

Synchronization and desynchronization of self-sustained oscillators by common noise

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We consider the effect of external noise on the dynamics of limit cycle oscillators. The Lyapunov exponent becomes negative under influence of small white noise, what means synchronization of two or more identical systems subject to common noise. We analytically study the effect of small nonidentities in the oscillators and in the noise, and derive statistical characteristics of deviations from the perfect synchrony. Large white noise can lead to desynchronization of oscillators, provided they are nonisochronous. This is demonstrated for the Van der Pol–Duffing system.

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Synchronization of oscillators by periodic signals is a quite understood phenomenon. It can be easily detected by looking on whether the oscillations follow the forcing (e.g., they attain the same frequency) or not. Much less evident is the effect of synchronization by external noise. Here also a nonlinear system can follow or not follow the noisy force. Although this effect can be hardly seen for one system, it can be unveiled by taking two or several identical systems driven by the same noise. In such a setup one easily detects synchrony (asynchrony) through identity (nonidentity) of driven systems [1–6]. There are several fields where this effect has been observed, although under different names. In neurophysiology one describes identical responses of a neuron to a repeated noisy driving as “reliability” [7]. In recent experiments with noise-driven ND:YAG lasers [8] this synchronization was called “consistency.” When the driving is not noisy but chaotic, one speaks on generalized synchronization [9]. The criterion for the synchronization is the negative largest Lyapunov exponent (LE) in the driven system. In many cases one can estimate the sign of LE from the noiseless limit; in particular for experiments described in [7,8] the noiseless system is stable, therefore one expects also negative LE and synchronization for small noise; in the case of generalized synchronization the driven system is often chosen to be chaotic, here for small driving the LE is positive.

A nontrivial dependence on the noise intensity is observed for limit-cycle oscillators that have zero LE for vanishing forcing. Calculations of the LE for different types of noise (see [1–3,10,11]) have demonstrated that small noise plays an ordering role, shifting the LE to negative values and thus synchronizing the oscillators. In the present paper we extend the theory based on the phase approximation of the dynamics, recently discussed in [10,11], to the nonideal situations of two types: slightly nonidentical oscillators and slightly nonidentical noise. Furthermore, we present a numerical study of a realistic Van der Pol–Duffing model and show that the results of the phase approximation are of limited validity. Although in this approximation the LE is negative, in the full system there is a range of noise intensities where the LE is positive. This means the existence of noise-induced desynchronization.

A self-sustained oscillator with a small external force can be adequately described within the phase approximation

[12], where only variations of the phase are considered. With a stochastic force the equation for the phase reads

$$\dot{\varphi} = \omega + \varepsilon f(\varphi) \xi(t), \quad (1)$$

where ε is a small noise amplitude and $f(\varphi)$ is normalized: $\bar{f}^2 = (2\pi)^{-1} \int_0^{2\pi} f^2(\varphi) d\varphi = 1$. Here we assume for simplicity that the noisy force is one component (scalar noise in terms of [11]). The Lyapunov exponent for the noisy dynamics is defined as $\lambda = \langle d\dot{\varphi}/d\varphi \rangle = \langle \varepsilon f'(\varphi) \xi \rangle$. Below we consider the case of white Gaussian noise $\langle \xi(t) \xi(t+t') \rangle = 2\delta(t')$ which allows us to apply the Fokker-Planck theory. Here, as it has been recently discussed in [10,11], the LE is negative $\lambda = -\varepsilon^2 \overline{f'^2} < 0$. Physically, this means that two or many oscillators driven by the same noise will be synchronized and attain the same randomly varying in time phases. The synchronization will be not perfect if the oscillators in the ensemble are slightly different and/or if the noise driving them is not exactly the same. Below we develop a quantitative theory of these effects.

