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# PHASE SYNCHRONIZATION OF REGULAR AND CHAOTIC SELF-SUSTAINED OSCILLATORS

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## Abstract

In this review article we discuss effects of phase synchronization of nonlinear self-sustained oscillators. Starting with a classical theory of phase locking, we extend the notion of phase to autonoumous continuous-time *chaotic* systems. Using as examples the well-known Lorenz and Rössler oscillators, we describe the phase synchronization of chaotic oscillators by periodic external force. Both statistical and topological aspects of this phenomenon are discussed. Then we proceed to more complex cases and discuss phase synchronization in coupled systems, lattices, large globally coupled ensembles, and of space-time chaos. Finally, we demonstrate how the synchronization effects can be detected from observations of real data.

## 1. Introduction

Synchronization, a basic nonlinear phenomenon, discovered at the beginning of the modern age of science by Huygens [27], is widely encountered in various fields of science, often observed in living nature [25, 24] and finds a lot of engineering applications [7, 8]. In the classical sense, synchronization means adjustment of frequencies of self-sustained oscillators due to a weak interaction [49].

The history of synchronization goes back to the 17th century when the famous Dutch scientist Christiaan Huygens [27] reported on his observation of synchronization of two pendulum clocks. Systematic study of this phenomenon, experimental as well as theoretical, was started by Edward Appleton [3] and Balthasar van der Pol [70]. They showed that the frequency of a triode generator can be entrained, or synchronized, by a weak external signal with slightly different frequency. These studies were of high practical importance because such generators became basic elements of radio communication systems.

Next impact to the development of the theory of synchronization was given by the representatives of the Russian school. Andronov and Vitt [2, 1] further developed methods of van der Pol and generalized his results. The case of $n : m$ external synchronization was studied by Mandelshtam and Papaleksi [40]. Mutual synchronization of two weakly nonlinear oscillators was analytically treated by Mayer [41] and Gaponov [23]; relaxation oscillators were studied by Bremsen and Feinberg [9] and Teodorchik [68]. An important step was done by Stratonovich [64, 65] who developed a theory of external synchronization of a weakly nonlinear oscillator in the presence of random noise.

Development of rigorous mathematical tools of the synchronization theory started with Denjoy works on circle map [19] and with treatment of forced relaxation oscillators by Cartwright and Littlewood [11, 12]. Recent development has been highly influenced by Arnold [4] and by Kuramoto [34].

In the context of interacting *chaotic* oscillators, several effects are usually referred to as"synchronization" [49, 47, 35]. Due to a strong interaction of two (or a large number) of identical chaotic systems, their states can coincide, while the dynamics in time remains chaotic [22, 51]. This effect is called "complete synchronization" of chaotic oscillators. It can be generalized to the case of non-identical systems [51, 38, 39], or that of the interacting subsystems [46, 31]. Another well-studied effect is the "chaos–destroying" synchronization, when a periodic external force acting on a chaotic system destroys chaos and a periodic regime appears [36], or, in the case of an irregular forcing, the driven system follows the behavior of the force [32]. This effect occurs for a relatively strong forcing as well. A characteristic feature of these phenomena is the existence of a threshold coupling value depending on the Lyapunov exponents of individual systems [22, 51, 6, 20].

In this article we concentrate on the recently described effect of *phase synchronization* of chaotic systems, which generalizes the classical notion of phase locking. Indeed, for periodic oscillators only the relation between phases is important, while no restriction on the amplitudes is imposed. Thus, we define phase synchronization of chaotic system as an appearance of a certain relation between the phases of interacting systems or between the phase of a system and that of an external force, while the amplitudes can remain chaotic and are, in general, non-correlated. This type of synchronization has been observed in experiments with electronic chaotic oscillators [52, 45], plasma discharge [69], and electrochemical oscillators [28].

## 2. Synchronization of periodic oscillators

### 2.1. PHASE LOCKING

In this section we remind basic facts on the synchronization of periodic oscillations (see, e.g., [43]). Stable periodic oscillations are represented by a stable limit cycle in the phase space. The motion of the phase point along the cycle can be parametrized by the phase $\phi(t)$, it's dynamics obeys

$$\frac{d\phi}{dt} = \omega_0, \tag{1}$$

where $\omega_0 = 2\pi/T_0$, and $T_0$ is the period of the oscillation. It is important that starting from any monotonically growing variable $\theta$ on the limit cycle (so that at one rotation $\theta$ increases by $\Theta$), one can introduce the phase satisfying Eq. (1). Indeed, an arbitrary $\theta$ obeys $\dot{\theta} = \gamma(\theta)$ with a periodic "instantaneous frequency" $\gamma(\theta + \Theta) = \gamma(\theta)$. The change of variables $\phi = \omega_0 \int_0^\theta [\gamma(\theta)]^{-1} d\theta$ gives the correct phase, with the frequency $\omega_0$ being defined from the condition $2\pi = \omega_0 \int_0^\Theta [\nu(\theta)]^{-1} d\theta$. A similar approach leads to correct angle-action variables in Hamiltonian mechanics. We have performed this simple consideration to underline the fact that the notions of the phase and of the phase synchronization are universally applicable to any self-sustained periodic behavior independently on the form of the limit cycle.

From (1) it is evident that the phase corresponds to the zero Lyapunov exponent, while negative exponents correspond to the amplitude variables. Note that we do not consider the equations for the amplitudes, as they are not universal.

When a small external periodic force with frequency $\nu$ is acting on this periodic oscillator, the amplitude is relatively robust, so that in the first approximation one can neglect variations of the amplitude to obtain for the phase of the oscillator $\phi$ and the phase of the external force $\psi$ the equations

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(\phi, \psi) \ , \qquad \frac{d\psi}{dt} = \nu \ , \tag{2}$$

where $G(\cdot, \cdot)$ is $2\pi$-periodic in both arguments and $\varepsilon$ measures the strength of the forcing. For a general method of derivation of Eq. (2) see [34]. The system (2) describes a motion on a 2-dimensional torus that appears from the limit cycle under periodic perturbation (see Fig. 1a,b). If we pick up the phase of oscillations $\phi$ stroboscopically at times $t_n = n\frac{2\pi}{\nu}$, we get a circle map

$$\phi_{n+1} = \phi_n + \varepsilon g(\phi_n) \ , \tag{3}$$

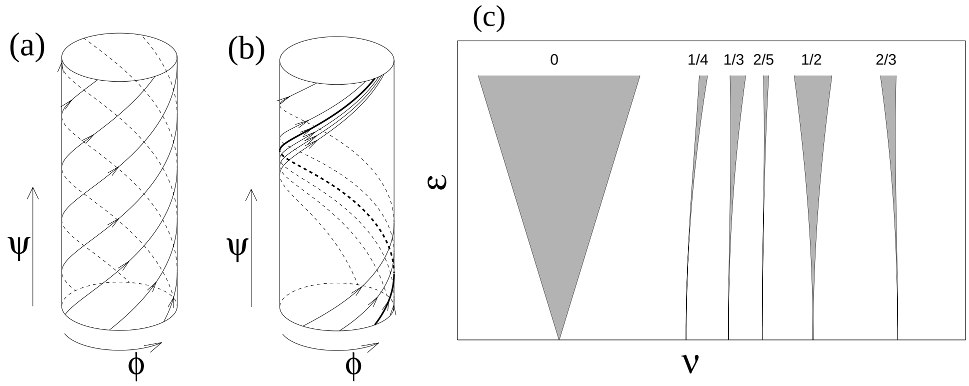


*Figure 1.* Quasiperiodic (a) and periodic flow (b) on the torus; a stable periodic orbit is shown by the bold line. (c): The typical picture of Arnold tongues (with winding numbers atop) for the circle map.

where the $2\pi$-periodic function $g(\phi)$ is defined via the solutions of the system (2). According to the theory of circle maps (cf. [43]), the dynamics can be characterized by the winding (rotation) number

$$\rho = \lim_{n\to\infty} \frac{\phi_n - \phi_0}{2\pi n} ,$$

which is independent on the initial point $\phi_0$ and can take rational and irrational values. If it is irrational, then the motion is quasiperiodic and the trajectories are dense on the circle. Otherwise, if $\rho = p/q$, there exists a stable orbit with period $q$ such that $\phi_q = \phi_0 + 2\pi p$. The latter regime is called *phase locking* or *synchronization*. In terms of the continuous-time system (2), the winding number is the ratio between the mean derivative of the phase $\phi$ and the forcing frequency $\nu$

$$\rho = \frac{\left\langle \frac{d\phi}{dt} \right\rangle}{\nu} = \frac{\omega}{\nu} . \qquad (4)$$

For $\rho$ irrational and rational one has, respectively, a quasiperiodic dense orbit and a resonant stable periodic orbit on the torus (Fig. 1a,b).

