Continuous approach for the random-field Ising chain

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We study the random-field Ising chain in the limit of strong exchange coupling. In order to calculate the free energy we apply a continuous Langevin-type approach. This continuous model can be solved exactly, whereupon we are able to locate the crossover between an exponential and a power-law decay of the free energy with increasing coupling strength. In terms of magnetization, this crossover restricts the validity of the linear scaling. The known analytical results for the free energy are recovered in the corresponding limits. The outcomes of numerical computations for the free energy are presented, which confirm the results of the continuous approach. We also discuss the validity of the replica method which we then utilize to investigate the sample-to-sample fluctuations of the finite size free energy.

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I. INTRODUCTION

A first stage extension of the usual Ising model to the description of impure systems is to allow for fluctuations of either the exchange coupling (spin glasses) or the magnetic field. In this paper we are concerned with the latter case, which is referred to as the random-field Ising model. The magnetic field is assumed to be constant, i.e., it acts as a quenched disorder. Despite its simplicity this model gives rise to a variety of disorder induced phenomena and is thus one of the basic models for the statistical description of disordered systems. The random-field Ising model is relevant to many applications, ranging from the description of impure materials to the problem of protein folding (see [1], and references therein). It also shares many properties with spin glasses, such as frustration paired with a slow approach to equilibrium [2]. Roughly speaking the competition of the two effects, the spin alignment with the disordered field versus the avoidance of interfaces, leads to a frustration. Qualitatively the resulting effect on the thermodynamics is well understood, mostly on the basis of Imry-Ma–like arguments [3] which analyze energetically favorable configurations. However, quantitative results exist mainly for the one-dimensional case that is considered in the following. By assuming the canonical ensemble we are especially interested in the free energy, which is often calculated by means of the replica method. In the one-dimensional case considered here, however, frustration gives rise to a nonanalytical scaling of the free energy as a function of the thermal correlation length which cannot be described by the standard replica approach. This effect has been analyzed in [4,5], where the authors distinguish the case of a magnetic field with zero average from the case of a nonvanishing average.

We develop here a plain continuous approach, which yields an analytical expression for the free energy that incorporates these two cases. Moreover, we find the dependence on the average of the applied magnetic field explicitly, which allows the calculation of the magnetization by differentiating the free energy. Thus we are able to describe in a closed form the crossover between the two scaling relations of [4,5]. In terms of the magnetization as a function of the average magnetic field, this crossover sets a limit to the linear scaling: If the ratio of the average magnetic field to the strength of the exchange coupling is of order of the inverse Imry-Ma length, the competing impact of the constant and the disordered part of the magnetic field gives rise to a nonlinear behavior of the magnetization.

The setup of this paper is as follows: After a review of the transfer matrix formalism we introduce the continuous approach, which allows an analytical calculation of the free energy. The results are analyzed in Sec. IV, followed by a discussion of the magnetization in Sec. V. A comparison with numerical outcomes is given in Sec. VI. Finally, we show in Sec. VII the results of the replica method, that are valid for large enough average magnetic fields.

II. THE GENERAL SETUP

We consider the Ising chain in a random field $\tilde{h}_m$, defined by the following Hamiltonian:

$$\mathcal{H} = -\tilde{J}\sum_{m=1}^{N-1} \sigma_m\sigma_{m+1} - \sum_{m=1}^{N} \tilde{h}_m\sigma_m, \quad \text{with } \sigma_m = \pm 1. \tag{1}$$

The strong uniform exchange coupling is given by $\tilde{J} \gg kT$. The quenched random field is given by independent, identically distributed random numbers with an average $\tilde{h}$, i.e., it can be written as

$$\tilde{h}_m = \tilde{\eta}_m + \tilde{h} \text{ with } \langle\tilde{\eta}_m\rangle = 0, \quad \langle\tilde{\eta}_m\tilde{\eta}_n\rangle = 2\tilde{D}\delta_{mn}.$$

Because of the symmetry of the problem it is sufficient to consider the case $\tilde{h} \geq 0$, which is adopted in the sequel. For

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the analytical approach below we assume a normal distribution of $\tilde{\eta}$, however, the numerical results in Sec. VI show the validity of the main results also for other distributions.