The evolution of N slightly different limit cycle oscillators can be described by the following generalization of (1)

$$\dot{\varphi}_j = \omega + \sigma_j + \varepsilon f(\varphi_j) \xi(t), \quad j = 1, 2, \dots, N, \quad (2)$$

where σ_j are deviations of frequencies from the mean frequency, $\sum_{j=1}^N \sigma_j = 0$. Note that the differences in functions f can be neglected due to smallness of ε . We expect the states of the oscillators to be close if the mismatch is small compared to the LE $|\sigma_j| \ll |\lambda| \ll 1$, then it is appropriate to introduce new variables $\varphi = N^{-1} \sum_{j=1}^N \varphi_j$ and $\theta_j = \varphi_j - \varphi$, $j = 1, 2, \dots, N-1$. Then system (2) for small θ_j can be written as

$$\dot{\varphi} = \omega + \varepsilon f(\varphi) \xi(t), \quad (3)$$

$$\dot{\theta}_j = \sigma_j + \varepsilon f'(\varphi) \theta_j \xi(t). \quad (4)$$

Noting that the deviations θ_j with different j are independent, we can study the evolution of each deviation θ_j separately and drop index j . Thus the evolution of φ and θ is the same as for two slightly different oscillators.

The following from the (3),(4) Fokker-Planck equation for the probability density distribution $W(\varphi, \theta, t)$ reads

$$\frac{\partial W}{\partial t} + \omega \frac{\partial W}{\partial \varphi} + \sigma \frac{\partial W}{\partial \theta} - \varepsilon^2 \hat{Q}^2 W = 0, \quad (5)$$

where $\hat{Q}g \equiv (\partial/\partial\varphi)(f(\varphi)g) + (\partial/\partial\theta)[f'(\varphi)\theta g]$. Performing expansion of the stationary solution in powers of ε^2 [we can here consider σ as of the same order as ε^2 , due to a possibility to renormalize θ in (5)] we obtain in the zeroth order $W_0 = w(\theta)$ and in the first order

$$\omega \frac{\partial W_1}{\partial \varphi} + \sigma \frac{\partial w}{\partial \theta} - \varepsilon^2 \hat{Q}^2 w = 0. \quad (6)$$

Substituting for $\hat{Q}^2 w(\theta)$ and integrating Eq. (6) over $\varphi \in [0, 2\pi)$, we obtain (due to 2π periodicity of W_1 in φ)

$$\sigma \frac{dw}{d\theta} = \varepsilon^2 f'^2 \left(\theta^2 \frac{d^2 w}{d\theta^2} + 4\theta \frac{dw}{d\theta} + 2w \right). \quad (7)$$

For $\sigma=0$ the solution of (7) is a δ function. When $\sigma \neq 0$, this equation can be rewritten as

$$x^2 \frac{d^2 w}{dx^2} + (4x-1) \frac{dw}{dx} + 2w = 0, \quad (8)$$

where $x \equiv \varepsilon^2 f'^2 \sigma^{-1} \theta = |\lambda| \sigma^{-1} \theta$. Solving this differential equation by virtue of the substitution $w(x) = h(x)/x^2$ and accounting for the normalization condition $\int_{-\infty}^{+\infty} w(\theta) d\theta = (2\pi)^{-1}$, we find

$$w(\theta) = \begin{cases} \frac{|\sigma|}{2\pi|\lambda|\theta^2} \exp\left(-\frac{\sigma}{|\lambda|\theta}\right), & \sigma\theta > 0; \\ 0, & \sigma\theta \leq 0. \end{cases} \quad (9)$$

This function is infinitely smooth at $\theta=0$. Noteworthy, for any pair of oscillators driven by the same noise, the phase of the faster oscillator never lags behind that of the slower one.

One can also evaluate the moments

$$\langle |\theta|^k \rangle = \left(\frac{|\sigma|}{|\lambda|} \right)^k \Gamma(1-k), \quad (10)$$

and the most probable value $\theta_{mp} = \sigma/(2|\lambda|)$. From this formula we see again that the phase difference θ is small provided $|\sigma| \ll |\lambda|$. Formula (10) gives finite moments for $k < 1$ only. Higher moments diverge due to the power-law distribution of θ ; to obtain finite moments one has to go beyond the linear in θ approximation even for small mismatches σ .

