

Controlling Synchronization in an Ensemble of Globally Coupled Oscillators

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We propose a technique to control coherent collective oscillations in ensembles of globally coupled units (self-sustained oscillators or maps). We demonstrate numerically and theoretically that a time delayed feedback in the mean field can, depending on the parameters, enhance or suppress the self-synchronization in the population. We discuss possible applications of the technique.

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Populations of a large number of coupled oscillators are abundant in physics, chemistry, and biology. The examples of nontrivial collective dynamics in such ensembles include synchronous regimes in arrays of Josephson junctions [1] and lasers [2], coordinated firing of cardiac pacemaker cells [3], synchronous emission of light pulses by a population of fireflies [4] and emission of chirps by a population of crickets [5], synchronization in ensembles of electrochemical oscillators [6] and in neuronal populations [7,8]. A general phenomenon in such populations of periodic, noisy, and chaotic oscillators and maps is the appearance of the collective synchrony, studied theoretically [9–11], as well as experimentally [6,12]. A proper model for many of these systems is that of global (each-to-each) coupling between the elements. Consider an ensemble of N units (with $N \rightarrow \infty$ in the thermodynamic limit); coupling within each pair of units is quantified by the parameter ε . Each unit can be regarded as driven by the force εX , where $X = N^{-1} \sum_{i=1}^N x_i$ is the *mean field* and x_i is an observable of the i th unit. Onset of synchronization in the population with the increase of the coupling parameter ε beyond a critical value ε_{cr} manifests itself via the appearance of nonzero (macroscopic) oscillations of the mean field; on the contrary, the variance of X is small (it vanishes in the thermodynamic limit) if the coupling strength is below the critical value, $\varepsilon < \varepsilon_{cr}$. The synchronization transition is often considered in analogy to phase transitions, with the variance of X playing the role of the order parameter [13].

In this Letter we suggest a technique for controlling the synchronous collective dynamics in an ensemble of globally coupled systems. For this purpose the *time delayed* mean field is fed back into the ensemble. By choosing the amplification ε_f and the delay τ in the feedback loop we can both enhance and suppress collective oscillations. For accomplishing this neither information on the details of the individual oscillators and their interactions nor the access to their parameters is needed. Only the macroscopic properties of the collective dynamics determine the feasibility of the control. Below we demonstrate the technique on several representative examples and develop a theory. For other applications of a time-delay control see [14].

To introduce the method and to illustrate its possible application in neuroscience, we first consider a realistic model of collective rhythmical activity in a population of neurons, where individual neurons are described by the Hindmarsh-Rose equations [15]:

$$\begin{aligned} \dot{x}_i &= y_i - x_i^3 + 3x_i^2 - z_i + 3 + \varepsilon X + \varepsilon_f X(t - \tau), \\ \dot{y}_i &= 1 - 5x_i^2 - y_i, \quad \dot{z}_i = 0.006 \cdot [4(x_i + 1.56) - z_i]. \end{aligned} \quad (1)$$

We have simulated the dynamics of the ensemble of 2000 identical neurons in the regime of chaotic bursting (Fig. 1). The global coupling ($\sim \varepsilon$) leads to a rather irregular mean field, though it has a strong T -periodic component. In spite of the irregularity, the mean field is suppressed when the control ($\sim \varepsilon_f$) is imposed. We emphasize that the feedback control does not affect oscillations in individual neurons [compare the dotted lines in panels (b) and (c) of Fig. 1], but destroys the synchrony between the

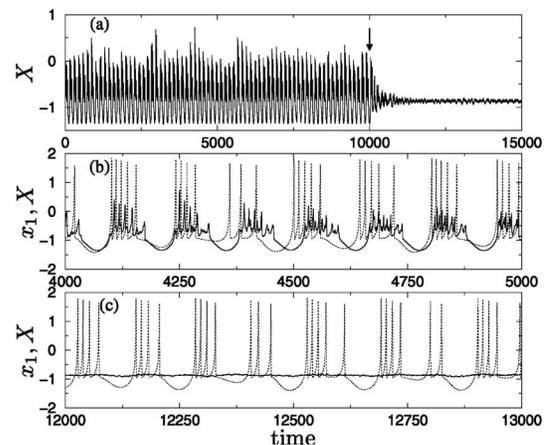


FIG. 1. (a) Suppression of synchrony in the population of Hindmarsh-Rose neurons (1) for $\varepsilon = \varepsilon_f = 0.08$, $\tau = 147 \approx T/2$. The arrow indicates when the control is switched on. (b),(c) Bursting of an individual neuron (dotted lines) and the mean field (solid lines) without and with delayed feedback, respectively. Note that only the collective dynamics is influenced by the feedback; dynamics of individual units is practically unchanged.

neurons, so that their spikes do not sum up coherently to a macroscopic mean field.