The calculation of the partition function can be accomplished by considering the spin-dependent partition function $Z_\sigma(n)$ of a chain with length n. The spatial evolution of the latter is determined by a matrix product as

$$\begin{pmatrix} Z_+(n+1) \\ Z_-(n+1) \end{pmatrix} = e^J \begin{pmatrix} e^{\eta_n+h} & \varepsilon e^{\eta_n+h} \\ \varepsilon e^{-\eta_n-h} & e^{-\eta_n-h} \end{pmatrix} \begin{pmatrix} Z_+(n) \\ Z_-(n) \end{pmatrix}$$

$$\equiv e^{nJ} \prod_{m=1}^{n} \begin{pmatrix} e^{\eta_m+h} & \varepsilon e^{\eta_m+h} \\ \varepsilon e^{-\eta_m-h} & e^{-\eta_m-h} \end{pmatrix} \begin{pmatrix} Z_+(1) \\ Z_-(1) \end{pmatrix}, \quad (2)$$

where we denote $\eta_m = \beta\tilde{\eta}_m$, $h=\beta\tilde{h}$, $J=\beta\tilde{J}$, and $D=\beta^2\tilde{D}$ with the inverse temperature $\beta=(kT)^{-1}$. Moreover we have introduced the "coupling" parameter

$$\varepsilon \equiv e^{-2J} \ll 1,$$

and are mainly concerned in the limiting behavior for $\varepsilon \to 0$. The elements of the matrix product on the right-hand side (rhs) of Eq. (2) are directly connected to correlations, such as the thermal average $\overline{\sigma_n}^{\text{th}}|_{\sigma_1=1}$ of the spin at site n conditioned to the spin at site 1 pointing up[1] (see [5]). The free energy per spin $\tilde{F}(\varepsilon)$ for the infinite system corresponds to the Lyapunov exponent of the matrix product (2),

$$F \equiv \beta\tilde{F} = -\lim_{n\to\infty} \frac{1}{n} \ln \| Z_\sigma(n) \|$$

$$= -J - \lim_{n\to\infty} \frac{1}{n} \ln \mathrm{Tr} \prod_{m=1}^{n} \begin{pmatrix} e^{\eta_m+h} & \varepsilon e^{\eta_m+h} \\ \varepsilon e^{-\eta_m-h} & e^{-\eta_m-h} \end{pmatrix}. \quad (3)$$

Derrida and Hilhorst [4] and ten years later Mello and Robledo [5] derived analytical expressions for the free energy. In [4] the case $h>0$, $\langle e^{2\eta_m} \rangle > e^{2h}$ was considered, with the result

$$F(\varepsilon) - F(0) \approx -C\varepsilon^{2\alpha}, \quad \text{for } \varepsilon \to 0. \quad (4)$$

The exponent α is calculated as the positive root of $\langle e^{-2\alpha(\eta_m+h)} \rangle = 1$. The critical point $h=0$ was investigated in [5] for the weak disorder limit $D \ll 1$. Here a logarithmic singularity of F has been derived,

$$F(\varepsilon) - F(0) \approx \frac{-2D}{|\ln(\varepsilon/D)|} \approx -\frac{D}{J}. \quad (5)$$

The power-law scaling on the rhs of relation (5) can be understood by a simple argumentation like that of Imry-Ma. For this we sketch the discrete dynamics given by Eq. (2) for $h=0$, where we consider for simplicity the evolution of the variables $z_\pm(n) \equiv Z_\pm(n) e^{-nJ}$. The parameter ε can be seen as a small coupling between the dynamics of z_+ and z_-. According to Eq. (2) the coupling is effective if $z_\pm \simeq z_\mp \varepsilon^{-1}$. Consider

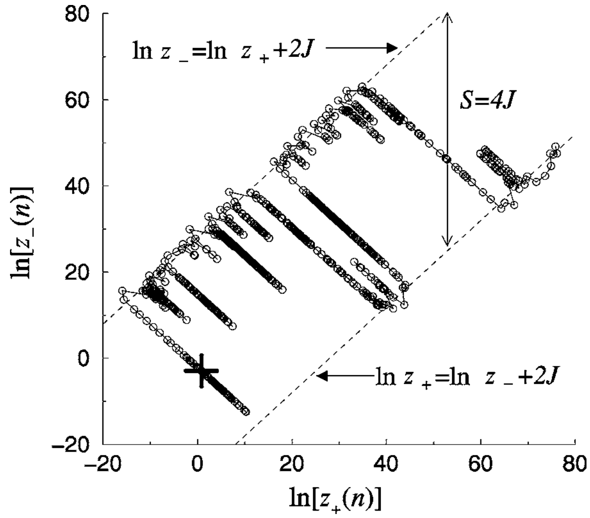

FIG. 1. Typical motion (starting at the cross) in the plane $\ln z_\pm$ for $J=14$ and $D=1$. The dashed lines indicate the boundaries due to the coupling ε.