Quite often N identical systems that are driven by a common external noise $\xi(t)$ experience also influences of different independent (e.g., thermal) noises $\eta_j(t)$. The phase dynamics in this case is given by

$$\dot{\varphi}_j = \omega + \varepsilon f(\varphi_j) \xi(t) + \gamma_j g_j(\varphi_j) \eta_j(t), \quad (11)$$

where $j=1, 2, \dots, N$, the functions f and g_j are normalized $\bar{f}^2 = g_j^2 = 1$, ε and γ_j are the noise amplitudes, and $\langle \xi(t)\xi(t+t') \rangle = 2\delta(t')$, $\langle \eta_j(t)\eta_k(t+t') \rangle = 2\delta_{jk}\delta(t')$, and $\langle \xi(t)\eta_j(t+t') \rangle = 0$. Similar to the case of mismatch, we can introduce a phase φ satisfying (3) and obtain for small deviations θ_j

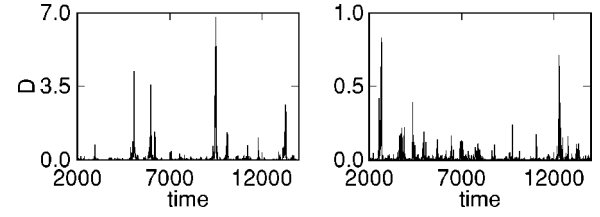


FIG. 1. The time dependencies of the difference $D \equiv \sqrt{(x_1 - x_2)^2 + (\dot{x}_1 - \dot{x}_2)^2}$ between two Van der Pol–Duffing oscillators (17). Left panel: common white noise acts on oscillators with small mismatch [Eq. (18) with $\sigma=10^{-4}$]. Right panel: two identical oscillators are driven by slightly different noises [Eq. (19) with $\gamma/\varepsilon=2 \cdot 10^{-4}$]. Parameters: $\mu=0.2$, $b=1$, $\varepsilon=0.2$.

$$\dot{\theta} = \varepsilon f'(\varphi) \theta \xi(t) + \gamma g(\varphi) \eta(t), \quad (12)$$

where we omitted index j . In this case the relevant Fokker-Plank equation takes the form

$$\frac{\partial W}{\partial t} + \omega \frac{\partial W}{\partial \varphi} - \gamma^2 g^2(\varphi) \frac{\partial^2 W}{\partial \theta^2} - \varepsilon^2 \hat{Q}^2 W = 0. \quad (13)$$

The stationary distribution can be found with the same approximative method as that of Eq. (5). Instead of (7) we now obtain

$$\gamma^2 \frac{d^2 w}{d\theta^2} + \varepsilon^2 f'^2 \left(\theta^2 \frac{d^2 w}{d\theta^2} + 4\theta \frac{dw}{d\theta} + 2w \right) = 0, \quad (14)$$

where due to the condition $\bar{g}^2=1$ the dependence on the function g disappears. With rescaling $x \equiv \varepsilon \gamma^{-1} \sqrt{f'^2} \theta = \sqrt{|\lambda|} \gamma^{-1} \theta$ the last equation can be rewritten as

$$(x^2 + 1) \frac{d^2 w}{dx^2} + 4x \frac{dw}{dx} + 2w = 0, \quad (15)$$

and solved by virtue of the same substitution $w(x) = h(x)/x^2$. Accounting for the normalization condition, we find the solution

$$w(\theta) = \frac{\gamma}{2\pi^2 \sqrt{|\lambda|}} \left[1 + \frac{|\lambda|}{\gamma^2} \theta^2 \right]^{-1} \quad (16)$$

in the form of the Cauchy distribution. Similar to (9) it has a power-law tail that indicates large fluctuations even for small values of γ . In both cases of small mismatch and small non-identity of noise these fluctuations have a form of intermittent bursts (see Fig. 1, cf. [10]), similar to other cases of imperfect synchronization [3].