Next, we consider an ensemble of $N = 2000$ nonidentical chaotic Rössler oscillators coupled via the mean field:

$$\begin{aligned}\dot{x}_i &= -\omega_i y_i - z_i + \varepsilon X, \\ \dot{y}_i &= \omega_i x_i + a y_i + \varepsilon_f X(t - \tau), \\ \dot{z}_i &= 0.4 + z_i(x_i - 8.5),\end{aligned}\quad (2)$$

where $a = 0.15$ and the distribution of ω_i is Gaussian with the mean $\omega_0 = 1$ and the standard deviation $\Delta\omega = 0.02$. Without feedback ($\varepsilon_f = 0$) the system undergoes the Kuramoto synchronization transition at $\varepsilon_{cr} \approx 0.05$ [10]. Above the transition threshold X is approximately periodic with the period $T \approx 6$; below the threshold one observes small irregular fluctuations of X that are due to the finite size of the ensemble. To model a realistic situation, where the additional feedback field may affect the oscillators in another way than the internal coupling, we include the control term in the second equation of (2). To characterize the efficiency of the control, we present in Fig. 2 the dependencies of the enhancement factor $E = (\text{Var}(X_f)/\text{Var}(X))^{1/2}$ and the suppression factor $S = (\text{Var}(X)/\text{Var}(X_f))^{1/2}$ on τ , ε_f ; X and X_f are the mean fields in the absence and presence of the feedback.

The efficiency of control depends on the regularity of the mean field determined by the regularity of individual elements and by the population size N . So, the suppression in a population of coherent Rössler systems (2) is less effective than in a population of limit cycle oscillators (in our simulation we used the FitzHugh–Nagumo systems), but more effective than in a population of funnel Rössler systems [parameters $a = 0.25$, $\omega_0 = 0.97$ in (2)], which can be considered as “very noisy” oscillators.

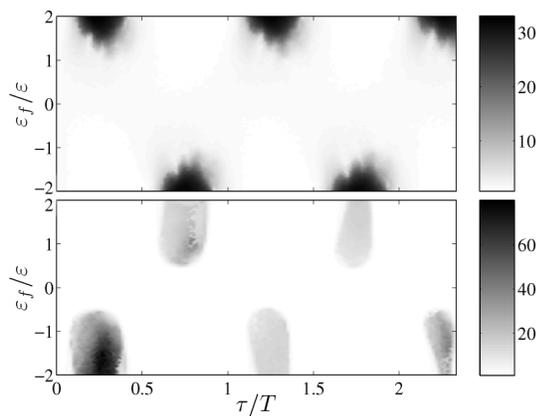


FIG. 2. Enhancement factor E (top panel, $\varepsilon = 0.03$) and suppression factor S (bottom panel, $\varepsilon = 0.1$) of the mean field oscillations in system (2) in a gray scale coding. Control is achieved within the “tongues” located around $\tau \approx \text{const} + nT/2$. For the given control scheme, no tongue exists at $\tau \approx 0$, so that the “trivial” feedback without delay does not suffice.

Considering the effect of the population size N , we have observed that in system (2) with $N = 2000$ the suppression factor S is ≈ 2 times higher than for the case $N = 500$. Indeed, in the finite-size population the mean field below the synchronization threshold can be treated as a noise with the variance $\sim N^{-1}$ [16]. On the other hand, for $\varepsilon > \varepsilon_{cr}$, $\text{Var}(X)$ does not practically depend on N , so the maximal possible suppression $S \sim \sqrt{N}$. Discussing the amount of intervention into the system, we note that although $\text{Var}(X)$ is small when the synchrony is suppressed, $\langle X \rangle$ does not necessarily vanish. In order to have a noninvasive control, i.e., a completely vanishing control term in the asynchronous state, one can use a differential feedback in the spirit of [14]. We demonstrate this by considering the feedback control of an ensemble of globally coupled maps modeling the chaotic bursting of neurons [11]:

$$\begin{aligned}x_i(t+1) &= 4.3[1 + x_i^2(t)]^{-1} + y_i(t) + \varepsilon X \\ &\quad + \varepsilon_f X(t - \tau) - \varepsilon'_f X(t), \\ y_i(t+1) &= y_i(t) - 0.01[x_i(t) + 1].\end{aligned}\quad (3)$$

