastic phase oscillators (cf. Ref. [27]) and discuss the results in terms of effective phase theory.

A kick applied at a time $t'$ with scalar strength $k$, representing a brief stimulus, is introduced to our basic model (1) by

$$\dot{\theta}(t) = h(\theta(t)) + g(\theta(t))\xi(t) + k\delta(t' - t). \tag{42}$$

The phase resetting curve is computed by a comparison of the kicked and unkicked stochastic phase oscillator. For this, the quantities appearing in Eq. (41) are interpreted as follows: Quantity $T$ is given by the mean period $T = 2\pi/\omega$ computed for the unkicked oscillator. Quantity $T_k(\alpha)$ is computed for the kicked oscillator as the mean first passage time of the unwound protophase starting at $\theta = \alpha + k$ (value just after the kick) to reach the boundary $\theta = \alpha + 2\pi$.

By the above interpretation of Eq. (41), the phase resetting curve can be calculated using the mean first passage times. From Eq. (28) it is deduced that $T(\alpha, \beta) + T(\beta, \delta) = T(\alpha, \delta)$. This is related to the Markov property of $\theta(t)$, and it allows us to express the mean first passage time as

$$T_k(\alpha) = T - T(\alpha, \alpha + k), \tag{43}$$

For Eq. (41), it follows that $\mathrm{PRC}(\alpha, k) = 2\pi T(\alpha, \alpha + k)/T$. This can be rewritten in terms of $R(\theta)$ using Eq. (23) as

$$\mathrm{PRC}(\alpha, k) = 2\pi \int_{\alpha}^{\alpha+k} R(\theta)\; d\theta. \tag{44}$$

This is the exact formula of phase resetting curve for a general stochastic phase oscillator (1).

It is seen that the phase resetting curve

$$\Delta\varphi(\alpha, k) = 2\pi \int_{\alpha}^{\alpha+k} P(\theta)\; d\theta \tag{45}$$

derived for the current model does not correspond to that of the stochastic phase oscillator, whereas the phase resetting curve

$$\Delta\psi(\alpha, k) = 2\pi \int_{\alpha}^{\alpha+k} R(\theta)\; d\theta \tag{46}$$

derived for the first passage model (21) yields the correct formula. Let us explain why the current model fails: In Sec. 3, it was seen that the current velocity is given by $H(\theta) = J/P$ which leads to the stationary solution (12). However, the stationary state is broken in the phase resetting procedure, because the time evolution of $\theta(t)$ starts from the definite value $\theta = \alpha + k$. Consider for example the moment right after the resetting of a stochastic phase oscillator with additive noise where $P(\theta) = \delta(\theta - \alpha - k)$. The probability flux is calculated by integrating Eq. (4), and one obtains $2\pi J(\alpha + k) = h(\alpha + k)$ which does not coincide with the stationary flux $\omega/2\pi$. Deviations to the stationary solution (12) are most prominent in the excitable regime where $2\pi J(\alpha + k) = h(\alpha + k) < 0$. In this case, the time-dependent current model

$$\dot{\theta} = \frac{J(\theta, t)}{P(\theta, t)} \tag{47}$$

has non-monotonic dynamics. Therefore, it does not yield a good phase description.

The phase resetting curve of stochastic phase oscillators can be understood in a more intuitive way. Again, the basic idea is that a stimulus kicks the system from a phase $\theta_1$ to a phase $\theta_2 = \theta_1 + k$. Let us compute the average unwound protophases $\Theta_{1,2}(t) = \langle\theta(t)\rangle$ with respect to the two probability densities $P_{1,2}(\theta, t = 0) = \delta(\theta_{1,2} - \theta)$ (with and without kick). Asymptotically,

[7] For one-dimensional systems, $\mathbf{k} = k$ is scalar.