The main synchronization region where $\omega = \nu$ corresponds to the winding number 1 (or, equivalently, 0 if we apply (mod $2\pi$) operation to the phase; for frequencies this means that we consider the difference $\omega - \nu$), other synchronization regions are usually much more narrow. A typical picture of synchronization regions, called also “Arnold tongues”, for the circle map (3) is shown in Fig. 1c.

The concept of phase synchronization can be applied only to *autonomous continuous-time* systems. Indeed, if the system is discrete (i.e. a mapping), its period is an integer, and this integer cannot be adjusted to some other integer in a continuous way. The same is true for forced continuous-time oscillations (e.g., for the forced Duffing oscillator): here the frequency of oscillations is completely determined by that of the forcing and cannot be adjusted to some other value. We can formulate this also as follows: in discrete or forced systems there is no zero Lyapunov exponent, so there is no corresponding marginally stable variable (the phase) that can be affected by small external perturbations.

The synchronization condition (4) does not mean that the difference between the phase $\phi$ of an oscillator and that of the external force $\psi$ (or between phases of two oscillators) must be a constant, as is sometimes assumed (see, e.g. [66]). Indeed, (2) implies, that to enable $\phi - \psi = const$, the function $G$ should depend not on separate phases but only on their difference: $G(\phi, \psi) = G(\phi - \psi)$. Denoting this phase difference as $\varphi = \phi - \psi$ we can rewrite Eq. (2) as

$$\frac{d\varphi}{dt} = \omega_0 - \nu + \varepsilon q(\varphi) \ . \tag{5}$$

In the synchronous state this equation should have (at least one) stable point. This happens if the frequency mismatch (detuning) is small enough, $\varepsilon q_{min} < \nu - \omega_0 < \varepsilon q_{max}$, and this condition determines the synchronization (phase-locking, mode-locking) region on the $(\omega, \varepsilon)$ plane. Within this region, the phase difference remains constant, $\psi = \delta$, and the value of this constant depends on the detuning, $\delta = q^{-1}[(\nu - \omega_0)/\varepsilon]$ (here the stable branch of the inverse function should be chosen). Generally, the coupling function $G(\phi, \psi)$ cannot be reduced to a function of the phase difference $\varphi$. Then, even in a synchronous regime $\varphi$ is not constant but fluctuates, although these fluctuations are bounded. Thus, we can define phase locking according to relation

$$|\phi(t) - \psi(t) - \delta| < \text{const} \ , \tag{6}$$

from which the condition of frequency locking $\langle \dot{\phi} \rangle = \nu$ naturally follows. The latter definition of phase locking will be used in the treatment of chaotic oscillations below, but even for periodic regimes it has an advantage when the forced oscillations are not close to the original limit cycle.

The winding number is a continuous function of system parameters; typically it looks like a devil's staircase. Take the main phase-locking region. Continuity means that near the de-synchronization transition the mean oscillation frequency is close to the external one. As the external frequency $\nu$ is varied, the de-synchronization transition appears as saddle-node bifurcation, where a stable $p/q$ - periodic orbit collides with the corresponding unstable one, and both disappear. Near this bifurcation point, similarly to the type-I intermittency [5], a trajectory of the system spends a large time

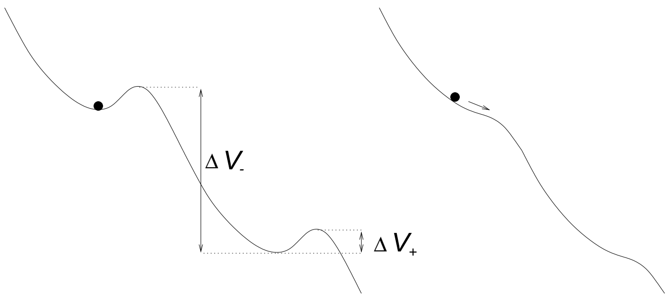


*Figure 2.* Phase as a particle in an inclined potential, inside and outside of the synchronization region.

in the vicinity of the just disappeared periodic orbits; in the course of time evolution the long epochs when the phases are locked according to (6), are interrupted with relatively short time intervals where a phase slip (at which the phase difference gains $2\pi$) occurs.

### 2.2. EFFECT OF NOISE

The most simple way to model a noisy environment is to add a noisy term to the first of Eqs. (2), or, to have the simplest possible situation, to Eq. (5):

$$\frac{d\varphi}{dt} = \omega_0 - \nu + \varepsilon q(\varphi) + \xi(t) \; . \tag{7}$$

The dynamics of the phase can be treated as the dynamics of an overdamped particle in a potential

$$V(\varphi) = (\nu - \omega_0)\varphi - \varepsilon \int^{\varphi} q(x)\; dx \; .$$

The average slope of the potential is determined by the mismatch of frequencies of the autonomous oscillator and external force; the depth of the minima (if they exist) is determined by the amplitude of the forcing, see Fig. 2. Without noise, the particle would either rest in a minimum, or slide downwards along the potential, if there are no local minima; this corresponds to a synchronous and non-synchronous states, respectively.

Suppose first that the noise is small and bounded, then its influence results in fluctuations of the particle around a stable equilibrium, i.e. in fluctuations of the phase difference around some constant value. We thus have a situation of phase locking in the sense of relation (6); here the observed frequency coincides with that of the external force.

Contrary to this, if the noise is unbounded (e.g., Gaussian), there is always a probability for the particle to overcome a potential barrier $\Delta V$ and to hop in a neighboring minimum of the potential. The time series looks

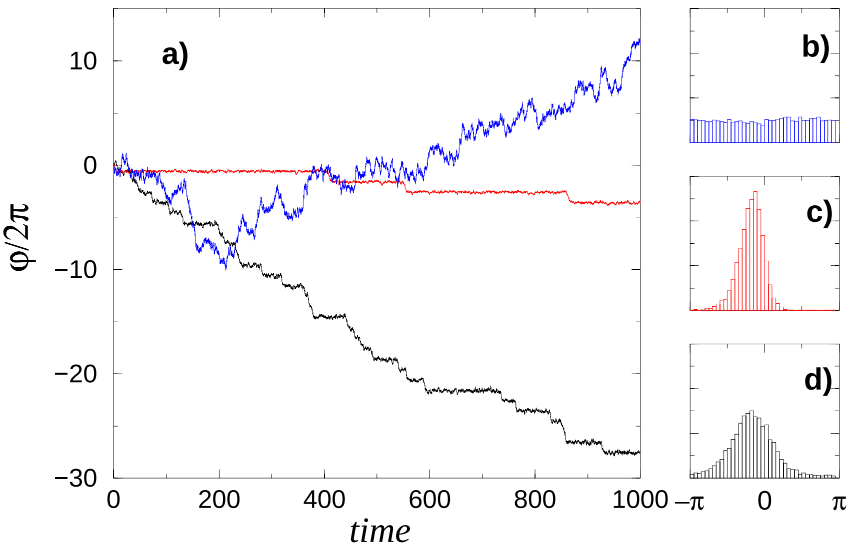


*Figure 3.* Fluctuation of the phase difference in a noisy oscillator (a). Without forcing, the behavior of $\varphi$ is diffusive: it performs a motion that reminds a random walk (upper curve); the distribution of $\varphi \bmod 2\pi$ is shown in (b), it is practically uniform. External forcing with non-zero detuning suppresses the diffusion, the phase of the oscillator is nearly locked (middle curve), but sometimes phase slips occur; the respective distribution (c) becomes rather narrow and unimodal. Stronger noise (bottom curve) causes more phase slips, so that there are only rather short epochs where $\varphi$ oscillates around a constant level; the distribution of $\varphi \bmod 2\pi$ remains nevertheless unimodal (d).

as a sequence of these phase slips (see Fig. 3) and relation (6) does not hold. Nevertheless, at least for small noise, the phase synchronization is definitely detectable, although it is not perfect: between slips we observe epochs of phase locking. Averaged locally over such an epoch, the frequency of the oscillator coincides with that of the external force. The observed frequency that is computed via averaging over a large period of time differs from that of the external force, but this difference is small if the slips are rare.