Here t is the discrete time; the mean field X is defined as above. Each neuron exhibits irregular bursts, whereas the collective oscillation is highly regular. The delayed feedback with $\varepsilon'_f = 0$, as in (1) and (2), we call direct, whereas the feedback with $\varepsilon'_f = \varepsilon_f$ we call differential. Both control schemes efficiently control the synchronization transition in this system. In Fig. 3(a) we show the results for the differential feedback: by an appropriate choice of τ and ε_f one shifts the curves, altering the synchronization threshold ε_{cr} . Suppression can be achieved for $\tau \approx T/2$ and $\tau \approx T$; the suppression regions have the form of closed islands (cf. Fig. 2 and the theoretical consideration below). With the discrete-time model (3) we can study very large ensembles; the obtained dependence of the suppression factor on the population size [Fig. 3(b)] fits well the expected law $S \sim \sqrt{N}$, both for the direct feedback scheme and for the differential one. Hence, as the variance of the control signal scales as

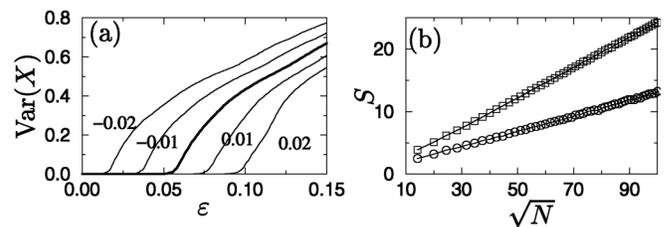


FIG. 3. (a) Variance of the mean field for system (3) in dependence on the coupling ε for different values of the feedback parameter $\varepsilon_f = \varepsilon'_f$ (shown at the correspondent curves) and constant delay $\tau = 30 \approx T/2$. Bold line shows the synchronization transition without feedback. (b) Efficiency of the control in dependence on the population size N for $\varepsilon'_f = 0$ (circles) and $\varepsilon'_f = \varepsilon_f$ (boxes).

N^{-1} , the differential control scheme is noninvasive in the thermodynamic limit (the direct control is generally invasive as the feedback signal tends to a constant).

Next, we discuss two important points for the practical implementation of the method. (i) The delay τ in the introduced global feedback is not equivalent to the internal delays τ_{int} , playing an important role in the analysis of neural interactions (see, e.g., [8,17,18,]). Usually $\tau_{\text{int}} \ll T$, where T is the period of collective oscillations, while τ must be of the order of T . Our technique works in the case of delayed internal interactions as well; this was checked in experiments with model (3) with a delay in the internal coupling between the neurons. The results are very similar to those presented in Fig. 3. (ii) The method works if there is a latency in the observation of the mean field. In this case the delay time τ in the direct feedback is simply a sum of the observational and additional delays.

For a theoretical description of the time delayed feedback control we exploit the noisy phase dynamics model (cf. [8,9,17])

$$\begin{aligned} \dot{\phi}_i &= \omega_i + \frac{\varepsilon}{N} \sum_1^N h[\phi_j(t) - \phi_i(t)] \\ &+ \frac{\varepsilon_f}{N} \sum_1^N f[\phi_j(t - \tau) - \phi_i(t)] + \xi_i(t), \end{aligned} \quad (4)$$

where ϕ_i are the phases of units, $\xi_i(t)$ is Gaussian noise with $\langle \xi_i(t) \xi_j(t') \rangle = 2D \delta(t - t') \delta_{ij}$; h and f are 2π -periodic functions describing coupling in the ensemble and the feedback. Such a model describes weakly interacting noisy limit cycle oscillators with different natural frequencies ω_i , as well as nonidentical chaotic oscillators of the Rössler type [10]. Considering the limit $N \rightarrow \infty$, we drop the indices and introduce the probability distributions of the phases $\rho(\phi, \omega, t)$ and of the natural frequencies $g(\omega)$, then ρ obeys the Fokker-Planck equation $(\partial \rho / \partial t) = -(\partial / \partial \phi)[\rho v] + D(\partial^2 \rho / \partial \phi^2)$, where v is given by