now the logarithms $\ln z_\pm$: These variables perform a random walk with the increments $\pm\eta_m$ (note that $\ln z_+ + \ln z_- = \text{const}$ for $\varepsilon=0$), where the coupling effectively reflects the motion at the boundaries $\ln z_\pm = \ln z_\mp + 2J$ in the direction of growing $z_\mp$. That is, the motion takes place within a strip of width $S=4J$ with positive shifts at the boundaries. A typical realization of the motion in the plane of $\ln z_\pm$ is plotted in Fig. 1. Because $z_+(n)$ is connected to the probability that the spin σ_n points up, the intervals with $\ln z_+ > \ln z_-$ correspond to domains with spins pointing up. The variance of $\ln z_\pm(n)$ grows due to the random field η_n as $4Dn$. Thus, for $n \simeq 4J^2D^{-1} \equiv L_{\text{IM}}$ the deviations are of order $4J$ which induces a flip to a domain with spins pointing down. In Fig. 1 this corresponds to a change of the motion from the boundary $\ln z_+ = \ln z_- + 2J$ to the opposite one. The length L_{IM} just corresponds to the Imry-Ma length of domains at low temperatures. Now each reflection due to the coupling ε leads to an increase of $\ln z_\pm$ which is of order $S=4J$. The length between such reflections is of order L_{IM}. Thus, the contribution of the coupling to the free energy per spin can be estimated as $-SL_{\text{IM}}^{-1} = -DJ^{-1}$ which corresponds to the scaling in (5).

In the following sections we complete the results (4) and (5) with an analytical form for the free energy which incorporates both the cases $h=0$ and $h>0$ and which delivers an explicit expression for the constant C in Eq. (4). This allows us to describe a crossover in the small ε behavior, whose location depends on the average magnetic field h.

III. CONTINUOUS APPROACH

We assume the case of a weak field and strong coupling, where the values of D, h, and ε are bounded by a small number δ:

$$D, h, \varepsilon < \delta \ll 1.$$

In practice our results remain valid for relatively large fields, h, $D \lesssim 0.1$, as is confirmed by the numerical computations in

[1]The thermal average of a quantity u refers to the average over the canonical ensemble, which we denote as $\bar{u}^{\text{th}}$. The sample average $\langle * \rangle$, on the other hand, refers to the average with respect to the quenched disorder distribution.

Sec. VI. In the following, especially the limit $\varepsilon \to 0$ will be of interest. With these prerequisites we propose the following continuous analog of the discrete evolution which is defined by the matrix product in (2):

$$\frac{d}{dx}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \xi(x)+h & \varepsilon \\ \varepsilon & -\xi(x)-h \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{6}$$

The disordered part of the magnetic field is assumed to be a δ-correlated Gaussian random process, also known as white noise, which is defined by

$$\langle \xi \rangle = 0, \quad \langle \xi(x)\xi(x') \rangle = 2D\,\delta(x-x'). \tag{7}$$

Equation (6) is formally equivalent to coupled linear Langevin equations where we adopt the Stratonovich interpretation. Hence the probability density of suitably chosen variables can be found by means of the Fokker-Planck equation [13]. This enables us to calculate the Lyapunov exponent analytically, as is shown in the next section.

In order to justify our model we integrate Eq. (6):

$$\begin{pmatrix} w_1(x+1) \\ w_2(x+1) \end{pmatrix} = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} + \int_x^{x+1} \begin{pmatrix} \xi(t)+h & \varepsilon \\ \varepsilon & -\xi(t)-h \end{pmatrix} \times \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} dt.$$

By iterating this integral equation one obtains an expansion, which can be summed up as follows:

$$\begin{pmatrix} w_1(x+1) \\ w_2(x+1) \end{pmatrix} = \begin{pmatrix} e^{\eta_x+h} & \varepsilon e^{\eta_x+h} \\ \varepsilon e^{-\eta_x-h} & e^{-\eta_x-h} \end{pmatrix}\begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} + O(\delta^2)\mathbf{v}.$$

Here $\mathbf{v}$ is a fluctuating vector of order $\sqrt{w_1^2+w_2^2}$. The random numbers η_x are given by

$$\eta_x = \int_x^{x+1} \xi(t)dt.$$

Their statistical properties follow from Eqs. (7), i.e., they are normal distributed and obey

$$\langle \eta_x \rangle = 0, \quad \langle \eta_x \eta_{x+k} \rangle = 2D\,\delta_{k,0}. \tag{8}$$

This establishes the equivalence of Eqs. (2) and (6), given that δ is small and that the random field is distributed according to a Gaussian.