In the thermodynamical limit $N \rightarrow \infty$ one can also evaluate the ensemble averages for moments of differences for slightly nonidentical oscillators

$$\langle |\theta|^k \rangle_{\text{ens}} = \frac{\Gamma(1-k)}{|\lambda|^k} \int_{-\infty}^{+\infty} |\sigma|^k F(\sigma) d\sigma,$$

and for oscillators driven by different noises

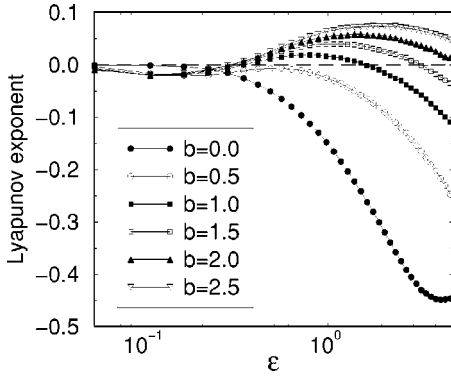


FIG. 2. For the Van der Pol–Duffing oscillator (17) driven by white Gaussian noise, the dependencies of the LE on the noise amplitude ε are plotted for $\mu=0.2$ and different values of b .

$$\langle |\theta|^k \rangle_{\text{ens}} = \frac{1}{|\lambda|^{k/2} \cos(\pi k/2)} \int_0^{+\infty} \gamma^k G(\gamma) d\gamma.$$

Here $F(\sigma)$ and $G(\gamma)$ are the distributions of σ_j and γ_j , correspondingly.

Although a small noise in all considered cases synchronizes the self-sustained oscillators, a desynchronization is possible for large noise intensities. This has been demonstrated in [1,2] for a noise in the form of a sequence of random pulses. In [10] a positive LE has been reported for a discontinuous integrate-and-fire neural model. Here we demonstrate that a desynchronization by noise is possible for white Gaussian noise source and a smooth oscillator, provided the latter has a sufficient degree of nonisochronicity.

As a model we use a standard Van der Pol–Duffing oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x + bx^3 = \varepsilon \xi(t), \quad (17)$$

where $\xi(t)$ is normalized white Gaussian noise. Here μ describes closeness to the Hopf bifurcation point and the “Duffing parameter” b describes nonisochronicity of oscillations. In Fig. 2 we show the dependencies of the LE on the noise amplitude ε for $\mu=0.2$ and different values of b . One can see

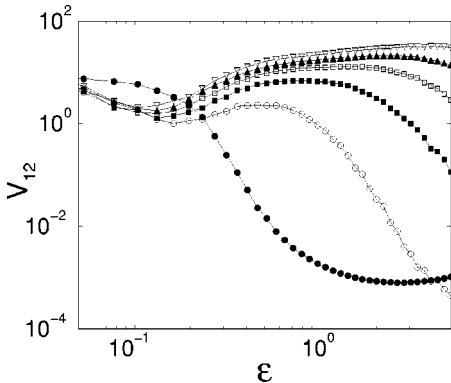


FIG. 3. The dependencies $V_{12}(\varepsilon)$ are plotted for $\mu=0.2$ and $\sigma=0.002$ for the pair of nonidentical Van der Pol–Duffing oscillators under common white Gaussian noise (18). The values of b are marked as in Fig. 2.

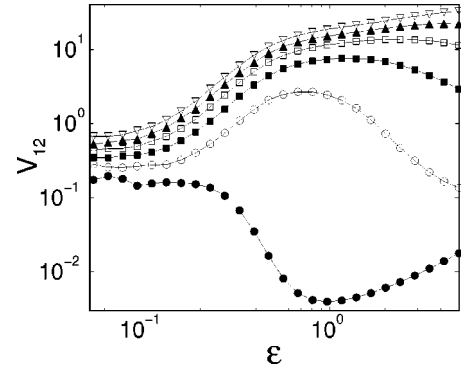


FIG. 4. The dependencies $V_{12}(\varepsilon)$ are plotted for $\mu=0.2$ and $\gamma/\varepsilon=0.01$ for the pair of identical Van der Pol–Duffing oscillators driven by different white Gaussian noises (19). The values of b are marked as in Fig. 2.

that at $b \geq 0.5$ (of course, this critical value depends on μ) positive LEs appear in a certain range of ε , while the asymptotic law $\lim_{\varepsilon \rightarrow 0} \lambda/\varepsilon^2 = \text{const} < 0$ is valid for all b . The region of positive LEs extends for large b .

