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Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators

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ABSTRACT

We analyze a minimal model of a population of identical oscillators with a nonlinear coupling—a generalization of the popular Kuramoto model. In addition to well-known for the Kuramoto model regimes of full synchrony, full asynchrony, and integrable neutral quasiperiodic states, ensembles of nonlinearly coupled oscillators demonstrate two novel nontrivial types of partially synchronized dynamics: self-organized bunch states and self-organized quasiperiodic dynamics. The analysis based on the Watanabe–Strogatz ansatz allows us to describe the self-organized bunch states in any finite ensemble as a set of equilibria, and the self-organized quasiperiodicity as a two-frequency quasiperiodic regime. An analytic solution in the thermodynamic limit of infinitely many oscillators is also discussed.

1. Introduction

A model of all-to-all coupled limit cycle oscillators describes many natural phenomena in physics, chemistry, biology and social sciences. For ensembles of weakly interacting units, the description of the dynamics is often provided by the paradigmatic Kuramoto model of globally coupled phase oscillators [1,2]. This model explains self-synchronization and appearance of a collective mode in an ensemble of generally non-identical elements—the problem relevant for Josephson junction and laser arrays [3,4], electrochemical reactions [5], neuronal dynamics [6], social behavior [7–9], etc. According to the analysis, first performed by Kuramoto, the transition to synchrony occurs at a certain critical value of the coupling constant that is roughly proportional to the width of the distribution of natural frequencies. The ordered phase is characterized by a non-zero mean field (collective mode) that maintains the collective synchrony. Many relevant references to the Kuramoto model can be found in [10,11].

The physical reason for the Kuramoto transition is an attractive force between the oscillators. In the present context the terms “attraction” and “repulsion” mean that the interaction between two oscillators tends to lock them in phase or in antiphase, respectively. Note that for two coupled oscillators both attraction and repulsion synchronize the oscillators in the sense that their frequencies become adjusted and eventually coincide, and their phases become locked. The situation is different if many oscillators interact, since, obviously, they cannot arrange themselves in antiphase. Thus, in an ensemble, an attracting interaction tends to adjust the phases of elements so that they all are in phase, i.e. form a cluster, and their mean field (order parameter) is large; this state is called a synchronous one. On the contrary, a repulsive interaction tends to distribute the phases uniformly, so that the mean field vanishes; this state is called an asynchronous one. We emphasize that in a particular case of identical oscillators, studied below, synchronization cannot be characterized by the adjustment of frequencies (since they are identical from the very beginning), but only by the adjustment of phases.

In this paper, following our brief communication [12], we describe and systematically discuss an extension of the Kuramoto model that demonstrates a novel transition from the synchronous state to a regime of partial synchronization, when the system settles in a self-organized fashion at the border between stable synchrony and asynchrony. Remarkably, the transition can be observed already in a population of ***identical*** units. Partial synchronization has been previously studied for coupled integrate-and-fire oscillators [13,14]; however, this system does not exhibit a transition from full to partial synchrony. We would also like to mention the work [15], where a similar phenomenon was observed numerically.

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The main physical effect that is responsible for the transition to partial synchrony is ***nonlinearity of coupling***. We will discuss this crucial issue in more detail in Section 2, while here we just briefly present the main idea. Roughly speaking, a nonlinear coupling means that the response of an oscillator to a strong forcing cannot be simply "upscaled" from its response to a weak forcing. For the simplest example, let us assume that weak forcing leads to "attraction", while strong forcing leads to "repulsion" (another case of nonlinear coupling which remains attractive for all nonlinearities have been considered recently in [16,17]). Now recall that in a large, globally coupled ensemble each oscillator can be considered as forced by the mean field; the strength of the forcing is determined by the product of the coupling constant and the amplitude of the mean field (see [1,2] and discussion below). Hence, with an increase of the coupling constant, the forcing increases as well, unless at some point the interaction changes from the attractive to the repulsive one, and the initially synchronous state dissolves. However, this would reduce the mean field and, therefore, the effective forcing. It means that the interaction would again become attractive and therefore a completely asynchronous state would be also impossible. As a result, an intermediate state sets in, where the interaction is tuned exactly to the border between attraction and repulsion. This neutral state is a self-organized partially coherent state with the order parameter (amplitude of the mean field) between zero and one. Below we demonstrate that there are two types of such states, which we call "self-organized quasiperiodic state" and "self-organized bunch state", respectively. We shall show that in the former case the dynamics is generally quasiperiodic.

Nonlinearity of the coupling appears to be a rather general phenomenon. One mechanism of its appearance was mentioned in the previous paragraph: if a response to strong forcing cannot be simply "upscaled" from the response of a very small forcing, this means that the coupling is nonlinear. Another mechanism works if the oscillators are coupled not directly to each other (like neurons which communicate via synaptic connections) but interact through a dynamical mediator (e.g., Josephson junctions or electrochemical oscillators can be coupled via a common load). If the equations for the mediator are nonlinear, the coupling is nonlinear as well (see discussion in [12]).