IV. THE FREE ENERGY

The free energy is defined by the Lyapunov exponent of the matrix product in Eq. (2). In other words, it is given by the dominating exponential growth of the components $Z_+(n)$, $Z_-(n)$. Thus, in terms of the continuous model, the free energy can be calculated by means of the exponential growth rate of the components $w_1(x)$, $w_2(x)$ as follows:

$$F = -J - \lim_{x\to\infty} \frac{1}{x} \ln \sqrt{w_1^2 + w_2^2}.$$

Apart from the constant J this is just the Lyapunov exponent connected to Eq. (6), which has been analytically obtained in [6]. Adopting these results we obtain the following result for the free energy:

$$F(\varepsilon) = -J - \frac{\varepsilon}{2}\frac{K_{1-\kappa}(\varepsilon/2D) + K_{1+\kappa}(\varepsilon/2D)}{K_\kappa(\varepsilon/2D)}. \tag{9}$$

Here K_κ denotes the Macdonald (modified Bessel) function (cf. [7]) and

$$\kappa = \frac{h}{2D}.$$

Hence there are two relevant scaling parameters given by ε/D and κ, respectively. In order to compare this result with the relations (4) and (5) we write the expansion of the Macdonald function [7] for the following limiting cases:

$$\varepsilon/D \to 0$$

$$F \approx -J - \frac{2D}{|\ln(\varepsilon/2D)|}\left[1 + \frac{\kappa^2}{3} ln^2(\varepsilon/2D)\right], \quad \text{for } \varepsilon/D \gtrsim e^{-1/\kappa}, \tag{10a}$$

$$F \approx -J - h - 2h\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}\left(\frac{\varepsilon}{4D}\right)^{2\kappa}, \quad \text{for } \varepsilon/D \lesssim e^{-1/\kappa}, \quad \kappa < 1, \tag{10b}$$

$$\varepsilon/D \to \infty$$

$$F \approx -J - \sqrt{h^2+\varepsilon^2} - D. \tag{11}$$

For $\kappa=0$ the relation (10a) corresponds exactly to the result (5) of Mello and Robledo [5], whereas for $\varepsilon/D \lesssim e^{-1/\kappa}$ and $\kappa<1$ the scaling (4) obtained by Derrida and Hilhorst is recovered from (10b) with an explicit expression for the constant C not calculated in [4]. In particular, the relation, which determines the exponent α in Eq. (4), is fulfilled by using the normal distribution of η_x and Eqs. (8):

$$\langle e^{-2\alpha(\eta_x+h)} \rangle = (4\pi D)^{-1/2}\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{4D} - 2\alpha(x+h)\right]dx = 1 \quad \text{for } \alpha = \kappa.$$

The connection with the parameter λ, which appears in Eq. (4.9) of [5], is established by identifying $\lambda = D/\varepsilon$. Finally the weak disorder limit (11) is consistent with the result (4.23a) in [5].

In the strong coupling limit (10) there is an interesting crossover between an inverse logarithmic and a power-law scaling about $\varepsilon/D = e^{-1/\kappa}$. For a clearer representation we write Eqs. (10) in terms of the coupling strength $J = |\ln\sqrt{\varepsilon}| \gg 1$, $\kappa < 1$:

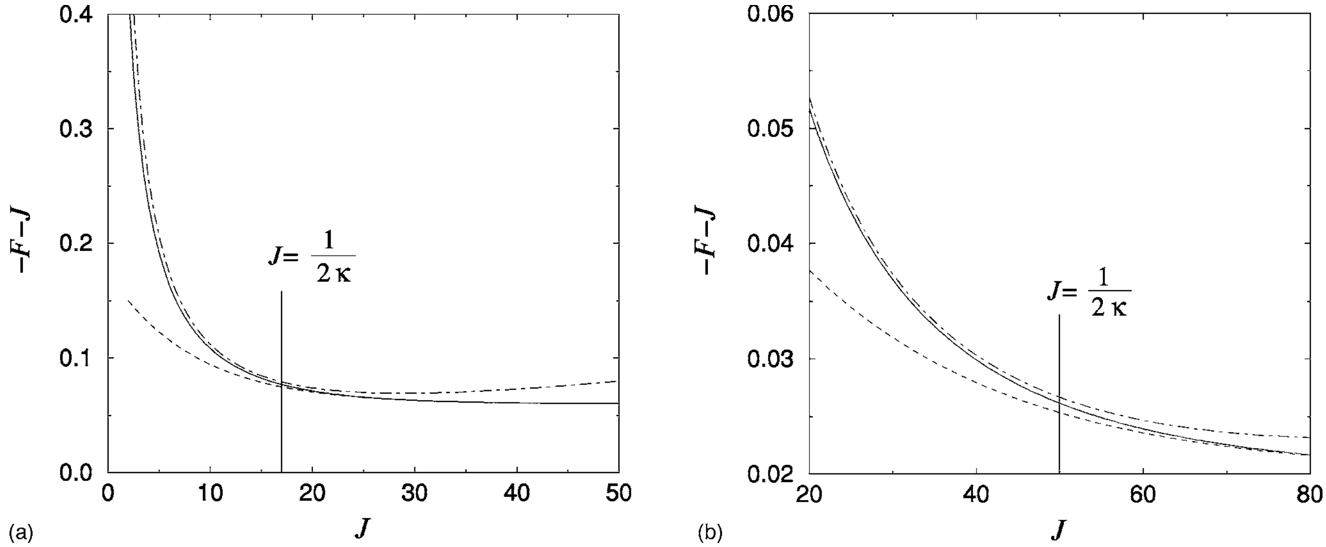

FIG. 2. The free energy as a function of J; $D=1$. The solid line corresponds to formula (9), the dot-dashed line to (12) for $J \lesssim (2\kappa)^{-1}$, and the dashed line to (12) for $J \gtrsim (2\kappa)^{-1}$. The vertical lines indicate the crossover region $J \simeq (2\kappa)^{-1}$. Two values of κ are considered: (a) $\kappa=0.03$, i.e., $(2\kappa)^{-1}=17$, (b) $\kappa=0.01$, i.e., $(2\kappa)^{-1}=50$.

$$F \approx -J + \begin{cases} -\dfrac{D}{J}\left(1+\dfrac{4\kappa^2}{3}J^2\right) & \text{for } J \lesssim (2\kappa)^{-1} \\ -h - \dfrac{2h}{(4D)^{2\kappa}}\dfrac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}e^{-4\kappa J} & \text{for } J \gtrsim (2\kappa)^{-1} \end{cases}. \tag{12}$$

Thus for $\kappa>0$ the power-law decrease of the correction converts to an exponential decrease when $J\to\infty$. In Fig. 2 the exact scaling (9) is compared with the approximations (12). The crossover is clearly visible for the two chosen values of κ. Let us now rewrite the free energy for the case $J \lesssim (2\kappa)^{-1}$ in terms of the physical entities used in the Hamiltonian (1):

$$\widetilde{F} \approx -\widetilde{J} - \frac{\widetilde{D}}{\widetilde{J}} - \frac{\widetilde{J}}{3\widetilde{D}}\widetilde{h}^2. \tag{13}$$

Thus, for $\beta\widetilde{J} \gg 1$ and $\widetilde{J} < \widetilde{D}/\widetilde{h}$ the free energy is independent of the temperature, i.e., its value results from a minimization of the energy. Hence the strong coupling limit corresponds to the zero temperature limit when the applied field $\widetilde{h}$ does not exceed $\widetilde{D}/\widetilde{J}$. This is tantamount to the extension of the Imry-Ma argument for finite temperatures by neglecting entropy changes due to domain flips. The crossover in (12) marks the onset of deviations from that picture, induced by a large enough applied field. The role of the crossover in terms of the magnetization is discussed in the next section.

For $\kappa>1$, the nonanalytic behavior of $F(\varepsilon)$ for $\varepsilon\to 0$ disappears, i.e., contributions $\sim\varepsilon^2$ dominate. This allows us a complete statistical description of the finite-length free energy by means of the replica method, as is shown below in Sec. VII. Another interesting transition occurs at $\kappa=0.5$, i.e., $h=D$, where the scaling of $F(\varepsilon)\sim\varepsilon^{2\kappa}$ becomes singular. If one considers the nearest neighbor correlation $L^{-1}\sum_{m=1}^{L-1}\overline{\sigma_m\sigma_{m+1}}^{\text{th}}$, which is given by $\partial F/\partial J$, the transition to $\kappa<0.5$, i.e., $h<D$, means that the correlations approach to unity with growing interaction J is slower than in the ordered zero-field chain ($h=D=0$) where the corresponding exponent is $\kappa_0=0.5$. In other words, the ordering effect of the average field h and the disordering due to the variance D balance for $h=D$, where the typical interaction strength $J=0.5$ of the zero-field chain is recovered. For a discussion of the role of frustation effects see Sec. VI of Ref. [4].