Phase locking in noisy systems can be also understood in a statistical sense, as an existence of a preferred value of the phase difference $\varphi \bmod 2\pi$. Indeed, the particle spends most of the time around a position of stable equilibrium, then rather quickly it jumps to a neighboring equilibrium, where the phase difference differs by a multiple of $2\pi$. This can be reflected by distribution of $\varphi \bmod 2\pi$: a non-synchronous state would have a broad distribution, whereas synchronization would correspond to a unimodal distribution (Fig. 3).

The synchronization transition in noisy oscillators appears as a continuous decrease of characteristic time intervals between slips, and is smeared:

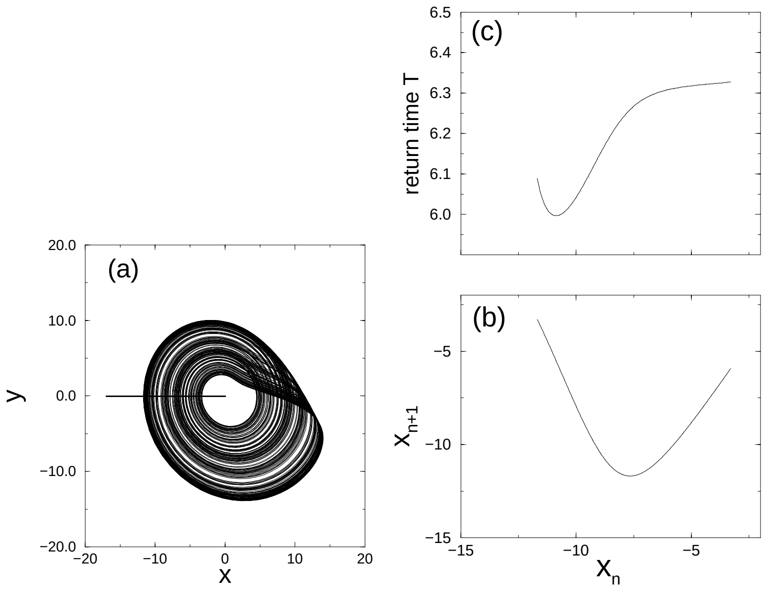


*Figure 4.* Projection of the phase potrait of the Rössler system (a). The horizontal line shows the Poincaré section that is used for computation of the amplitude mapping (b) and dependence of the return time (rotation period) on the amplitude (c).

we cannot unambiguously determine the border of this transition.

## 3. Phase and frequency of a chaotic oscillator

### 3.1. DEFINITION OF THE PHASE

The first problem in extending the basic notions from periodic to chaotic oscillations is to define properly a phase. There seems to be no unambiguous and general definition of phase applicable to an arbitrary chaotic process. Roughly speaking, we want to define phase as a variable which is related to the zero Lyapunov exponent of a continuous-time dynamical system with chaotic behavior. Moreover, we want this phase to correspond to the phase of periodic oscillations satisfying (1).

To be not too abstract, we illustrate a general approach below on the well-known Rössler system. A projection of the phase portrait of this autonomous 3-dimensional system of ODEs (see Eqs. (17) below) is shown in Fig. 4.

Suppose we can define a Poincaré map for our autonomous continuous-time system. Then, for each piece of a trajectory between two cross-sections with the Poincaré surface we define the phase just proportional to time, so that the phase increment is $2\pi$ at each rotation:

$$\phi(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \qquad t_n \le t < t_{n+1}. \tag{8}$$

Here $t_n$ is the time of the $n$-th crossing of the secant surface. Note that for periodic oscillations corresponding to a fixed point of the Poincaré map, this definition gives the correct phase satisfying Eq. (1). For periodic orbits having many rotations (i.e. corresponding to periodic points of the map) we get a piecewise-linear function of time, moreover, the phase grows by a multiple of $2\pi$ during the period. The second property is in fact useful, as it represents in a proper way the organization of periodic orbits inside the chaos. The first property demonstrates that the phase of a chaotic system cannot be defined as unambiguously as for periodic oscillations. In particular, the phase crucially depends on the choice of the Poincaré surface.

Nevertheless, defined in this way, the phase has a physically important property: its perturbations neither grow nor decay in time, so it does correspond to the direction with the zero Lyapunov exponent in the phase space. We note also, that this definition of the phase directly corresponds to the special flow construction which is used in the ergodic theory to describe autonomous continuous-time systems [15].

For the Rössler system (Fig. 4(a)) a proper choice of the Poincaré surface may be the halfplane $y = 0,\ x < 0$. For the amplitude mapping $x_n \to x_{n+1}$ we get a unimodal map (Fig. 4(b), the map is essentially one-dimensional, because the coordinate $z$ for the Rössler attractor is nearly constant on the chosen Poincaré surface). In this and in some other cases the phase portrait looks like rotations around a point that can be taken as the origin, so we can also estimate the phase as the angle between the projection of the phase point on the plane and a given direction on the plane (see also [48, 26]):

$$\phi_P = \arctan(y/x)\ . \tag{9}$$

Note that although the phase $\phi$ and its estimate $\phi_P$ do not coincide microscopically, i.e on a time scale less than the average period of oscillation, they have equal average growth rates. In other words, the mean frequency defined as the average of $d\phi_P/dt$ over large period of time coincides with a straightforward definition of the mean frequency via the average number of crossings of the Poincaré surface per unit time.

### 3.2. LOCKING-BASED DEFINITION OF FREQUENCY FOR SYSTEMS WITH ILL-DEFINED PHASE

Phase can be straightforwardly introduced if one can find a two-dimensional projection of the attractor in which all trajectories revolve around some origin. This is typically the case for systems exhibiting a transition to chaos via a cascade of period doubling bifurcations; e.g., for the Rössler oscillator. For such projections one can define phase according to Eq. (8), or estimate it according to Eq. (9) or using the Hilbert transform (see section 8 below). Sometimes, a proper projection can be achieved with a coordinate transformation (e.g., using the symmetry properties of the attractor, as in the Lorenz system) [50, 49]. Estimation of the average frequency of individual oscillators $\langle\dot{\phi}\rangle$ then allows one to characterize the degree of synchronization. Contrary to these cases of well-defined phase, chaotic oscillators with "wild", non-revolving trajectories are often termed as those with ill-defined phase. Here only indirect indications for phase synchronization exist (based, e.g., on the ensemble averages [48, 50]), but no direct calculation of the phase and the frequency can be performed.