The paper is organized as follows. We introduce a minimal model with nonlinear coupling in Section 2. The self-organized bunch state is described numerically and theoretically in Section 3. In Section 4 we present results of numerical simulations, illustrating the properties of the self-organized quasiperiodicity (SOQ) and of the transition to this partially synchronized state. The theory of the effect is developed in Section 5; this theory is heavily based on the seminal paper by Watanabe and Strogatz [18] where a full analysis of linearly coupled identical phase oscillators has been performed. We first describe the approach of [18] and then show how it can be extended to the case of nonlinear coupling. In a particular case, which is, however, relevant for large ensembles with homogeneous initial conditions, the resulting equations are rather simple and their bifurcation analysis can be performed analytically. We discuss our results in Section 6.

2. A minimal model for an ensemble of nonlinearly coupled identical phase oscillators

2.1. Linear vs. nonlinear coupling: A generalized Kuramoto model

We start by a brief description of the popular Kuramoto model of sine-coupled identical phase oscillators:

$$\dot{\varphi}_k = \omega + \frac{\varepsilon}{N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_k + \beta), \tag{1}$$

where φ_k is the phase of the k-th oscillator, N is the number of oscillators in the ensemble, ε is the strength of the interaction between each pair of oscillators, β is the phase shift, inherent to coupling, and ω is a natural frequency. The model with $\beta \neq 0$ is also called the Sakaguchi–Kuramoto model [19]. The model (1) can be reformulated in terms of the complex mean field, defined via

$$Z_1 = M e^{i\Theta} = \frac{1}{N}\sum_{k=1}^{N} e^{i\varphi_k}, \tag{2}$$

where M and Θ are the amplitude and the phase of the mean field. M is also called the order parameter of the synchronization transition; obviously, it varies from zero in the absence of synchrony to one, if all elements have identical phases. By means of simple manipulations, Eq. (1) can be re-written in the form

$$\dot{\varphi}_k = \omega + \varepsilon M \sin(\Theta - \varphi_k + \beta). \tag{3}$$

To explain the notion of the nonlinear coupling, let us interpret each of the Eq. (3) as an equation of a phase oscillator, driven by a harmonic force with amplitude $\delta = \varepsilon M$ and phase Θ. Although the interaction is described by a nonlinear sine-function, we denote the coupling in this model as linear, since all the parameters in Eq. (3) are independent of δ. Generally, the response of an oscillator to the forcing can depend on the amplitude of the latter. It means that the oscillator frequency as well as the parameters of the coupling function in Eq. (3) can depend on $\delta = \varepsilon M$. Considering these dependencies as a minimal possible nonlinear effect, we generalize Eq. (3) to

$$\dot{\varphi}_k = \omega(\varepsilon M) + R(\varepsilon M)\varepsilon M \sin(\Theta - \varphi_k + \beta(\varepsilon M)), \tag{4}$$

where the functions ω, R, and β tend to some constants as $\varepsilon M \to 0$. Model (4) is minimal, because here the dependence on φ_k remains purely harmonic; for a general nonlinear coupling see the discussion in [12].

To summarize, we call the coupling nonlinear, if the parameters of the coupling function depend on the amplitude of the force that acts on the oscillator. We emphasize that the dependencies themselves can be linear.

2.2. Effect of nonlinearity on the stability of the self-consistent synchronous solution

Let us again interpret the basic model (4) as equations for oscillators, driven by the force with amplitude $\delta = \varepsilon M$ and frequency ω_{ext}, so that the driving phase is $\Theta = \omega_{ext} t$. Considering one oscillator and dropping the index, we write:

$$\dot{\varphi} = \omega(\delta) + R(\delta)\delta \sin(\omega_{ext} t - \varphi + \beta(\delta)). \tag{5}$$

Now we discuss synchronization properties of this nonlinearly forced oscillator. Writing an equation for the phase difference $\varphi - \Theta$, we easily find the phase locking region (Arnold tongue) on a plane of parameters (δ, ω_{ext}):

$$\omega(\delta) - R(\delta)\delta \le \omega_{ext} \le \omega(\delta) + R(\delta)\delta. \tag{6}$$

In addition to this standard problem of synchronization by an external force, in our context we have to take into account the *self-consistency condition*, namely, that the force itself is produced by the ensemble of synchronized oscillators. This means that $\varphi = \Theta$ and $\dot{\varphi} = \omega_{ext}$, which being substituted in (5) gives

$$\omega_{ext} = \omega(\delta) + R(\delta)\delta \sin\beta(\delta). \tag{7}$$

This condition defines a line $\delta = \delta(\omega_{ext})$ within the synchronization region (6) of a harmonically driven individual oscillator; this line corresponds to possible fully synchronized states of the ensemble. Stability of these states is determined by the condition $\frac{d\dot{\varphi}}{d\varphi} = -R(\delta)\delta\cos\beta(\delta) < 0$.

