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Physica A 351 (2005) 126–132
PHYSICA A
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Synchronization of self-sustained oscillators by common white noise

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Received 18 October 2004
Available online 1 January 2005

Abstract

We study the stability of self-sustained oscillations under the influence of external noise. For small-noise amplitude a phase approximation for the Langevin dynamics is valid. A stationary distribution of the phase is used for an analytic calculation of the maximal Lyapunov exponent. We demonstrate that for small noise the exponent is negative, which corresponds to synchronization of oscillators.

PACS: 05.40.−a; 02.50.Ey; 05.45.Xt

Keywords: Noise; Synchronization; Lyapunov exponent

1. Introduction

The main effect of noise on periodic self-sustained oscillations is phase diffusion: the oscillations are no more periodic but possess finite correlations [1,2]. However, noise can play also an ordering role, e.g., it can lead to a synchronization. If two identical (or slightly different) systems are driven by the same noise, then their states can be synchronized by this action. This effect depends on the sign of the largest

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2004.12.014

Lyapunov exponent that measures stability of the motion. For deterministic periodic self-sustained oscillations the largest Lyapunov exponent is zero, it corresponds to a perturbation along the trajectory in the phase space. In driven systems the largest Lyapunov exponent may become negative, what would lead to a synchronization: both systems driven by the same noise forget their initial conditions and eventually evolve to a same state. This problem was first formulated in Refs. [3,4], where the Lyapunov exponent has been calculated for a self-sustained quasiharmonic oscillator driven by a random sequence of pulses. In this paper, we consider general dynamical systems driven by Gaussian white noise. We note that the effect considered is a particular case of synchronization in noisy systems, for more general aspects of this phenomenon see Refs. [5,6].

Our approach is based on the reduction of the dynamics to a phase equation. This is valid if the action of noise on the oscillation amplitude is small. We will derive the Langevin equation for the phase and will find a stationary distribution of it. The Lyapunov exponent is represented via an integral of this distribution. We will demonstrate that for small noise the exponent is negative, i.e., small noise always leads to synchrony.

2. Basic model

We start with general stochastic equations for the dynamics of an N-dimensional oscillatory system x_j, $j = 1, \ldots, N$, in the presence of uncorrelated forces $\xi_k(t)$, $k = 1, \ldots, M \leqslant N$:

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = f_j(\mathbf{x}) + \sum_{k=1}^{M} Q_{jk}(\mathbf{x})\xi_k(t) . \tag{1}$$

If in the noiseless system there exists a limit cycle $\mathbf{x}^0 = \mathbf{x}^0(t + 2\pi/\omega_0)$, it can be parameterized by the phase variable $\varphi(\mathbf{x}^0)$ [7], which grows linearly in time: $\dot{\varphi} = \omega_0$. For a stable limit cycle the phase, satisfying the same equation, can be introduced also in its vicinity. In the presence of noise the evolution of the phase in a small vicinity of the cycle is governed by equations

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega_0 + \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{\partial\varphi(\mathbf{x})}{\partial x_j} Q_{jk}(\mathbf{x})\bigg|_{\mathbf{x}=\mathbf{x}^0(\varphi)} \xi_k(t) . \tag{2}$$

The deviations from the cycle are small in two cases: (i) if the noise intensity is small, or (ii) if the leading negative Lyapunov exponent is large whereas the noise is moderate. Below we normalize time in such a way that the frequency of the limit cycle is one. A particular form of the stochastic equation for the phase depends on how the noise enters the original system (1). If there is a single noise source, i.e., only for one k $\xi_k \neq 0$, then

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 + \varepsilon f(\varphi)\xi(t) , \tag{3}$$

where $\xi(t)$ is a δ-correlated Gaussian noise with $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t'+t)\rangle = 2\delta(t')$, parameter ε describes the noise intensity (as a result of time normalization $\varepsilon \sim \omega_0^{-1/2}$). $f(\varphi)$ is a normalized periodic function of the phase: $f(\varphi) = f(\varphi + 2\pi)$, $\int_0^{2\pi} f^2(\varphi)\,\mathrm{d}\varphi = 2\pi$. A more complex equation appears if there are several noise sources in the original system (we call this multi-component case):

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 + \sum_{k=1}^{M} \varepsilon_k f_k(\varphi)\xi_k(t), \quad \langle\xi_i(t)\xi_k(t'+t)\rangle = 2\delta(t')\delta_{ik} . \tag{4}$$