Nere we describe a method [59], based on the use of *auxiliary limit cycle oscillators*, that allows one to estimate the average frequency of the observed signals for situations with ill-defined phase. To introduce the method let us consider an ensemble of *uncoupled* limit cycle oscillators with natural frequencies $\omega_k$ distributed in an interval $[\omega_{min}, \omega_{max}]$. Let each oscillator of this ensemble be driven by a common periodic force of a frequency $\nu \in [\omega_{min}, \omega_{max}]$. It is well-known that the force synchronizes those elements of the ensemble which have frequencies close to $\nu$. This can be demonstrated by plotting the frequencies of the driven limit cycle oscillators $\Omega_k$, called hereafter the observed frequencies, vs. the natural frequencies $\omega_k$: the synchronization manifests itself in the appearance of a horizontal plateau (more precisely, one expects to observe a devil's staircase structure with infinitely many plateaus), where the frequency of entrained elements is equal to $\nu$. Hence, an *unknown* frequency of the drive can be revealed by the analysis of the $\Omega_k$ vs. $\omega_k$ plot. The idea of our approach is to use the ensemble of auxiliary oscillators as a *device for measuring the frequency of complex signals.*

A simple implementation of the method is to drive the array of Poincaré oscillators with a signal $X(t)$, which frequency we would like to determine

$$\dot{A}_k = (1 + i\omega_k)A_k - |A_k|^2 A_k + \varepsilon X(t) \ . \tag{10}$$

Separating the real amplitude $R$ and the phase $\phi$ from the complex amplitude $A = Re^{i\phi}$ we obtain for the phase $\dot{\phi}_k = \omega_k - R_k^{-1}\varepsilon X(t)\sin\phi_k$. Noting that for small $\varepsilon$ the amplitude $R$ is close to unity, and neglecting its fluctuations, we can write equations for our measuring oscillators as pure phase equations:

$$\dot{\phi}_k = \omega_k - \varepsilon X(t)\sin\phi_k, \tag{11}$$

and the observed frequencies are $\Omega_k = \langle\dot{\phi}_k\rangle$. Note that Eqs. (11) become exact if one writes a higher order nonlinearity $|A|^p A$ in (10) and considers the limit $p \to \infty$. In calculations below we normalize the signal $X(t)$ to have zero mean and unit variance so that the coupling constant $\varepsilon$ is the only parameter of the method (the mean value can slightly influence the result).

To show how the method works we consider a model quasiharmonic process with mean frequency $\omega_0$ and slowly varying amplitude and phase: $X(t) = 2(1 + a(t)) \cos(\omega_0 t - \theta(t))$. Substituting this in (11) and averaging over the period of fast oscillations $2\pi/\omega_0$, we obtain for the slowly varying phase difference $\psi = \phi - \omega_0 t + \theta$ the equation

$$\dot{\psi} = \omega - \omega_0 + \dot{\theta}(t) - \varepsilon(1 + a(t)) \sin\psi \ ,$$

for a harmonic signal ($\dot{\theta} = a = 0$) it has for $\varepsilon \geq |\omega - \omega_0|$ the synchronized solution $\psi_0 = \arcsin((\omega - \omega_0)/\varepsilon)$. For weak modulation we can linearize around this state and obtain for the deviations $\delta\psi$:

$$\frac{d(\delta\psi)}{dt} = \dot{\theta} - a(t)(\omega - \omega_0) - \sqrt{\varepsilon^2 - (\omega - \omega_0)^2}\,\delta\psi \ .$$

Assuming that $\theta$ and $a$ are independent random processes, we can express the power spectrum of the phase fluctuations through the spectra of these processes:

$$S_{\delta\psi}(\sigma) = \frac{\sigma^2 S_\theta(\sigma) + (\omega - \omega_0)^2 S_a(\sigma)}{\varepsilon^2 - (\omega - \omega_0)^2 + \sigma^2} \ .$$

One can see that the fluctuations are small only in the middle of the synchronization region (for $\omega \approx \omega_0$); here only the phase fluctuations $S_\theta$ contribute. Modeling $S_\theta$ by the Lorentz-like spectrum

$$S_\theta = \frac{2\Delta V_\theta}{(\sigma^2 + \Delta^2)\pi} \ ,$$

where $V_\theta$ and $\Delta$ are the variance and the characteristic maximal frequency of fluctuations of $\theta$, we obtain

$$V_{\delta\psi} = \int_0^\infty S_{\delta\psi}(\sigma) d\sigma = V_\theta \Delta(\varepsilon + \Delta)^{-1} \ .$$

This final formula shows that good synchronization (i.e. small variance of $\delta\psi$) can be achieved if $\varepsilon$ is sufficiently larger than $\Delta$, i.e. if the coupling constant is larger than the characteristic frequency of phase fluctuations. From the other side, in the limit $\varepsilon \to \infty$ the dependence of the observed frequency $\Omega$ on the natural one disappears, and the measured frequency is the Rice frequency of the process $X(t)$.

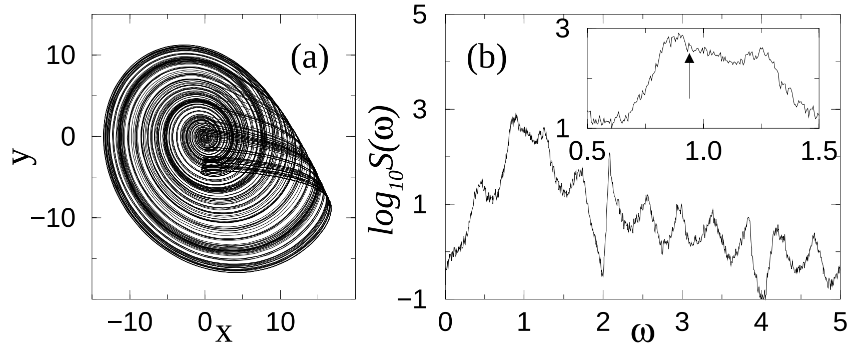


*Figure 5.* (a) Funnel attractor in the Rössler system (12). (b) Power spectrum of $x(t)$. The arrow shows the characteristic frequency as determined according to Fig. 6.

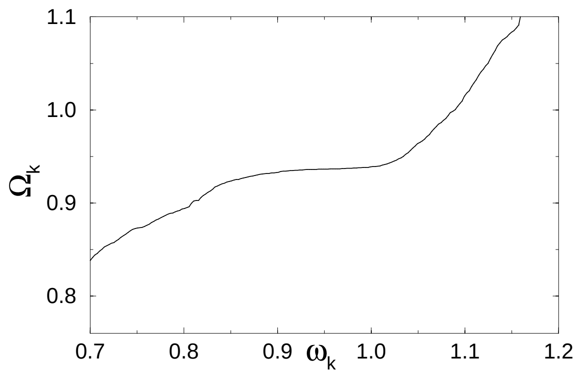


*Figure 6.* Output of the frequency measuring "device" (11) as a function of the natural frequencies $\omega_k$ for the Rössler oscillator (12). The height of the plateau determines the characteristic frequency of this chaotic drive $\Omega^{(p)} = 0.94$.

To illustrate the approach we consider the Rössler system with a funnel attractor (Fig. 5a):

$$\begin{aligned} \dot{x} &= -y - z \ , \\ \dot{y} &= x + 0.4y \ , \\ \dot{z} &= 0.25 + z(x - 8.5) \ . \end{aligned} \tag{12}$$

Clearly we cannot find an origin around which all trajectories revolve. The power spectrum for the variable $x$ (Fig. 5b) is broad; it does not contain a dominating maximum. Due to these properties, there is no direct way to introduce the phase for this system and to characterize its synchronization [50]. In our method, we use the (normalized) signal $x(t)$ to drive system (11) with $\varepsilon = 0.5$. The frequencies of the oscillators in the measuring device driven by $X(t) = x(t)$ are shown in Fig. 6. The resulting plateau in the $\Omega_k$ vs. $\omega_k$ plot gives $\Omega^p \approx 0.94$ where "p" stands for "plateau". Practically, the middle point of the plateau $\Omega_k^p$ was determined from the minimum of the running variance $\sum_{j=k-L}^{k+L}(\Omega_j - \bar{\Omega}_k)^2$, where $\bar{\Omega}_k = (2L+1)^{-1}\sum_{j=k-L}^{k+L}\Omega_j$. Variation of $L$ from 3 to 10 gave no essential difference. This method provides also a smoothening of the $\Omega_k$ vs. $\omega_k$ curve. One can see that this characteristic frequency cannot be directly associated with a peak in the power spectrum (Fig. 5b). We also see that Fig. 6 does not show the devil's staircase structure, but only one, smeared plateau. This is due to the chaotic nature of the process $x(t)$, so that, similar to the case of narrow-band noisy signals, the high-order phase-locked regions are not observed [37, 49].