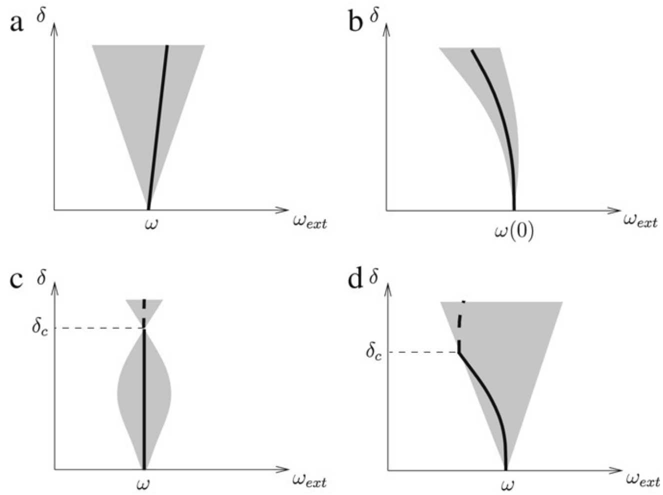

Fig. 1. Determination of self-consistent synchronous solutions for globally coupled oscillators. Arnold tongues (6) for individual oscillators, driven by a harmonic force with amplitude δ and frequency ω_{ext} are shown as shadowed regions. Bold (dashed) lines show the parameters, corresponding to stable (unstable) self-consistent solutions (7) for the ensemble dynamics. (a) Linear coupling. (b, c, d) Cases of nonlinear coupling, when different parameters of the model (5) depend on the amplitude δ of the force. (b) Nonlinearity in the natural frequency $\omega(\delta)$. (c) Nonlinearity in the effective forcing $R(\delta)$. Here at some critical value δ_c the tongue shrinks to zero. (d) Nonlinearity in the phase shift $\beta(\delta)$. Here at some critical value δ_c the line of stable self-consistent synchronous solutions, Eq. (7), reaches the boundary of the tongue; for $\delta > \delta_c$ these solutions are unstable.

For simplicity of presentations let us consider the effects of different possible nonlinearities separately. We start with the case of linear coupling, when $\omega = \text{const}$, $\beta = \text{const}$, $|\beta| \le \pi/2$, and $R = \text{const} > 0$; it is illustrated in Fig. 1(a). Here the line of stable self-consistent solutions is a straight line. Next we discuss the nonlinearity in the frequency, $\omega = \omega(\delta)$, while $\beta = \text{const}$, $|\beta| \le \pi/2$, and $R = \text{const} > 0$. Now the Arnold tongue is skewed (Fig. 1(b)), and the line of stable self-consistent solutions is skewed as well. However, it is easy to see that this line remains within the synchronization region. Thus, in both considered cases the stable synchronous solution exists for all amplitudes δ, only the frequency of this solution experiences a linear or nonlinear shift, respectively.

Two nontrivial cases leading to a breakdown of the collective synchrony are depicted in Fig. 1(c, d). In the first plot we illustrate a situation when $\omega = \text{const}$, $\beta = \text{const}$, $|\beta| \le \pi/2$, and $R(\delta)$ changes its sign with an increase of δ, e.g., $R(\delta) = a_1 - a_2\delta^2$, where $a_{1,2}$ are positive constants. Here at the critical strength of the forcing $\delta_c = \sqrt{a_1/a_2}$ the Arnold tongue shrinks to zero[1] and for $\delta > \delta_c$ the self-consistent synchronous solution becomes unstable, because the coupling becomes repulsive. The same may happen in the case when $\omega = \text{const}$, $R = \text{const} > 0$, but the phase shift β depends on the forcing strength δ; for definiteness in the example illustrated in Fig. 1(d) we assume that $|\beta| \le \pi/2$ for $\delta \to 0$ and monotonically decreases with δ. Now the Arnold tongue has a finite width for all amplitudes of the forcing, but the self-consistent solution (7) is stable only for $\delta < \delta_c$, where the critical value δ_c is determined from the condition $\beta(\delta) = \pm\pi/2$.

Now we come back to the coupled oscillator model (4) and recall that the amplitude of the forcing is $\delta = \varepsilon M$, where M is the mean field amplitude, defined by Eq. (2). Now the discussed above conditions for the loss of stability of a self-consistent synchronous solution with $\varphi_1 = \varphi_2 = \cdots = \varphi_N$ and $M = 1$ lead to the critical value of the parameter ε determined according to

$$R(\varepsilon_b) = 0 \quad \text{or} \quad \beta(\varepsilon_q) = \pm\pi/2. \tag{8}$$

Below we demonstrate that these two conditions determine critical values for transitions to bunch and quasiperiodic states, respectively. Therefore, we use indices b and q to denote the corresponding values of the critical coupling. The nontrivial dynamics beyond the criticality is illustrated by several examples in the next section.