Our goal is the analytical analysis of stability of solutions of stochastic equations (3) and (4) [8]. For this we consider the linearized Eq. (3) for a small deviation α:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \varepsilon\alpha f'(\varphi)\xi(t) . \tag{5}$$

The Lyapunov exponent measuring the average exponential growth rate of α can be obtained by averaging the corresponding velocity

$$\lambda = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \ln \alpha \right\rangle = \langle \varepsilon f'(\varphi)\xi(t)\rangle . \tag{6}$$

For the multi-component noise the corresponding expression reads

$$\lambda = \sum_{k=1}^{M} \langle \varepsilon_k f'_k(\varphi)\xi_k(t)\rangle . \tag{7}$$

Note that the Lyapunov exponent determines the asymptotic behavior of small perturbations, and in our case describes whether close initial points diverge or converge in course of the evolution. This process must not be monotonous, i.e., close trajectories can diverge at some time intervals while demonstrating asymptotic convergence, and vice versa.

3. Fokker–Planck equation and its stationary solution

The Fokker–Planck equation for the stochastic equation (3), interpreted in Stratonovich sense, reads [9,10]

$$\frac{\partial W(\varphi,t)}{\partial t} + \frac{\partial}{\partial\varphi}\left(W(\varphi,t) - \varepsilon^2 f(\varphi)\frac{\partial}{\partial\varphi}(f(\varphi)W(\varphi,t)) \right) = 0 . \tag{8}$$

In a stationary state the probability flux S is constant:

$$W(\varphi) - \varepsilon^2 f(\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}(f(\varphi)W(\varphi)) = S . \tag{9}$$

This allows us to express the solution for periodic boundary conditions as

$$W(\varphi) = C\int_{\varphi}^{\varphi+2\pi} \frac{\mathrm{d}\psi}{f(\varphi)f(\psi)} \exp\left(-\frac{1}{\varepsilon^2}\int_{\varphi}^{\psi} \frac{\mathrm{d}\theta}{f^2(\theta)} \right) , \tag{10}$$

where C is determined by the normalization condition:

$$C^{-1}=\int_0^{2\pi}\mathrm{d}\varphi\int_{\varphi}^{\varphi+2\pi}\mathrm{d}\psi\,\frac{\exp\left(-\frac{1}{\varepsilon^2}\int_{\varphi}^{\psi}\frac{\mathrm{d}\theta}{f^2(\theta)}\right)}{f(\varphi)f(\psi)}\,. \tag{11}$$

The probability flux reads

$$S=\left(1-\exp\left(-\frac{1}{\varepsilon^2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{f^2(\theta)}\right)\right)C\,. \tag{12}$$

The analogous expression for the multi-component noise is

$$\frac{\partial W(\varphi,t)}{\partial t}+\frac{\partial}{\partial\varphi}\left(W(\varphi,t)-\sum_{k=1}^{M}\varepsilon_k^2 f_k(\varphi)\frac{\partial}{\partial\varphi}(f_k(\varphi)W(\varphi,t))\right)=0\,. \tag{13}$$

Remarkably, this formula is equivalent to the single-component one (8), if one sets

$$f^2(\varphi)=\frac{\sum_{k=1}^{M}\varepsilon_k^2 f_k^2(\varphi)}{\sum_{k=1}^{M}\varepsilon_k^2},\quad \varepsilon^2=\sum_{k=1}^{M}\varepsilon_k^2\,. \tag{14}$$

Thus, the stationary solution presented above is valid in this case as well.

4. Lyapunov exponent

For the calculation of the Lyapunov exponent (6), (7) we have to find averages of the type $\langle F(\varphi)\xi(t)\rangle$. Such expressions for stochastic equations (3) and (4) with delta-correlated noise can be calculated using the Novikov–Furutsu formula:

$$\langle F(\varphi)\xi(t)\rangle=\varepsilon\langle F'(\varphi)f(\varphi)\rangle\,. \tag{15}$$

Writing the average as the integral over the stationary phase distribution we obtain for the single-component case

$$\lambda=\varepsilon^2\langle f''(\varphi)f(\varphi)\rangle=\varepsilon^2\,C\int_0^{2\pi}\mathrm{d}\varphi\int_{\varphi}^{\varphi+2\pi}\mathrm{d}\psi\,\frac{\frac{\partial^2 f(\varphi)}{\partial\varphi^2}}{f(\psi)}\exp\left(-\frac{1}{\varepsilon^2}\int_{\varphi}^{\psi}\frac{\mathrm{d}\theta}{f^2(\theta)}\right)\,. \tag{16}$$