### 3.3. DYNAMICS OF THE PHASE OF CHAOTIC OSCILLATIONS

In contrast to the dynamics of the phase of periodic oscillations, the growth of the phase in the chaotic case cannot generally be expected to be uniform. Instead, the instantaneous frequency depends in general on the amplitude. Let us hold to the phase definition based on the Poincaré map, so one can represent the dynamics as (cf. [52])

$$A_{n+1} = M(A_n) \ , \tag{13}$$

$$\frac{d\phi}{dt} = \omega(A_n) \equiv \omega_0 + F(A_n) \ . \tag{14}$$

As the amplitude $A$ we take the set of coordinates for the point on the secant surface; it does not change during the growth of the phase from 0 to $2\pi$ and can be considered as a discrete variable; the transformation $M$ defines the Poincaré map. The phase evolves according to (14), where the "instantaneous" frequency $\omega = 2\pi/(t_{n+1} - t_n)$ depends in general on the amplitude. Assuming the chaotic behavior of the amplitudes, we can consider the term $\omega(A_n)$ as a sum of the averaged frequency $\omega_0$ and of some effective noise $F(A)$; in exceptional cases $F(A)$ may vanish. For the Rössler attractor the "period" of the rotations (i.e. the function $2\pi/\omega(A_n)$) is shown in Fig. 4(c). This period is not constant, so the function $F(A)$ does not vanish, but the variations of the period are relatively small.

Hence, the Eq. (14) is similar to the equation describing the evolution of phase of periodic oscillator in the presence of external noise. Thus, the dynamics of the phase is generally diffusive: for large $t$ one expects

$$\langle(\phi(t) - \phi(0) - \omega_0 t)^2\rangle \propto D_p t \ ,$$

where the diffusion constant $D_p$ determines the phase coherence of the chaotic oscillations. Roughly speaking, the diffusion constant is proportional to the width of the spectral peak calculated for the chaotic observable [21].

Generalizing Eq. (14) in the spirit of the theory of periodic oscillations to the case of periodic external force, we can write for the phase

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(\phi, \psi) + F(A_n) \ , \qquad \frac{d\psi}{dt} = \nu \ . \tag{15}$$

Here we assume that the force is small (of order of $\varepsilon$) so that it affects only the phase, and the amplitude obeys therefore the unperturbed mapping $M$. This equation is similar to Eq. (7), with the amplitude-depending part of the instantaneous frequency playing the role of noise. Thus, we expect that in general the synchronization phenomena for periodically forced chaotic system are similar to those in noisy driven periodic oscillations. One should be aware, however, that the "noisy" term $F(A)$ can be hardly calculated explicitly, and for sure cannot be considered as a Gaussian $\delta$-correlated noise as is commonly assumed in the statistical approaches [65, 55].

## 4. Phase synchronization by external force

### 4.1. SYNCHRONIZATION REGION

We describe here the effect of phase synchronization of chaotic oscillations by periodic external force, taking as examples two prototypic models of nonlinear dynamics: the Lorenz

$$\begin{aligned} \dot{x} &= 10(y - x), \\ \dot{y} &= 28x - y - xz, \\ \dot{z} &= -8/3 \cdot z + xy + E \cos \nu t. \end{aligned} \tag{16}$$

and the Rössler

$$\begin{aligned} \dot{x} &= -y - z + E \cos \nu t \ , \\ \dot{y} &= x + 0.15y \ , \\ \dot{z} &= 0.4 + z(x - 8.5) \ . \end{aligned} \tag{17}$$

oscillators. In the absence of forcing, both are 3-dimensional dissipative systems which admit a straightforward construction of the Poincaré maps. Moreover, we can simply use the phase definition (9), taking the original variables $(x, y)$ for the Rössler system and the variables $(\sqrt{x^2 + y^2} - u_0, z - z_0)$ for the Lorenz system (where $u_0 = 12\sqrt{2}$ and $z_0 = 27$ are the coordinates of the equilibrium point, the "center of rotation"). The mean rotation frequency can be thus calculated as

$$\Omega = \lim_{t \to \infty} 2\pi \frac{N_t}{t} \tag{18}$$

where $N_t$ is the number of crossings of the Poincaré section during observation time $t$. This method can be straightforwardly applied to the observed

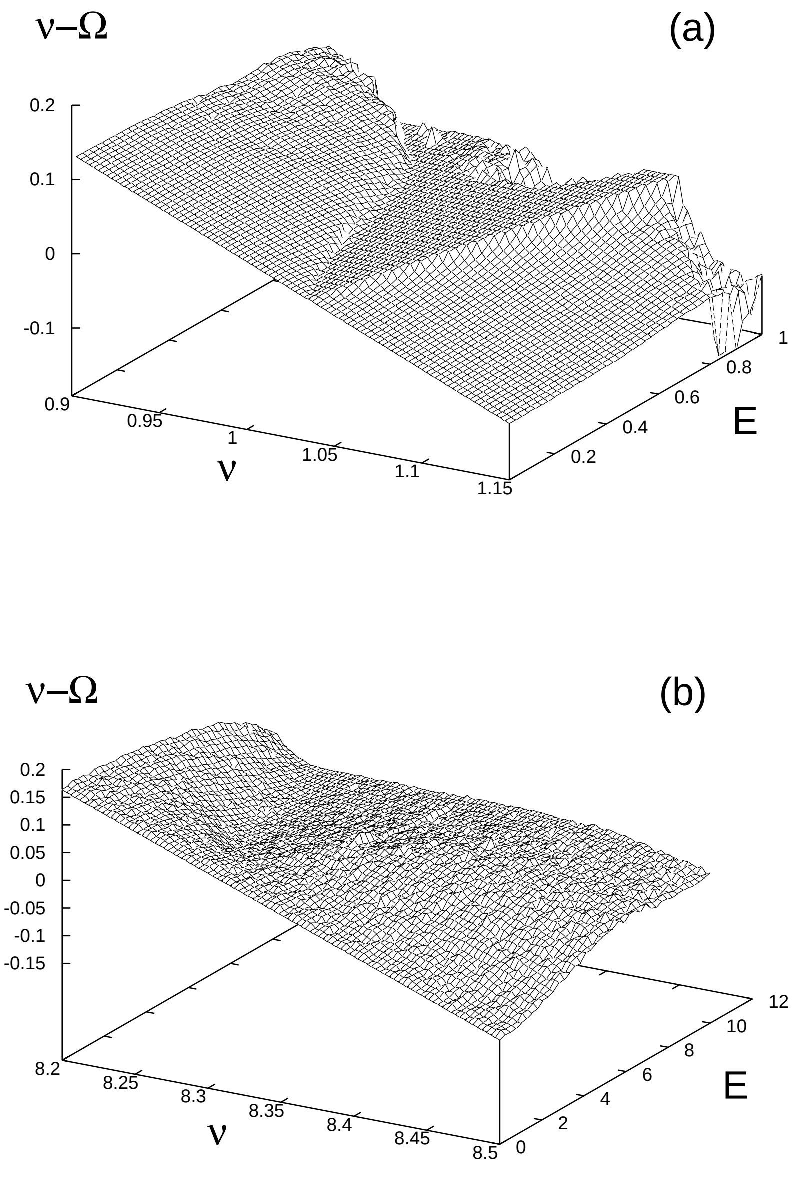


*Figure 7.* The phase synchronization regions for the Rössler (a) and the Lorenz (b) systems.

time series, in the simplest case one can, e.g., take for $N_t$ the number of maxima (of $x(t)$ for the Rössler system and of $z(t)$ for the Lorenz one).