2.3. Further generalization

The model (4) can be further generalized, if we take into account that a dependence on the forcing amplitude can appear in the equations for state variables. Then in the corresponding phase equation the parameters of the coupling can generally be functions of two arguments: the amplitude of the mean field M and of the bifurcation parameter (coupling constant ε), and not just functions of the product εM. In the following we analyze such a general model [12] for the case of identical oscillators:

$$\dot\varphi_k = \omega(\varepsilon, M) + R(\varepsilon, M)\varepsilon M \sin(\Theta - \varphi_k + \beta(\varepsilon, M)). \tag{9}$$

For this model, as it follows from the arguments above, the critical values of the bifurcation parameter ε are determined by the conditions similar to (8):

$$R(\varepsilon_b, 1) = 0 \quad \text{or} \quad \beta(\varepsilon_q, 1) = \pm\pi/2. \tag{10}$$

Finally, we note that the Kuramoto model (1) is a particular case of the Daido model [21–23]

$$\dot\varphi_k = \omega_k + N^{-1}\sum_j^N h(\varphi_j - \varphi_k), \tag{11}$$

where $h(\cdot)$ is an arbitrary 2π-periodic function. In its turn, model (9) is a particular case of the nonlinear generalization [12] of the Daido model; this generalization accounts for possible dependence of the interaction function on the generalized order parameters $Z_n = N^{-1}\sum_j^N e^{in\varphi_j}$.

2.4. Nonlinear coupling: An example

We further motivate our generalization of the Kuramoto model to Eq. (9) by the following example. We consider an ensemble of Landau–Stuart oscillators, each described by a complex variable a_k, coupled nonlinearly via the mean field $A = N^{-1}\sum_{k=1}^N a_k$:

$$\begin{aligned}\dot a_k &= (1 + i\omega)a_k - (1 + i\alpha)|a_k|^2 a_k \\ &\quad + (\mu_1 + i\mu_2)A - (\eta_1 + i\eta_2)|A|^2 A,\end{aligned} \tag{12}$$

where $\mu_{1,2}$ and $\eta_{1,2}$ are real parameters describing linear and nonlinear coupling, respectively. If these parameters are small, then in the first approximation we can neglect variations of the amplitudes of the oscillators. Looking for the solution in the form $a_k = e^{i\varphi_k}$ and $A = Me^{i\Theta}$, we obtain

$$\begin{aligned}\dot\varphi &= \omega - \alpha + M[(\mu_1 - \eta_1 M^2)\sin(\Theta - \varphi) \\ &\quad + (\mu_2 - \eta_2 M^2)\cos(\Theta - \varphi)].\end{aligned} \tag{13}$$

Introducing new notations $\omega - \alpha \to \omega$ and

$$R^2\varepsilon^2 = (\mu_1 - \eta_1 M^2)^2 + (\mu_2 - \eta_2 M^2)^2,$$

$$\tan\beta = \frac{\mu_2 - \eta_2 M^2}{\mu_1 - \eta_1 M^2}, \tag{14}$$

we finally obtain a particular case of Eq. (9), where the bifurcation parameter ε corresponds to one of the parameters $\mu_{1,2}$, $\eta_{1,2}$, or to their combination. The bifurcation values of parameters ε, β directly follow from relations (10).

[1] See [20] for an analysis of the closing of Arnold tongues for forced weakly nonlinear oscillators.

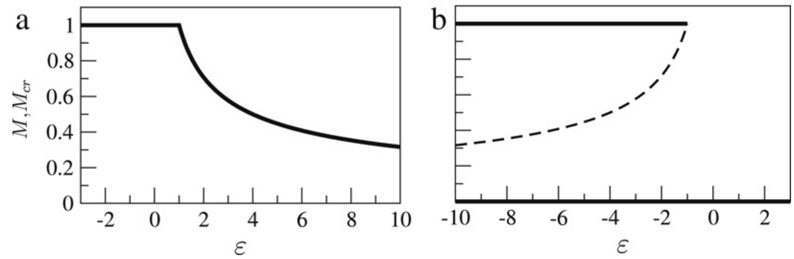

Fig. 2. Self-organized bunch states in the model (15). (a) Mean field amplitude M for positive μ_1 (here $\mu_1 = 1$). (b) Mean field amplitude M for negative μ_1 (here $\mu_1 = -1$). For $\varepsilon < \mu_1$ we observe multistability: the initial states with $M > M_{cr}(\varepsilon)$ synchronize; $M_{cr}(\varepsilon)$ is shown by a dashed line.