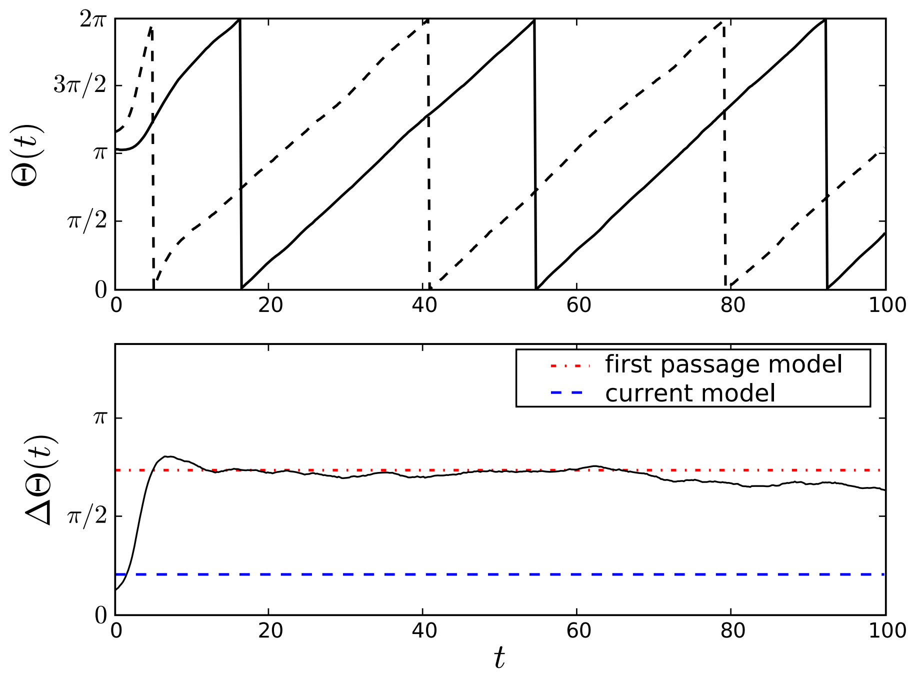


**Fig. 7.** Two trajectories of $\Theta(t)$, each one averaged over 1000 realizations (and then taken modulo $2\pi$), were computed starting at two distinct initial conditions $\theta_{1/2} = \theta_+ \pm 0.2$ (top plot, $+$: dashed, $-$: solid) for the Adler equation. They showed after an initial non-uniformity a constant growth rate, such that the initial phase difference $\Delta\Theta(0) = \theta_2 - \theta_1$ widens (bottom plot, solid line) to the value $\text{PRC} = \Delta\psi = Z(\theta_2) - Z(\theta_1)$ (red [dotted] line) as predictedby the first passage model; the corresponding prediction of the current model (blue [dashed] line) is not correct. The parameters were chosen as $a = 0.95$ and $\sigma = 0.2$.

the probabilities assume the same stationary state leading to a constant average velocity $\lim_{t\to\infty}\langle\dot{\theta}(t)\rangle = \omega$. Thus in the limit $t \to \infty$, the phase difference $\Delta\Theta(t) = \Theta_2(t) - \Theta_1(t)$ is a constant, and it is equal to the corresponding value of the phase resetting curve $\text{PRC}(\theta_1, k)$.

To give a numerical example, two groups, each containing 1000 realizations of the Adler equation, with one group starting at $\theta_1 = \theta_+ - 0.2$ and the other at $\theta_2 = \theta_+ + 0.2$ were calculated, where $\theta_+$ is the unstable fixed point $a = -\cos\theta_+$. For each group an average over realizations was performed to obtain the average dynamics $\langle\theta(t)\rangle$. While asymptotically uniformly rotating (cf. top plot of Fig. 7), initially $\Theta(t)$ has systematic non-uniformities such that initial phase difference $\Delta\Theta(0) = \theta_2 - \theta_1$ widens (cf. bottom plot of Fig. 7). Asymptotically, it reaches the value $\Delta\psi = Z(\theta_2) - Z(\theta_1)$ as correctly predicted by the stochastic phase resetting curve in Eq. (44) and that of the first passage model (46).

## 7 Conclusions

Two useful formulations of the effective phase theory are presented each one relying on different definitions of phase velocity. While the concept of average time spent in an interval leads to the current velocity, a definition based on mean first passage times leads to the first passage velocity. For noise-perturbed oscillations the two velocities converge as noise vanishes, whereas for noise-induced oscillations they do not. While the current model was previously shown to be useful for a characterization of continuous coupling of stochastic phase oscillators [9], we have argued in this article that the first passage model gives a correct description of the stochastic phase resetting curve. Importantly, a formula is presented that allows for the numerical estimation of the stochastic phase resetting curve from data (through transformation (34)). This offers a, to our knowledge, novel method of processing oscillatory data.

A natural extension of the effective phase theory would be to construct two-dimensional models of stochastic oscillations for which an effective phase variable and a corresponding effective radius variable may be found. The extension is not straight-forward: A deterministic current-type model for stochastic oscillations must have a meaning different from the one-dimensional case, because the correspondence between the distribution and the probability density (condition (ii)) cannot be drawn. For example, one could look for an effective limit-cycle oscillator, but as it turns out, the corresponding two-dimensional current-type model is Hamiltonian [28]. On the other hand, it might be possible to construct a first passage-type model exhibiting the correct phase resetting curve, and in this way the concept of isophases may be generalized [25].

Presently, it also remains unclear how an effective phase theory is constructed for oscillations under the influence of an external force consisting of both, a continuous and a pulse-like contribution, because respective effective models are incompatible (cf. Fig. 5). Furthermore, we expect the presented work to be useful for a characterization of finite ensembles of oscillators with either continuous or pulse-like coupling. The attempt to describe stochastic oscillations by a deterministic model shows clearly that there can be multiple solutions. Different aspects of the stochastic dynamics are represented by incompatible deterministic descriptions. Therefore, one has to chose, dependent on the concrete problem, which simplified description should be taken.

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