Dependence of the frequency $\Omega$ obtained in this way on the amplitude and frequency of the external force is shown in Fig. 7. Synchronization here corresponds to the plateau $\Omega = \nu$. One can see that the synchronization properties of these two systems differ essentially. For the Rössler system there exists a well-expressed region where the systems are perfectly locked. Moreover, there seems to be no amplitude threshold of synchronization (cf. Fig. 1c, where the phase-locking regions start at $\varepsilon = 0$). It appears that the phase locking properties of the Rössler system are practically the same as for a periodic oscillator. On the contrary, for the Lorenz system we observe the frequency locking only as a tendency seen at relatively large forcing amplitudes, as this should be expected for oscillators subject to a rather strong noise. In this respect, the difference between Rössler and Lorenz systems can be described in terms of phase diffusion properties (see Sect. 3.3). Indeed, the phase diffusion coefficient for autonomous Rössler system is extremely small $D_p < 10^{-4}$, whereas for the Lorenz system it is several oder of magnitude larger, $D_p \approx 0.2$ [50]. This difference in the coherence of the phase of autonomous oscillations implies different response to periodic forcing.

In the following sections we discuss the phase synchronization of chaotic oscillations from the statistical and the topological viewpoints.

### 4.2. STATISTICAL APPROACH

We define the phase of an autonomous chaotic system as a variable that corresponds to the zero Lyapunov exponent, i.e. to the invariance with respect to time shifts. Therefore, the invariant probability distribution as a function of the phase is nearly uniform. This follows from the ergodicity of the system: the probability is proportional to the time a trajectory is spending in a region of the phase space, and according to the definition (8) the phase motion is (piecewise) uniform. With external forcing, the invariant measure depends explicitly on time. In the synchronization region we expect that the phase of oscillations nearly follows the phase of the force, while without synchronization there is no definite relation between them. Let us observe the oscillator stroboscopically, at the moments corresponding to some phase $\psi_0$ of the external force. In the synchronous state the probability distribution of the oscillator phase will be localized near some preferable value (which of course depends on the choice of $\psi_0$). In the non-synchronous state the phase is spread along the attractor. We illustrate this behavior of the probability density in Fig. 8. One can say that synchronization means localization of the probability density near some preferable time-periodic state. In other words, this means appearance of the long-range correlation

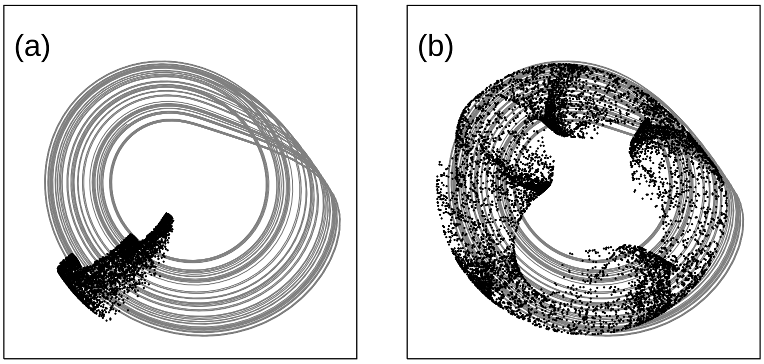


*Figure 8.* Distribution inside (a) and outside (b) the synchronization region for the Rössler system, shown with black dots. The autonomous Rössler attractor is shown with gray.

in time and of the significant discrete component in the power spectrum of oscillations.

Let us consider now the ensemble interpretation of the probability. Suppose we take a large ensemble of identical copies of the chaotic oscillator which differ only by their initial states, and let them evolve under the same periodic forcing. After the transient, the projections of the phase state of each oscillator onto the plane $x, y$ form the cloud that exactly corresponds to the probability density. Let us now consider the ensemble average of some observable. Without synchronization the cloud is spread over the projection of the attractor (Fig. 8b), and the average is small: no significant average field is observed. In the synchronous state the probability is localized (Fig. 8a), so the average is close to some middle point of the cloud; this point rotates with the frequency $\nu$ and one observes large regular oscillations of the average field. Hence, the synchronization can be easily indicated through the appearance of a large (macroscopic) mean field in the ensemble. Physically, this effect is rather clear: unforced chaotic oscillators are not coherent due to internal chaos, thus the summation of their fields yields a small quantity. Being synchronized, the oscillators become coherent with the external force and thereby with each other, so the coherent summation of their fields produces a large mean field.

We can look on the probability also from the ergodic point of view, where instead of taking an ensemble one takes separated in time states of one system. The described above coherence that appears due to phase synchronization is now coherence in time. It can be revealed by calculating the autocorrelation function or the power spectrum. Synchronization means that correlations in time are large and a significant discrete peak appears in the spectrum of oscillations.

An important consequence of the statistical approach described above is that the phase synchronization can be characterized without explicit computation of the phase and/or the mean frequency: it can be indicated implicitly by the appearance of a macroscopic mean field in the ensemble of oscillators, or by the appearance of the large discrete component in the spectrum. Although there may be other mechanisms leading to the appearance of macroscopic order, the phase synchronization appears to be one of the most common ones.

## 5. Phase synchronization in coupled systems

Now we demonstrate the effects of phase synchronization in coupled chaotic oscillators. We start with the simplest case of two interacting systems, and then briefly discuss oscillator lattices, globally coupled systems, and space-time chaos.

### 5.1. SYNCHRONIZATION OF TWO INTERACTING OSCILLATORS

We consider here two non-identical coupled Rössler systems

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2} y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + a y_{1,2}, \\ \dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c), \end{aligned} \tag{19}$$

where $a = 0.165$, $f = 0.2$, $c = 10$. The parameters $\omega_{1,2} = \omega_0 \pm \Delta\omega$ and $\varepsilon$ determine the mismatch of natural frequencies and the coupling, respectively.

Again, like in the case of periodic forcing, we can define the mean frequencies $\Omega_{1,2}$ of oscillations of each system, and study the dependence of the frequency mismatch $\Omega_2 - \Omega_1$ on the parameters $\Delta\omega, \varepsilon$. This dependence is shown in Fig. 9 and demonstrates a large region of synchronization between two oscillators.

It is instructive to characterize the synchronization transition by means of the Lyapunov exponents (LE). The 6-order dynamical system (19) has 6 LEs (see Fig. 10). For zero coupling we have a degenerate situation of two independent systems, each of them has one positive, one zero, and one negative exponent. The two zero exponents correspond to the two independent phases. With coupling, the phases become dependent and the degeneracy must be removed: only one LE should remain exactly zero. We observe, however, that for small coupling also the second zero Lyapunov exponent remains extremely small (in fact, numerically indistinguishable from zero). Only at relatively stronger coupling, when the synchronization sets on, the second LE becomes negative: now the phases are dependent

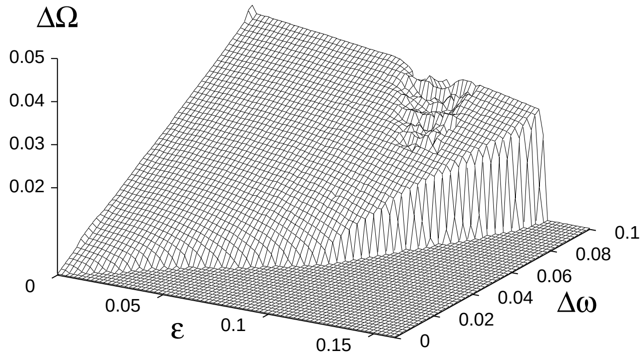


*Figure 9.* Synchronization of two coupled Rössler oscillators; $\omega_0 = 1$.