3. Self-organized bunch states

In the case when the nonlinearity in the coupling does not lead to the phase shift, but affects only the effective coupling strength R (like in Fig. 1(c)), the complete analysis of the dynamics is quite simple. For a particular example we consider purely real coupling in Eq. (13), i.e. $\mu_2 = \eta_2 = 0$. Let us fix μ_1 and take η_1 as a bifurcation parameter, i.e. $\varepsilon = \eta_1$. From Eq. (14) it follows that $\beta = 0$, the amplitude function takes the form $R\varepsilon = \mu_1 - \varepsilon M^2$, and the phase equation (9) reads

$$\dot{\varphi}_k = (\mu_1 - \varepsilon M^2) M \sin(\Theta - \varphi_k). \tag{15}$$

Here we set the frequency ω to zero, which can be always done by a transformation to the reference frame, rotating with velocity ω.

From (15) we easily obtain the equation for the evolution of the complex order parameter Z_1 (see Eq. (2)):

$$\frac{dZ_1}{dt} = \frac{1}{2}(\mu_1 - \varepsilon |Z_1|^2)(Z_1 - Z_1^* Z_2), \tag{16}$$

where Z_2 is the second generalized order parameter

$$Z_2 = M_2 e^{i\Theta_2} = \frac{1}{N}\sum_k e^{i2\varphi_k}. \tag{17}$$

(This order parameter is in fact the complex amplitude of the second harmonic of the distribution of the phases. It vanishes for a uniform distribution and is large if the distribution has two humps shifted by π.) Thus, the evolution of M obeys

$$\frac{dM}{dt} = \frac{1}{2}(\mu_1 - \varepsilon M^2) M (1 - M_2 \cos(\Theta_2 - 2\Theta)). \tag{18}$$

If we exclude a situation when the phases are initially organized in several clusters, then $M_2 \leq 1$ reaches unity only together with M for the fully synchronous state $\varphi_1 = \cdots = \varphi_N$. Therefore $(1 - M_2 \cos(\Theta_2 - 2\Theta)) > 0$ for $M < 1$ and $(1 - M_2 \cos(\Theta_2 - 2\Theta)) = 0$ if $M = M_2 = 1$. This consideration allows us to give a qualitative picture of the dynamics.

Consider first the case $\mu_1 > 0$. Here the critical value of parameter ε at which the steady state $M = 1$ bifurcates is $\varepsilon_b = \mu_1$. For $\varepsilon < \varepsilon_b$ the fully synchronous state with $M = 1$ is stable, whereas for $\varepsilon > \varepsilon_b = \mu_1$, a new stable state with

$$M = \sqrt{\frac{\varepsilon_b}{\varepsilon}} \quad \text{for } \varepsilon \geq \varepsilon_b, \tag{19}$$

appears according to Eq. (18), see Fig. 2(a). Thus, the system organizes itself in such a way that the coupling always vanishes and the system stays at the border between attraction and repulsion. We call this regime *self-organized bunch state*. Obviously, the above consideration can be done for the full equation (12): if with variation of a coupling parameter the nonlinear term compensates the linear one, the system settles on the border of stability of the synchronous regime, and the synchrony gets destroyed, cf. [24].

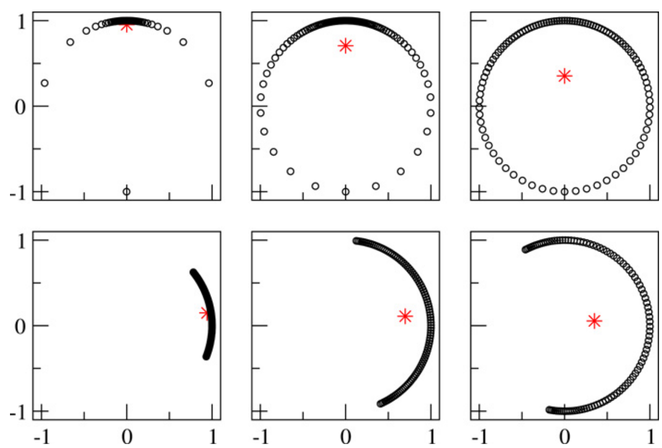

Fig. 3. Snapshots for the model (15) in the self-organized bunch state. Top and bottom row correspond to nearly uniform and nearly identical initial conditions, respectively (see text for details). Left, middle, and right column correspond to different values of coupling: $\varepsilon = 1.1$, $\varepsilon = 2$, and $\varepsilon = 8$. Individual oscillators are shown by circles; mean field is shown by a red star. For better visibility, only 100 oscillators out of $N = 1000$ are shown.

In the self-organized bunch state oscillators do not rotate (in the frame moving with frequency ω) and are generally non-uniformly distributed around the unit circle: this "static" distribution has a fixed mean field amplitude M but all other generalized order parameters Z_m, $m \geq 2$ are arbitrary; they depend on the initial conditions. This is illustrated in Fig. 3, where we show snapshots for model (15) for 3 different values of coupling and two different sets of initial conditions: a nearly uniform and a nearly identical initial distribution of the phases φ_k. For a nearly uniform distribution we took the phases as $\varphi_k = 2\pi(k-1)/N$ and added a perturbation 10^{-4} to one phase φ_1; by the nearly identical initial conditions we mean a uniform distribution of phases along the arc 0.02π. Parameters of the model are $N = 1000$ and $\mu_1 = 1$.