V. THE MAGNETIZATION

We start with relation (12) for the free energy in the strong coupling limit, where we neglect for simplicity the gamma functions which appear for $J \gtrsim \kappa$, and calculate the magnetization as the derivative $-\partial F/\partial h$:

$$\langle\overline{\sigma_m}^{\text{th}}\rangle \approx \frac{2J}{3D}h \quad \text{for } J \lesssim D/h, \tag{14a}$$

$$\langle\overline{\sigma_m}^{\text{th}}\rangle \approx 1 - 2[(h/D)(\ln 4D + 2J) - 1]e^{-(\ln 4D+2J)h/D} \quad \text{for } J \gtrsim D/h. \tag{14b}$$

Note that on the rhs of Eq. (14b) the expression within the brackets is positive for strong couplings and $J \gtrsim D/h$, so that the magnetization does not exceed unity.

Thus the crossover separates two distinct scaling regions of the magnetization: For $h<D/J$, which comprises the limit $h\to 0$, the magnetization scales linearly with the applied field h, wheras for $h>D/J$ the nonlinear terms dominate. By replacing the quantities in Eq. (14a) by those used in the Hamiltonian (1), we obtain

$$\langle\overline{\sigma_m}^{\text{th}}\rangle \approx \frac{2\widetilde{J}}{3\widetilde{D}}\widetilde{h} \quad \text{for } \widetilde{h} < \widetilde{D}/\widetilde{J}. \tag{15}$$

In accordance with the discussion of the free energy (13) in the previous section there is no temperature dependence in

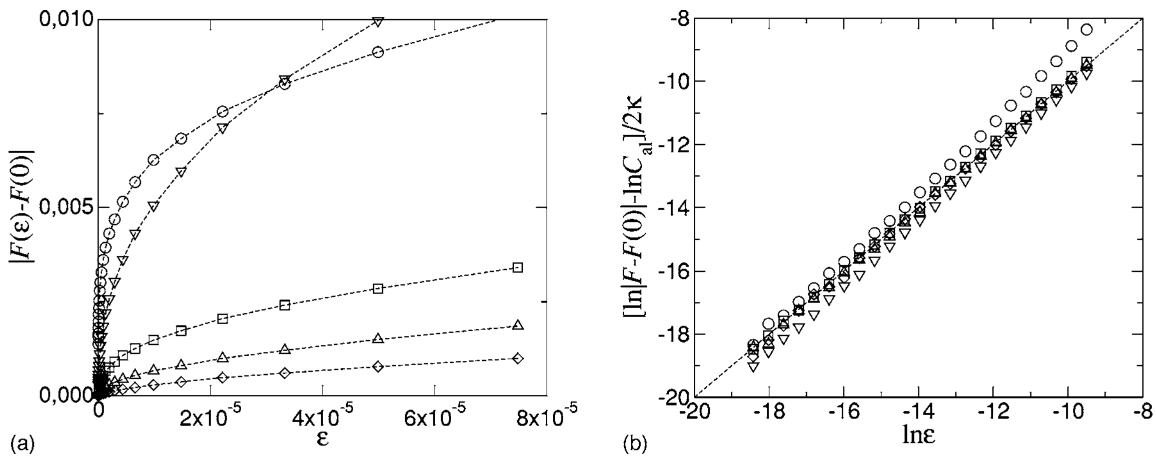

FIG. 3. (a) The free energy as a function of ε for $h=0.02$ (circles), $h=0.04$ (squares), $h=0.06$ (diamonds) for the rectangular distribution, and for $h=0.05$ (triangles up) for the two-point distribution; all for $D=0.1$. Also shown is the result for $h=0.4$ and $D=1$ (triangles down). The dashed lines are to improve readability. (b) The same as (a) but in rescaled coordinates. The dashed line shows the scaling corresponding to the result (10b) of the continuous approach.

relation (15). Thus the spin configuration is dominated by a minimization of the total energy, where the usual temperature dependence of the magnetization is replaced by the dependence on the variance $\tilde{D}$ of the disordered field.