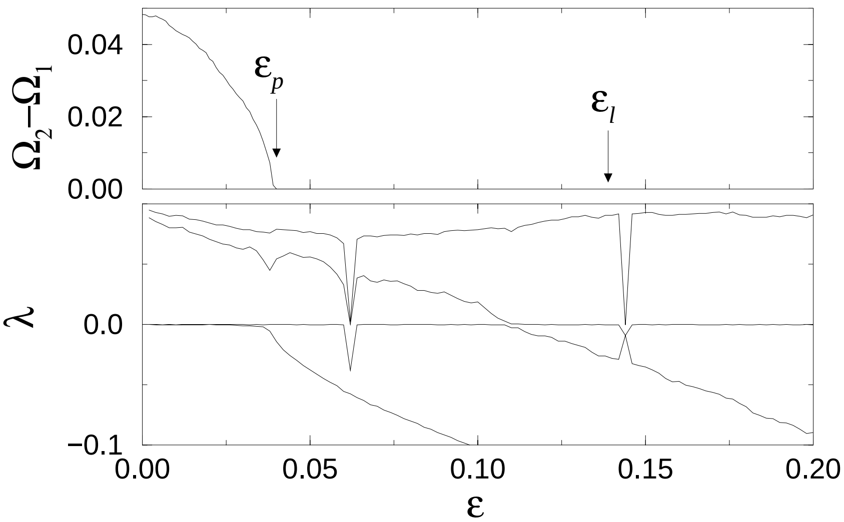


*Figure 10.* The Lyapunov exponents $\lambda$ (bottom panel, only the 4 largest LEs are depicted) and the frequency difference vs. the coupling $\varepsilon$ in the coupled Rössler oscillators; $\omega_0 = 0.97$, $\Delta\omega = 0.02$. Transition to the phase ($\varepsilon_p$) and to the lag synchronization ($\varepsilon_l$) are marked.

and a relation between them is stable. Note that the two positive exponents remain positive which means that the amplitudes remain chaotic and independent: the coupled system remains in the state of hyperchaos.

With the increase of coupling one of the positive LE becomes smaller. Physically this means that not only the phases are locked, but the difference between the amplitudes is suppressed by coupling as well. At a certain coupling only one LE remains positive, so one can expect synchronization both in phases and amplitudes. As the systems are not identical (due to the frequency mismatch), their states cannot be identical: $x_1(t) \neq x_2(t)$. However, almost perfect correspondence between the time-shifted states of the systems can be observed: $x_1(t) \approx x_2(t-\Delta t)$. This phenomenon is called "lag synchronization" [58]. With further increase of the coupling $\varepsilon$ the lag $\Delta t$ decreases and the states of two systems become nearly identical, like in case of complete synchronization.

### 5.2. SYNCHRONIZATION IN A POPULATION OF GLOBALLY COUPLED CHAOTIC OSCILLATORS

A number of physical, chemical and biological systems can be viewed at as large populations of weakly interacting non-identical oscillators [34]. One of the most popular models here is an ensemble of globally coupled nonlinear oscillators (often called "mean-field coupling"). A nontrivial transition to self-synchronization in a population of periodic oscillators with different natural frequencies coupled through a mean field has been described by Kuramoto [34, 33]. In this system, as the coupling parameter increases, a sharp transition is observed for which the mean field intensity serves as an order parameter. This transition owes to a mutual synchronization of the periodic oscillators, so that their fields become coherent (i.e. their phases are locked), thus producing a macroscopic mean field. In its turn, this field acts on the individual oscillators, locking their phases, so that the synchronous state is self-sustained. Different aspects of this transition have been studied in [61, 17, 18], where also an analogy with the second–order phase transition has been exploited.

A similar effect can be observed in a population of *non-identical chaotic* systems, e.g. the Rössler oscillators

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon X, \\ \dot{y}_i &= \omega_i x_i + a y_i, \\ \dot{z}_i &= 0.4 + z_i(x_i - 8.5), \end{aligned} \tag{20}$$

coupled via the mean field $X = N^{-1}\sum_1^N x_i$. Here $N$ is the number of elements in the ensemble, $\varepsilon$ is the coupling constant, $a$ and $\omega_i$ are parameters of the Rössler oscillators. The parameter $\omega_i$ governs the natural frequency of an individual system. We take a set of frequencies $\omega_i$ which are Gaussian-distributed around the mean value $\omega_0$ with variance $(\Delta\omega)^2$. The Rössler system typically shows windows of periodic behavior as the parameter $\omega$ is changed; therefore we usually choose a mean frequency $\omega_0$ in a way that we avoid large periodic windows. In our computer simulations we solve numerically Eqs. (20) for rather large ensembles $N = 3000 \div 5000$.

With an increase of the coupling strength $\varepsilon$, the appearance of a non-zero macroscopic mean field $X$ is observed [48], as is shown in Fig. 11. This

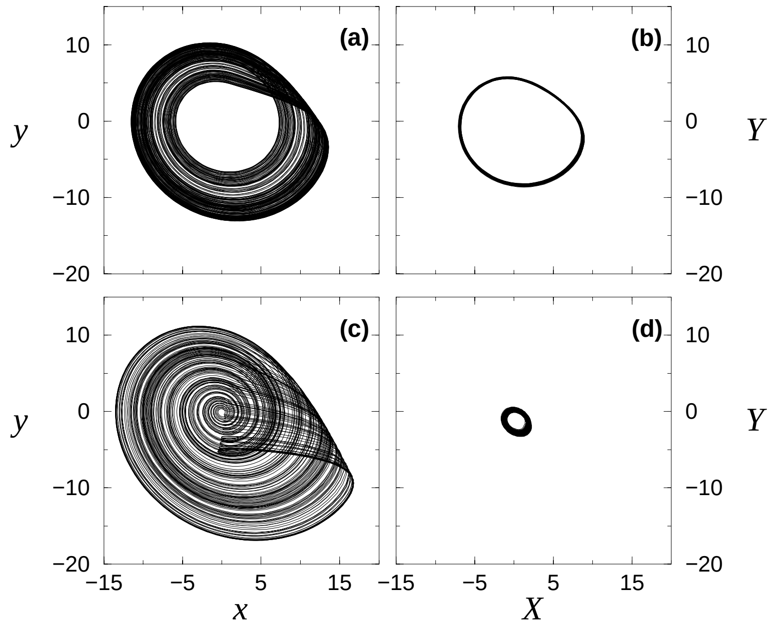


*Figure 11.* Projections of the phase portraits of the Rössler oscillators (left column) and of the mean fields $X = \langle x_i \rangle$, $Y = \langle y_i \rangle$ in ensemble of $N = 5000$ oscillators. (a): Phase-coherent Rössler attractor, $\omega_0 = 1$, $a = 0.15$. (b): Mean field in the ensemble of oscillators (a) with Gaussian distribution of frequencies $\Delta\omega = 0.02$ and coupling $\varepsilon = 0.1$. (c) Funnel attractor $\omega_0 = 0.97$, $a = 0.25$. (d): Mean field in the ensemble of oscillators (c) with Gaussian distribution of frequencies $\Delta\omega = 0.02$ and coupling $\varepsilon = 0.15$.

indicates the phase synchronization of the Rössler oscillators that arises due to their interaction via mean field. This mean field is large, if the attractors of individual systems are phase-coherent (parameter $a = 0.15$) and the phase is well-defined. On the contrary, in the case of the funnel attractor $a = 0.25$, when the oscillations look wild and the imaging point makes large and small loops around the origin, there seems to be no way to choose the Poincaré section unambiguously; in this case the mean field is rather small. Nevertheless, in both cases synchronization transition is clearly indicated by the onset of the mean field, without computation of the phases themselves. Finally, we note that phase synchronization in a globally coupled ensemble of chaotic oscillators have been observed experimentally in [30, 29].