Consider now the case $\mu_1 < 0$. For $\varepsilon > \mu_1$ only the asynchronous state $M = 0$ is stable, whereas for $\varepsilon < \mu_1$ the system is bistable, i.e. both the fully synchronous ($M = 1$) and the asynchronous ($M = 0$) states are possible. Indeed, if the initial configuration has the mean field amplitude $M > M_{cr}$ such that $\mu_1 - \varepsilon M^2 > 0$, then, according to Eq. (18), the system synchronizes. This yields the value of the critical mean field amplitude $M_{cr} = \sqrt{\mu_1/\varepsilon}$, see Fig. 2(b). Here, again, besides the bistability in M, there exists a multistability with respect to initial conditions for asynchronous solutions.

4. Self-organized quasiperiodic states: Numerical illustration

In this Section we analyze the case illustrated in Fig. 1(d), when the breakdown of synchrony occurs due to the nonlinear phase shift $\beta(\varepsilon, M)$. A complete description of the arising dynamical state will be given in Section 5, where the theory is developed. Here we provide a numerical illustration of the effect. For simulations we

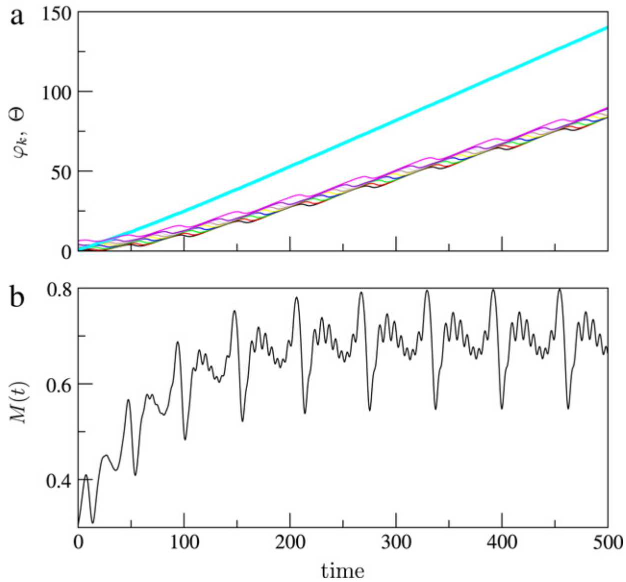

Fig. 4. Dynamics of 10 oscillators according to Eq. (20) for $\varepsilon = 0.4$ and initial conditions $\varphi_k = \frac{k^2}{100}2\pi$. (a) Individual phases φ_k (solid lines) and the phase of the mean field Θ (upper, bold, curve). (b) Time evolution of the amplitude of the mean field $M(t)$.

consider a particular case of the general model (4) with $R = 1$, $\omega = 0$ and $\beta(\varepsilon, M) = \beta_0 + \varepsilon^2 M^2$, i.e. the following model:

$$\dot{\varphi}_k = \varepsilon M \sin(\Theta - \varphi_k + \beta_0 + \varepsilon^2 M^2), \tag{20}$$

and simulate it for $\beta_0 = 0.475\pi$. According to the stability analysis above, the state of full synchrony $\varphi_1 = \cdots = \varphi_N$ becomes unstable at $\varepsilon_c = \sqrt{0.025\pi} \approx 0.2802$, where $\beta(\varepsilon_c, 1) = \pi/2$. Contrary to the bunch solutions, analyzed in Section 3, now beyond the transition we generally observe time-dependent, partially synchronous solutions where both the order parameter M as well as the growth rate of individual phases $\dot{\varphi}_k$ vary in time. We first illustrate this in Fig. 4 for the case of $N = 10$ oscillators. As follows from numerics, after a transient the amplitude of the mean field M becomes a periodic function of time with some period T_M. However, the complex mean field $Z_1(t)$ is not periodic, because $\Theta(t + T_M) (\mathrm{mod}\ 2\pi) \neq \Theta(t)$. This is demonstrated in Fig. 5, where we plot the values $M_i = M(t_i)$ at the moments of time when $\Theta(t_i)(\mathrm{mod}\ 2\pi) = 0$. This Poincaré map plot proves that the dynamics of the system is quasiperiodic.