The corresponding result for the multi-component noise reads

$$\lambda=\sum_{k=1}^{M}\varepsilon_k^2\int_0^{2\pi}\frac{\mathrm{d}^2 f_k(\varphi)}{\mathrm{d}\varphi^2}f_k(\varphi)\,W(\varphi)\,\mathrm{d}\varphi\,. \tag{17}$$

Prior to the analysis of the obtained expressions we mention that in the limit of small noise the Lyapunov exponent is always negative: in the leading order in ε

$$\lambda\approx-\sum_{k=1}^{M}\frac{\varepsilon_k^2}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}f_k(\varphi)}{\mathrm{d}\varphi}\right)^2\mathrm{d}\varphi<0\,. \tag{18}$$

(M can be equal to 1, what corresponds to a single-component noise.)

4.1. Example: linearly polarized homogeneous noise

If in the original system the noise is additive and forces only one variable of the system, and the limit cycle is nearly a circle with nearly constant phase velocity on it, then one obtains a single-component stochastic phase equation with $f(\varphi) = \sqrt{2}\sin\varphi$. In this case

$$\int_{\varphi}^{\varphi+2\pi} \frac{\mathrm{d}\psi}{f(\psi)} \exp\left(-\frac{1}{\varepsilon^2}\int_{\varphi}^{\psi}\frac{\mathrm{d}\theta}{f^2(\theta)}\right) = \int_{\varphi}^{\pi+\pi[\frac{\varphi}{\pi}]} \mathrm{d}\psi\, \frac{\exp\left(\frac{\cot\psi-\cot\varphi}{2\varepsilon^2}\right)}{\sqrt{2}\sin\psi}\,,$$

where $[\cdots]$ denotes the integer part. For the chosen function $f(\varphi)$ the distribution has period π and for $\varphi \in [0,\pi)$

$$W(\varphi) = \frac{C}{2}\int_{\varphi}^{\pi} \mathrm{d}\psi\, \frac{\exp\left(\frac{\cot\psi-\cot\varphi}{2\varepsilon^2}\right)}{\sin\psi\,\sin\varphi}\,,$$

$$S = C = \left(\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{y}\mathrm{d}x\,\frac{\exp\left(\frac{x-y}{2\varepsilon^2}\right)}{\sqrt{1+x^2}\sqrt{1+y^2}}\right)^{-1},$$

$$\lambda = -\frac{\varepsilon^2 C}{2}\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{y}\mathrm{d}x\,\frac{\exp\left(\frac{x-y}{2\varepsilon^2}\right)}{(1+x^2)^{\frac{1}{2}}(1+y^2)^{3/2}}\,. \tag{19}$$

In the transformation to last expression, which makes the convergence of the integrals clear, the ansatz $(x,y) = (\cot\psi, \cot\varphi)$ has been used. A further simplification appears to be not possible, and formula (19) has been used for numerical calculation. The obtained dependence of the Lyapunov exponent on the noise intensity is presented in Fig. 1.

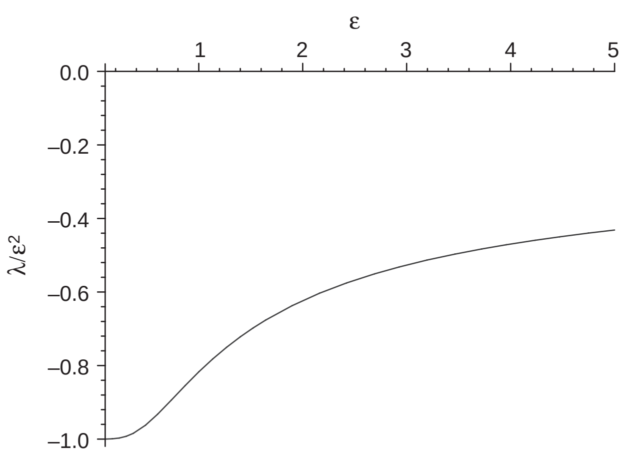

Fig. 1. Linearly polarized noise. Dependence of the Lyapunov exponent (normalized by ε^2) on the noise amplitude ε. For small and large noises λ_0 quadratically depends on ε, with different coefficients.