To characterize the synchronization-desynchronization transition in system (17) quantitatively, we have performed a numerical simulation of two weakly nonidentical Van der Pol–Duffing oscillators under common white Gaussian noise

$$\ddot{x}_{1,2} - \mu(1 - x_{1,2}^2)\dot{x}_{1,2} + (1 \pm \sigma)x_{1,2} + bx_{1,2}^3 = \varepsilon \xi(t), \quad (18)$$

and of two identical Van der Pol–Duffing oscillators driven with slightly different noisy forces

$$\ddot{x}_{1,2} - \mu(1 - x_{1,2}^2)\dot{x}_{1,2} + x_{1,2} + bx_{1,2}^3 = \varepsilon \xi(t) \pm \gamma \eta(t). \quad (19)$$

The quality of synchronization have been measured by the average difference $V_{12} = \langle (x_1 - x_2)^2 + (\dot{x}_1 - \dot{x}_2)^2 \rangle$. In dependence on the noise amplitude ε , this quantity has a maximum in the region of positive values of LE (see Figs. 3 and 4).

We have also performed simulations with a large ensemble of slightly different oscillators driven by the same noise. Here the distribution of the systems states on the plane

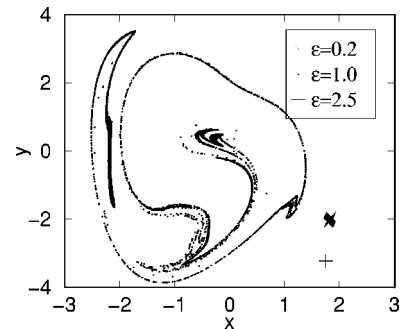


FIG. 5. The snapshots of the ensemble of 10 000 Van der Pol–Duffing oscillators with homogeneous distribution of σ_j within $[-0.01; 0.01]$ under common white Gaussian noise are presented at $\mu=0.2$ and $b=1$. The three chosen values of noise amplitude ε correspond to negative [$\varepsilon=0.2$, the states in the vicinity of the point (1.82; -2.07); and $\varepsilon=2.5$, the states in the vicinity of the point (1.76; -3.29)] and positive ($\varepsilon=1$) LEs.

(x, \dot{x}) at a certain moment of time is concentrated for a negative LE and is extended for a positive LE (see Fig. 5). These distributions correspond to different types of snapshot attractors in system (17) (see [13]).

Summarizing, in this paper we have studied synchrony of populations of limit cycle oscillators subject to common white noise. In the limit of small noise, where the phase approximation is valid, the LE is negative, which means synchronization of oscillators by noise. In this case we have performed an analytical treatment of nonideal situations of two types: nonidentical oscillators and nonidentical noise. In both cases the distribution of the phase difference between the systems has a power-law tail, which indicates on a strongly intermittent character of the synchronous state, where synchronous epochs and asynchronous bursts intermingle

For a large noise one has to go beyond the phase approximation; here analytical treatment of the LE is not possible.

Numerical analysis of the basic Van der Pol–Duffing model driven by Gaussian white noise has shown that for large nonisochronicity of oscillators for moderate noise levels the LE is positive. This means desynchronization of oscillations by a moderate noise.

One of possible applications of the theory above is to the reliability of neural oscillators. Neurons in a regime of periodic spiking can be considered as limit-cycle oscillators; here a small noise will synchronize them according to the general theory valid in the phase approximation. However, a large noise may lead to nonreliability, similar to what has been shown above for the Van der Pol–Duffing system. This problem is currently under investigation.

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