We emphasize that the features of the oscillations $M(t)$ strongly depend on the initial conditions. Numerically we found the oscillations of the mean field amplitude to be minimal for a nearly uniform initial distribution of the phases $\varphi_k(0)$. To be precise, we took initially the phases as $\varphi_k = 2\pi(k-1)/N$ and added a perturbation 10^{-4} to φ_1. This case has been studied in more details for different number of oscillators in the ensemble: $N = 10$, $N = 100$, and $N = 1000$. The results are presented in Fig. 6, where we show the averaged over time amplitude of the mean field amplitude $\overline{M} = T^{-1}\int_0^T \mathrm{d}t M(t)$, its root mean square rms(M), and the mean frequencies of one oscillator $\omega_{osc} = T^{-1}(\varphi_k(T) - \varphi_k(0))$ and of the mean field $\Omega = T^{-1}(\Theta(T) - \Theta(0))$ as functions of the coupling strength ε. As expected, for small coupling $\varepsilon < \varepsilon_c$ we observe a synchronous regime with $M = 1$ and $\omega_{osc} = \Omega$, whereas a transition to a time-dependent state occurs at ε_q. Beyond this point, the time-averaged mean field amplitude $\overline{M}$ decays monotonically. Its value between zero and one corresponds to a partially coherent state, where the phases of oscillators are scattered around the unit circle. However, the instantaneous distributions of phases are not uniform (cf. Figs. 3 and 4). The time variations of the mean field amplitude M are characterized by the root mean square rms(M). The variations are mostly pronounced close to the transition point, and decay rapidly with the ensemble size N. This allows us to hypothesize that, for the considered initial conditions, the amplitude of the mean field in the thermodynamical limit is constant. This constant simply corresponds to a value for which $\beta(\varepsilon, M) = \pi/2$, which in our case means $M = \varepsilon_q/\varepsilon$ (we do not show this curve in Fig. 6 since it overlaps with the numerical curves).

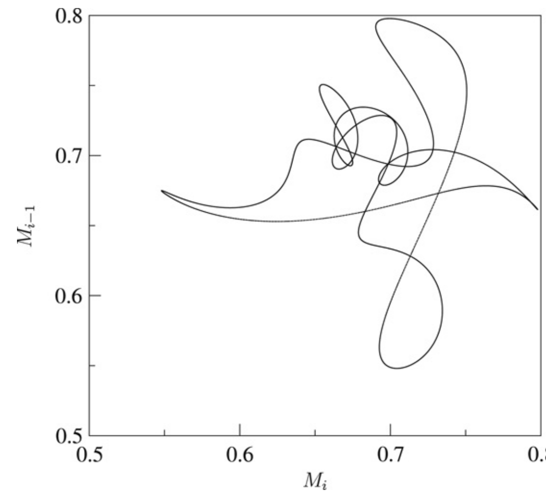

Fig. 5. The Poincaré map based on the dynamics of the mean field for ten oscillators. Points lie on a line, indicating that the dynamics is quasiperiodic with two incommensurate frequencies.

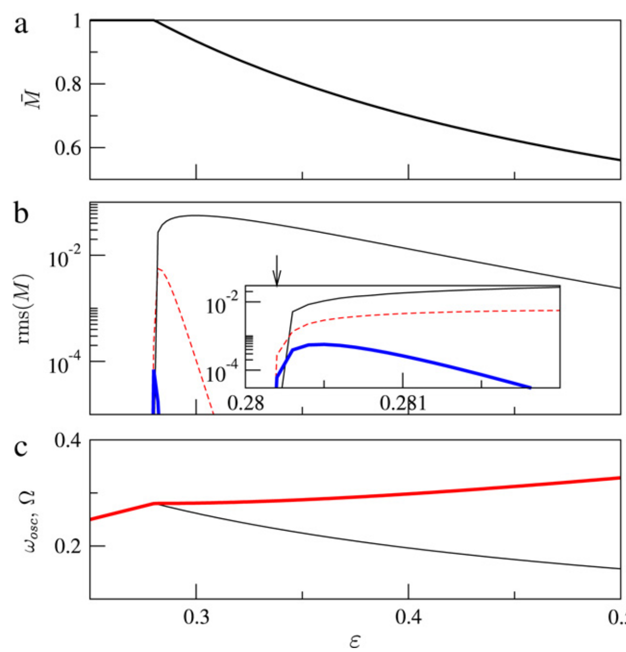

Fig. 6. Average value $\bar{M}$ (a) and root mean square (b) of the mean field amplitude and frequencies of oscillators ω_{osc} and of the mean field Ω (c) as functions of the coupling constant ε, for nearly uniform initial conditions. The results in (a) and (c) are shown for $N = 100$; the corresponding curves for $N = 10$ and $N = 1000$ differ by a third decimal and are therefore not shown here. Size effect manifests itself only in the fluctuations of the mean field (b); here the results for $N = 10$, $N = 100$, and $N = 1000$ are shown by solid, dashed and bold lines, respectively. The theoretical value of the critical coupling is indicated by an arrow. In (c) frequencies of the mean field and of one oscillator are shown by bold and solid lines, respectively.