$$\begin{aligned} v &= \omega + \varepsilon \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} g(\omega) h(\theta - \phi) \rho(\theta, \omega, t) d\omega \\ &+ \varepsilon_f \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} g(\omega) f(\theta - \phi) \rho(\theta, \omega, t - \tau) d\omega; \end{aligned}$$

its stationary solution $\rho_0 = 1/2\pi$ describes the asynchronous state with vanishing mean field. To analyze the stability of this solution we substitute $\rho = 1/2\pi + \sum_k [C_k(\omega) e^{\lambda_k t + ik\phi} + C_k^*(\omega) e^{\lambda_k^* t - ik\phi}]$. Preserving only linear in C_k terms we get

$$C_k(\omega) = -ik \frac{\varepsilon H_k^* + \varepsilon_f F_k^* e^{-\lambda_k \tau}}{\lambda_k + k^2 D + ik(\omega + \varepsilon H_0 + \varepsilon_f F_0)} B_k,$$

where H_k , F_k are coefficients of the Fourier series of functions h , f , and $B_k = \int_{-\infty}^{\infty} g(\omega) C_k(\omega) d\omega$. Substituting C_k in this expression, we obtain

$$1 = - \int_{-\infty}^{\infty} \frac{ik(\varepsilon H_k^* + \varepsilon_f F_k^* e^{-\lambda_k \tau}) g(\omega) d\omega}{\lambda_k + k^2 D + ik(\omega + \varepsilon H_0 + \varepsilon_f F_0)}.$$

The problem can be completely solved for the Lorentzian distribution of the frequencies $g(\omega) = \gamma / (\pi[\gamma^2 + (\omega - \omega_0)^2])$. In this case the integral can be calculated explicitly and we obtain transcendental equations $\lambda_k + k^2 D + \gamma + ik\tilde{\omega} + ik(\varepsilon H_k^* + \varepsilon_f F_k^* e^{\lambda_k \tau}) = 0$, where $\tilde{\omega} = \omega_0 + \varepsilon H_0 + \varepsilon_f F_0$. Remarkably, the equations for different Fourier modes of the distribution density are independent, so that the stability analysis can be performed separately for each of them (cf. [8]). Although the eigenvalues λ_k cannot be found explicitly, assuming $\lambda_k = -i\varphi/\tau$ and writing $F_k^* = |F_k| e^{i\alpha_k}$, one easily finds the k th stability region on the plane of the feedback parameters (τ, ε_f) in a parametric form:

$$\begin{aligned} \varepsilon_f &= [\gamma/k + kD + \varepsilon \text{Im}(H_k)] [|F_k| \cos(\varphi - \alpha_k)]^{-1}, \\ \tau &= \varphi k^{-1} [\varepsilon_f |F_k| \sin(\varphi - \alpha_k) + \tilde{\omega} + \varepsilon \text{Re}(H_k)]^{-1}. \end{aligned} \quad (5)$$

The overall stability of the asynchronous state is determined by the overlap of these domains for all modes.

Stability analysis of the asynchronous state should be complemented by the analysis of stability of the synchronous regime. The latter can be accomplished in an instructive case of identical oscillators without noise, i.e., for $g(\omega) = \delta(\omega - \omega_0)$ and $D = 0$. The frequency of the completely synchronous regime with the uniformly growing phase $\phi_i = \Omega t$ is determined from (4) by

$$\Omega = \omega_0 + \varepsilon h(0) + \varepsilon_f f(-\Omega\tau). \quad (6)$$