VI. NUMERICAL RESULTS

Here we check numerically how well the constant C in relation (4) can be approximated by the result

$$C_{\rm al} \equiv 2h(4D)^{-2\kappa}\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}$$

of the continuous approach [cf. Eq. (10b)]. We have computed the Lyapunov exponent of the matrix product in (3) numerically, where we chose two distributions of the random field: A rectangular distribution with a probability density $P(\xi) \propto \Theta(\sqrt{6D}-|\xi|)$, and a two-point distribution with $P(\xi) \propto \delta(\xi+\sqrt{2D})+\delta(\xi-\sqrt{2D})$. Two values of the disorder strength were considered: $D=0.1$ and $D=1$. The results for different values of h are shown in Fig. 3. In the rescaled plot the curves for $D=0.1$ collapse onto a straight line which corresponds to our analytical result. Hence in the parameter range considered the constant C is well approximated by $C_{\rm al}$ (the deviations are less than 7%, cf. Table I), even though we used relatively large values of the field: D, $h \lesssim 0.1$. The numerical results show further that though we assumed a normal distribution for the derivation of the free energy (9), the analytical findings are a good approximation for a wider range of distributions of the random field. However, for distributions with diverging cumulants, such as the Cauchy distribution, our approach is not appropriate. For $h=0.02$ the scaling (10b) holds only for very small $\varepsilon \lesssim 10^{-6}$, which is a consequence of the crossover between the inverse logarithmic and a power-law scaling in Eqs. (10). For larger field strengths, here $D=1$ and $h=0.4$, the numerical results deviate significantly from the analytical line. In Table I the numerical estimate $C_{\rm num}$ of the constant C is compared with $C_{\rm al}$ for different field strengths. Accordingly, our result $C_{\rm al}$ overestimates the true value of the constant C for larger field strengths ~ 1.

TABLE I. Comparison of the numerical estimates $C_{\rm num}$ with the analytical findings $C_{\rm al}$ for the constant C in Eq. (4).

D	h	$C_{\rm num}$	$C_{\rm al}$
0.1	0.02	0.057	0.054
0.1	0.04	0.145	0.146
0.1	0.06	0.283	0.301
1	0.4	0.473	0.583

VII. FINITE-LENGTH STATISTICS FOR $\kappa > 1$

As shown in Sec. IV, the nonanalytic behavior of $F(\varepsilon)$ weakens with increasing average of the magnetic field and vanishes at $h=2D$. Beyond this point, i.e., $h \geqslant 2D$, it should be possible to apply the replica method (see e.g., [8]). Hence one can describe analytically the statistics, i.e., sample to sample fluctuations, of the free energy F_L for large yet finite chain length L. We demonstrate this by employing the continuous approach. That is, we consider the quantity $\|w\| \equiv \sqrt{w_1^2+w_2^2}$ whose evolution is determined by Eq. (6). The exponential growth rates of its moments are related to the cumulants $C_n(L) \equiv \langle\langle F_L^n \rangle\rangle$ of the distribution of F_L as follows (see Ch. 4 of [9]):

$$\lim_{L\to\infty}\frac{1}{L}\ln\langle\|w\|^q\rangle = -Jq + \sum_{n=1}^{\infty}\frac{(-q)^n}{n!}\lim_{L\to\infty}L^{n-1}C_n(L). \tag{16}$$

The brackets $\langle * \rangle$ denote the average over realizations of the disorder $\xi(x)$. The lhs of Eq. (16) is also known as the generalized Lyapunov exponent or cumulant generating function for the local Lyapunov exponent, respectively (A more detailed treatise of this procedure can be found in [10]). For even $q=2n$ the generalized Lyapunov exponent is determined by the asymptotic growth of the $q+1$ mixed moments $\langle w_1^k w_2^{q-k}\rangle$ with $k=0,1,\ldots,q$. For odd $q=2n+1$, however, we generally have $\langle w_{1,2}^q\rangle \neq \langle |w_{1,2}^q|\rangle$, so the following method does not work. As shown in Appendix A, the spatial evolution of the mixed moments is given by a closed set of $q+1$ coupled linear differential equations

$$\frac{d}{dx}\langle w_1^k w_2^{q-k}\rangle = [(2k-q)h + (2k-q)^2 D]\langle w_1^k w_2^{q-k}\rangle + \varepsilon[(q-k) \times \langle w_1^{k+1} w_2^{q-k-1}\rangle + k\langle w_1^{k-1} w_2^{q-k+1}\rangle]. \tag{17}$$

The solution of this system leads to an eigenvalue problem, where the eigenvalue with the largest real part determines the asymptotic growth of $\langle \|w\|^q\rangle$. For finite $\varepsilon \ll 1$ it is possible to solve this problem by standard perturbation theory, which leads to

$$\lim_{L\to\infty}\frac{1}{L}\ln\langle \|w\|^q\rangle = hq + Dq^2 + \frac{\varepsilon^2}{4D}\frac{q}{(q-1+\kappa)} + O(\varepsilon^4) \quad \text{for even } q=2,4,\ldots\,. \tag{18}$$