Another important feature of the partially synchronous regime is a discrepancy between the frequency of the mean field Ω and that of individual units ω_{osc} (Fig. 6(c)). Note that both frequencies

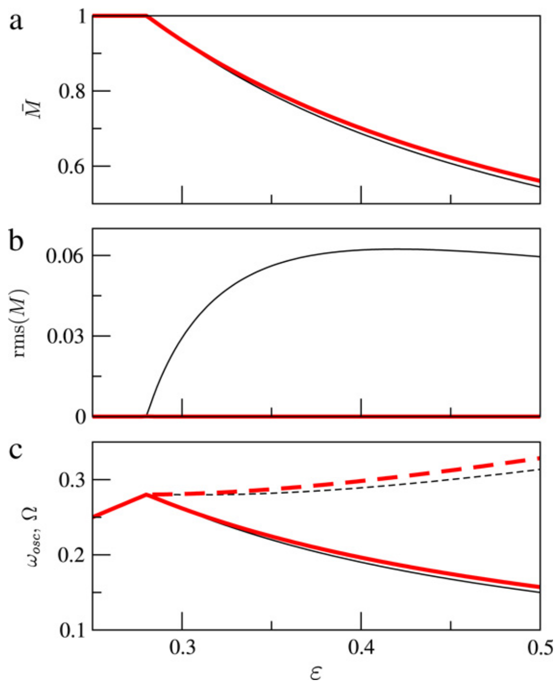

Fig. 7. Average value (a) $\bar{M}$ and root mean square (b) of the mean field amplitude and frequencies of oscillators ω_{osc} and of the mean field Ω (c), as function of the coupling strength ε for nearly identical (black solid line) and nearly uniform (red bold line) initial conditions, for $N = 1000$. In (c) dashed lines show the frequencies of the mean field and solid lines those of individual oscillators.

Fig. 8. Time dependence of the mean field amplitude (a, b), of the instantaneous frequency of the mean field (c, d), and of the phase of individual oscillators (e, f) for nearly uniform (left) and nearly identical (right) initial conditions; $N = 1000$. It is seen that uniform initial conditions lead to a harmonic mean field, whereas almost identical initial conditions yield a quasiperiodic mean field (see also Fig. 9). Individual oscillators are quasiperiodic in both cases.

seemingly smoothly depend on ε and therefore are, generally, incommensurate. It means that oscillators are generally in the quasiperiodic state.

Next we choose a different initial distribution of the phases, when they are almost identical, namely they are uniformly distributed along the arc 0.02π. In Fig. 7 we compare the results for this initial conditions with those for almost uniform distribution, for $N = 1000$. One can see that while for a nearly uniform initial distribution of phases $M \approx const$, for nearly identical initial phases there is a strong variation of M even in the thermodynamic limit. Another way of comparison is to look for time dependence (after a transient) of the mean field amplitude and of the instantaneous frequency, here for $\varepsilon = 0.5$ (Fig. 8). Similar to the case of 10 oscillators, presented in Fig. 5, the mean field itself is quasiperiodic. This can be confirmed by plotting the Poincaré map in Fig. 9. The map was constructed for $\Theta(t_i)(\text{mod } 2\pi) = 0$.

To illustrate the self-organized onset of quasiperiodic dynamics, we plot in Fig. 10 the transients, for both nearly uniform and nearly identical initial conditions. We see that in the former case the amplitude monotonically increases unless it approaches a constant value $\varepsilon_q/\varepsilon \approx 0.5605$. Similarly, the nonlinear phase shift β approaches $\pi/2$.

Finally, we note that computation of the Lyapunov exponents shows that the system has one negative and $N-1$ vanishingly small exponents. These results will be explained in Section 5.3.

5. Self-organized states: Theory

In this section we present a theoretical analysis of the self-organized collective dynamics. First we mention that a particular, however important, solution under the assumption of a harmonic mean field ($M = \text{const}$, $\Theta = \Omega t$) was treated in our previous publication [12]. Following the standard Kuramoto approach for linearly coupled oscillators (Eq. (1)) and writing self-consistent equations for the mean field [2,10], we have found in [12] the quantities $M(\varepsilon)$, $\Omega(\varepsilon)$, and $\omega_{osc}(\varepsilon)$ for the nonlinear case (Eq. (4)),

Fig. 9. Poincaré section for the mean field for the case of almost identical initial conditions, $N = 1000$, proves that the mean field is quasiperiodic.

for supercritical coupling $\varepsilon > \varepsilon_c$. Here we exploit the method developed by Watanabe and Strogatz (WS) [18], to analyze the ensembles of nonlinearly coupled oscillators in a general setting of a time-dependent mean field.

5.1. Watanabe–Strogatz theory

In the following analysis we concentrate on the effect of functions $R(\varepsilon, M)$ and $\beta(\varepsilon, M)$, and take $\omega(\varepsilon, M) = \text{const} = 0$. For $\omega = 0$ both the standard Kuramoto model (1) and the generalized model (9) can be rewritten as

$$\dot{\varphi}_k = g\cos\varphi_k + h\sin\varphi_k. \quad (21)$$

In the latter case the functions g, h are

$$g = R\varepsilon M\sin(\Theta + \beta), \qquad h = -R\varepsilon M\cos(\Theta + \beta). \quad (22)$$