This regime can lose stability in two ways. (i) All phases remain equal but the rotation becomes nonuniform in time; this corresponds to instability to a homogeneous perturbation $\phi_i = \Omega t + \Delta\phi(t)$. This perturbation satisfies $d\Delta\phi/dt + \varepsilon_f f'(-\Omega\tau)[\Delta\phi(t) - \Delta\phi(t - \tau)] = 0$, leading to the stability criterion $\tau \varepsilon_f f'(-\Omega\tau) + 1 > 0$. Together with (6) this allows one to represent the border of the stability domain in a parametric form:

$$\varepsilon_f = \tilde{\omega}_0 [\varphi f'(\varphi) - f(\varphi)]^{-1}, \quad \tau = -[\varepsilon_f f'(\varphi)]^{-1}, \quad (7)$$

where $\tilde{\omega}_0 = \omega_0 + \varepsilon h(0)$. (ii) Another instability mode corresponds to destruction of synchrony via ‘‘evaporation’’ of individual oscillators from the synchronous cluster. Here we have to consider a perturbation of a single oscillator, $\phi_i = \Omega t + \Delta\phi_i(t)$, which satisfies $d\Delta\phi_i/dt = -\Delta\phi_i[\varepsilon h'(0) + \varepsilon_f f'(-\Omega\tau)]$. Together with (6), the condition $\varepsilon h'(0) + \varepsilon_f f'(-\Omega\tau) > 0$ gives the stability domain

$$\begin{aligned} \tau &= [\varphi f'(\varphi)] [\varepsilon h'(0) f(\varphi) - \tilde{\omega}_0 f'(\varphi)]^{-1}, \\ \varepsilon_f &= -\varepsilon h'(0) [f'(\varphi)]^{-1}. \end{aligned} \quad (8)$$

Regions of stability of synchronous and asynchronous regimes are shown in Fig. 4. Let us for definiteness

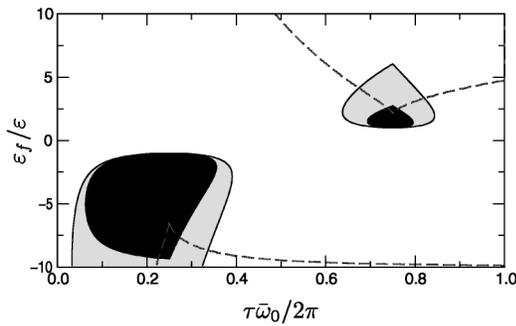


FIG. 4. Results of the theoretical analysis of the delayed feedback control in model (4) with $h(\phi) = \sin\phi$, $f(\phi) = \cos\phi$, $\omega_i = \omega_0$, and $D = 0$ for $\varepsilon/\omega_0 = 0.1$, cf. Fig. 2. Grey: regions of stability of the asynchronous state according to (5); black: regions of evaporation instability of the uniformly rotating synchronous state according to (8); these are regions of strong control. The uniformly rotating state is homogeneously unstable in regions marked by dashed lines (7).

discuss the suppression of synchrony. There exists a domain of parameters where the synchronous state is stable towards homogeneous perturbations and is unstable towards evaporation, and the asynchronous state is stable. This can be called a region of *strong control*, as here even initially complete synchrony can be suppressed. In another domain of *weak control* both the synchronous and asynchronous states are stable; here the asynchronous state can be maintained by the feedback control, but the synchronous state is not destroyed. Finally, there is a region where the uniformly rotating synchronous state is unstable towards homogeneous perturbations, and only complex synchronous states are possible. Stability of these regimes towards evaporation cannot be analyzed analytically but requires numerical evaluation of the corresponding Lyapunov exponent.

In summary, we have proposed a technique for controlling the synchrony in populations of globally coupled elements, which enables us to suppress or to enhance it. Tuning the amplification and the delay of the feedback loop we can effectively change the critical point ε_{cr} of the synchronization transition in such systems. We expect possible applications of our technique, e.g., in neuroscience, where it is often required to suppress a pathological activity of ensembles of neurons [19,20]. For example, it is widely believed that the onset of a rhythmic brain activity in the cases of the Parkinson's disease can be regarded as a synchronization transition in a large neuronal population; this viewpoint is used in developing the techniques for suppression of these rhythms by means of the deep brain stimulation [20]. The feedback control described in this paper might be more appropriate for this goal because it reduces the intervention into the system: the control signal is large only for the rather short transient time before the mean field oscillations are sup-

pressed; afterwards the control signal is of the order of the background noise. On the other hand, feedback enhancement of collective oscillations might be useful in the cases of failure of cardiac or neural pacemakers.

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