4.2. Example: superposition of two independent linearly polarized noise terms

If we use the same conditions as above (nearly circular limit cycle with uniform rotation on it) but consider the effect of two independent noisy forces acting on two variables shifted in phase by $\pi/2$, then naturally we get an equation with a multi-component noise with $f_1(\varphi)=\sqrt{2}\sin\varphi$ and $f_2(\varphi)=\sqrt{2}\cos\varphi$. The effective coupling function $f(\varphi)=\sqrt{1+\Delta\cos 2\varphi}$ and the noise intensity $\varepsilon^2=\varepsilon_1^2+\varepsilon_2^2$, where $\Delta\equiv(\varepsilon_2^2-\varepsilon_1^2)/(\varepsilon_2^2+\varepsilon_1^2)$ (evidently $\Delta\in[-1,1]$) should be inserted in the stationary distribution (10)–(12). From the expression for $f(\varphi)$ follows the symmetry $(\Delta,\varphi)\leftrightarrow(-\Delta,\varphi+\pi/2)$. In this case

$$\int\frac{\mathrm{d}\theta}{f^2(\theta)}=\frac{1}{\sqrt{1-\Delta^2}}\left(\pi\left[\frac{\theta}{\pi}\right]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right),$$

what gives the following expressions for the probability density, flux, and the normalization constant

$$W(\varphi)=C\int_{\varphi}^{\varphi+\pi}\mathrm{d}\psi\,\frac{\exp\left(-\frac{1}{\varepsilon^2\sqrt{1-\Delta^2}}\left(\pi[\frac{\theta}{\pi}]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right)\Big|_{\varphi}^{\psi}\right)}{\sqrt{1+\Delta\cos 2\varphi}\sqrt{1+\Delta\cos 2\psi}},$$

$$C=\left(2\int_0^{\pi}\mathrm{d}\varphi\int_{\varphi}^{\varphi+\pi}\mathrm{d}\psi\,\frac{\exp\left(-\frac{1}{\varepsilon^2\sqrt{1-\Delta^2}}\left(\pi\left[\frac{\theta}{\pi}\right]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right)\Big|_{\varphi}^{\psi}\right)}{\sqrt{1+\Delta\cos 2\varphi}\sqrt{1+\Delta\cos 2\psi}}\right)^{-1},$$

$$S=\left(1-\exp\left(-\frac{\pi}{\varepsilon^2\sqrt{1-\Delta^2}}\right)\right)C.$$

The final expression for the Lyapunov exponent reads

$$\begin{aligned}\lambda=&-2\varepsilon^2C\int_0^{\pi}\mathrm{d}\varphi\int_{\varphi}^{\varphi+\pi}\mathrm{d}\psi\,\frac{\sqrt{1+\Delta\cos 2\varphi}}{\sqrt{1+\Delta\cos 2\psi}}\\&\times\exp\left(-\frac{1}{\varepsilon^2\sqrt{1-\Delta^2}}\left(\pi\left[\frac{\theta}{\pi}\right]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right)\Big|_{\varphi}^{\psi}\right).\end{aligned}\tag{20}$$

The dependence on the noise intensity and the essential parameter Δ is shown in Fig. 2.

5. Conclusion

In this paper we have demonstrated a possibility of synchronization of two identical dynamical systems driven by the same white Gaussian noise, assuming that the phase approximation is valid. The quantitative characteristics of the

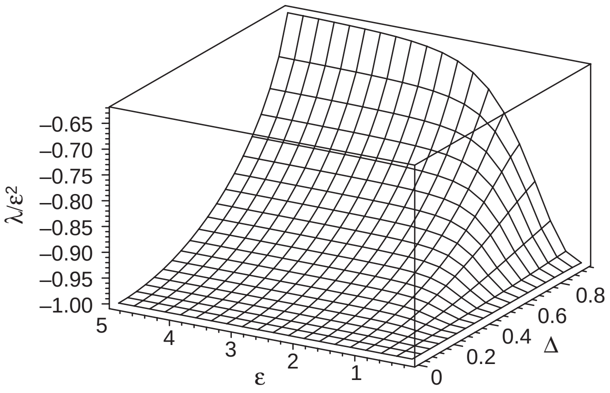

Fig. 2. Superposition of two independent linearly polarized noise terms. Dependence of the Lyapunov exponent (normalized by ε^2) on the noise amplitude ε and on the normalized ratio of noise intensities Δ.

synchronization is the Lyapunov exponent. Analytical expressions for this quantity have been found for a single-component and multi-component noisy forces. In all considered cases the Lyapunov exponent is negative. We expect that it is a general property for small noise. Numerical experiments with strong noise show that the Lyapunov exponent can be positive, and these results will be reported elsewhere.

Acknowledgements

D.S. Goldobin acknowledges support of the DAAD trilateral project "Germany-France-Russia", of the "Dinastija" foundation, and of the Moscow International Center for Fundamental Physics.

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