Usually one would assume this equation to be valid for all values of $q\in\mathbf{R}$. However, the small ε expansion breaks down at $q=1-\kappa$. That is, the continuation of Eq. (18) for $q\to 0$ exists only for $\kappa>1$. In this case a comparison of Eqs. (18) and (16) gives the first two limiting cumulants of F_L as follows:

$$C_1 = \langle F_L\rangle \xrightarrow[L\to\infty]{} -J - h - \frac{\varepsilon^2}{4D(\kappa-1)},$$

$$LC_2 \xrightarrow[L\to\infty]{} 2D - \frac{\varepsilon^2}{2D(\kappa-1)^2}. \tag{19}$$

Of course the expression for C_1 can also be obtained from an expansion of result (9) for $\kappa>1$, $\varepsilon\ll 1$. The decrease of $C_2 \sim L^{-1}$ reflects the self-averaging property of the free energy (cf. [15]). Higher cumulants, which determine the validity of a normal distribution of F_L, vanish in the limit $\varepsilon=e^{-2J}\to 0$. For the relation of the cumulants to multifractal properties see [11], and references therein. As discussed in [9], it is possible to relate the result (18) directly to fluctuations of the two-point correlation $\overline{\sigma_m\sigma_{m+L}}^{\text{th}}$.

VIII. CONCLUSION

The focus of this paper was the calculation of the free energy for the random-field Ising chain in the limit of strong coupling J. We have introduced a continuous Langevin-type approach which is exact in the weak-field limit. This allowed us to derive an analytical formula (9) which gives the free energy as a function of two scaling parameters ε/D and h/D, where h is the mean value of the magnetic field, D is the disorder strength, and $\varepsilon=e^{-2J}$ measures the coupling strength. This formula holds for all values of the ratio h/D and reproduces as special cases the known results [4,5]. In particular, we find an explicit expression for the constant C which appears in Eq. (80) of [4]. Thus we are able to describe analytically the effect of the average magnetic field. The interesting behavior appears when the average magnetic field becomes smaller than the disorder strength, i.e., for $h<2D$, where the free energy F scales nonanalytically with the coupling strength J. In this regime we have located the crossover of the correction to the infinite-coupling case between a power-law, $\delta F\sim J^{-1}$, and an exponential decay, $\delta F\sim e^{-4\kappa J}$. By calculating the magnetization, we have shown that this crossover separates the regions of linear and nonlinear scaling in the applied field h. In terms of the Imry-Ma length L_{IM} the linear scaling holds for $h/J<L_{\text{IM}}^{-1}$.

In order to check our analytical formula, we performed numerical computations of the free energy which were involved with the computation of the Lyapunov exponent of a random transfer matrix product. The numerical results are well fitted by our analytical formula also for relatively strong fields h, D and several distributions.

If the average magnetic field is stronger than the disorder, i.e., for $h>2D$, the free energy scales regularly with the coupling strength. In this regime standard perturbation techniques can be applied, which allowed us to describe analytically the statistics of the finite-length free energy.

Finally we hope that our method can also be employed to describe the long-time behavior in the Sinai diffusion problem [12].

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APPENDIX A: EVOLUTION OF MOMENTS

Here we derive Eq. (17) for the evolution of the $q+1$ moments $\langle w_1^k w_2^{q-k}\rangle$ with $k=0,1,\ldots,q$. Differentiating these moments with respect to the spatial variable x and applying (6) gives

$$\left\langle \frac{d}{dx} w_1^k w_2^{q-k}\right\rangle = (2k-q)\langle(\xi+h)w_1^k w_2^q\rangle + \varepsilon[(q-k) \times \langle w_1^{k+1} w_2^{q-k-1}\rangle + k\langle w_1^{k-1} w_2^{q-k+1}\rangle]. \tag{A1}$$

This equation still contains a term $\langle \xi w_1^k w_2^{q-k}\rangle$ which is multiplicative in the random process $\xi(x)$. Such averages can be evaluated with the help of the Furutsu-Novikov theorem [14] which yields

$$\langle \xi w_1^k w_2^{q-k}\rangle = (2k-q)D\langle w_1^k w_2^{q-k}\rangle.$$

After inserting this result into Eq. (A1) and utilizing the linearity of the averaging we finally obtain

$$\frac{d}{dx}\langle w_1^k w_2^{q-k}\rangle = [(2k-q)h + (2k-q)^2 D]\langle w_1^k w_2^{q-k}\rangle + \varepsilon[(q-k)\langle w_1^{k+1} w_2^{q-k-1}\rangle + k\langle w_1^{k-1} w_2^{q-k+1}\rangle]. \tag{A2}$$

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