## 6. Lattice of chaotic oscillators

If chaotic oscillators are ordered in space and form a lattice, usually it is assumed that only the nearest neighbors interact. Such a situation is relevant for chemical systems, where homogeneous oscillations are chaotic, and the diffusive coupling can be modeled with dissipative nearest neighbors

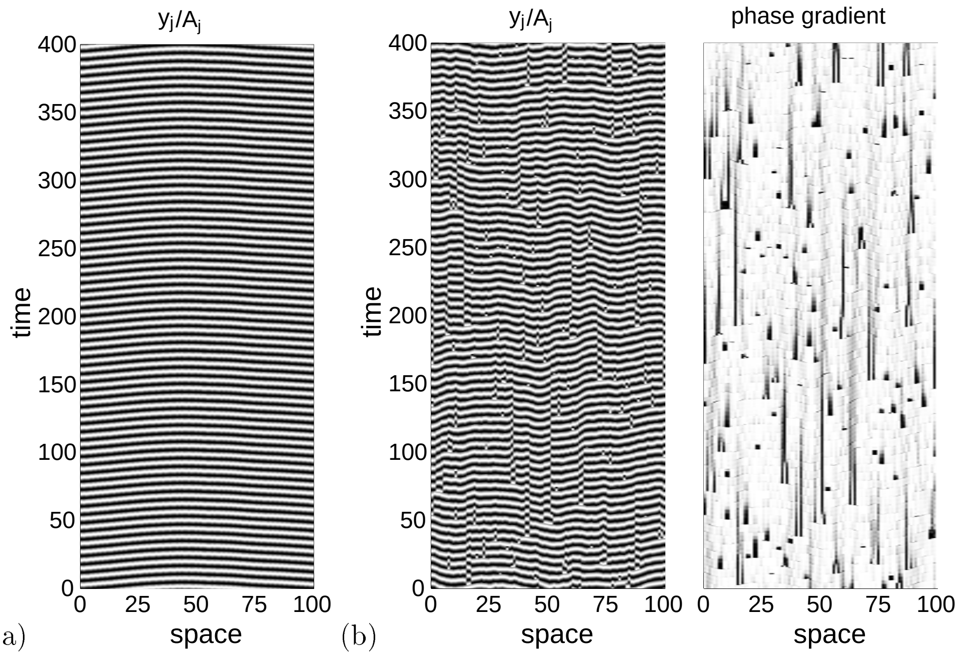


*Figure 12.* (a): Nearly homogeneous phase profile for the lattice of identical phase-coherent Rössler oscillators. (b) In a lattice of identical funnel Rössler oscillators $a = 0.23$, $\varepsilon = 0.05$ an inhomogeneous phase profile with numerous defects is observed. The defects are clearly seen in the right panel as large values of the gradient

interaction [10, 26]. In a lattice, one can expect complex spatio-temporal synchronization structures to be observed.

Consider as a model a 1-dimensional lattice of Rössler oscillators with local dissipative coupling:

$$\begin{aligned} \dot{x}_j &= -\omega_j y_j - z_j, \\ \dot{y}_j &= \omega_j x_j + a y_j + \varepsilon(y_{j+1} - 2y_j + y_{j-1}), \\ \dot{z}_j &= 0.4 + (x_j - 8.5) z_j. \end{aligned} \tag{21}$$

Here the index $j = 1, \ldots, N$ counts the oscillators in the lattice and $\varepsilon$ is the coupling coefficient. In a homogeneous lattice (i.e. for equal natural frequencies $\omega_j$) the observed regime significantly depends on the coherence properties of a single chaotic oscillator. If the Rössler oscillator is phase-coherent, all the phases nearly synchronize and a regular phase pattern (in fact, a nearly homogeneous phase distribution) is observed (Fig. 12a). In the case of funnel Rössler oscillator the phase at a individual element can "spontaneously" change by $\pi$, this prevents synchronization in the lattice (Fig. 12b).

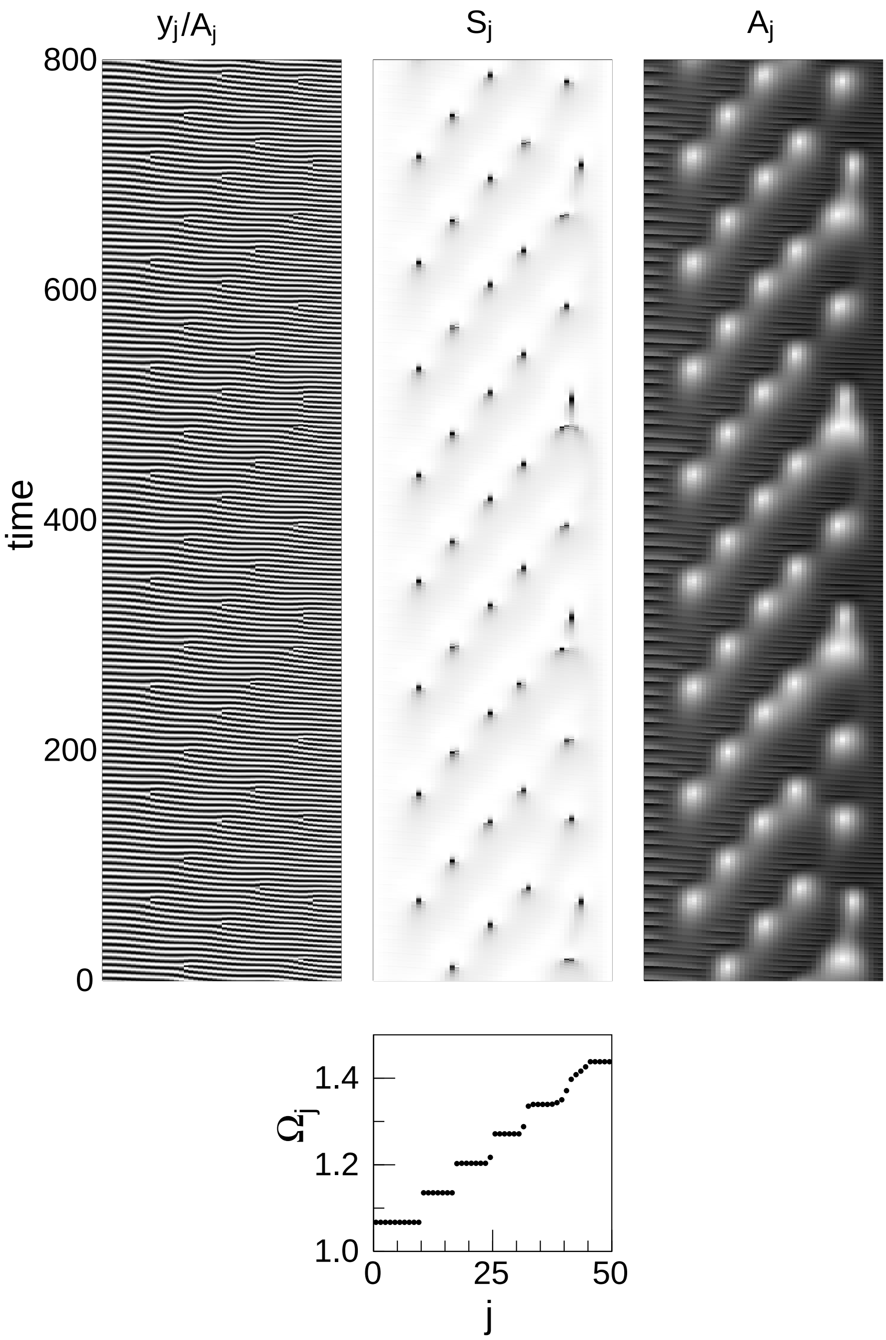


*Figure 13.* Clusters and space-time defects in a lattice of Rössler oscillators with a linear distribution of natural frequencies. On the bottom panel the observed frequencies are shown.

To study synchronization in a lattice of non-identical oscillators, we introduce a linear distribution of natural frequencies $\omega_j$

$$\omega_j = \omega_1 + \delta(j-1) \tag{22}$$

where $\delta$ is the frequency mismatch between neighboring sites. Depending on the values of $\delta$ we observed two scenarios of transition to synchronization [42]. For small $\delta$, the transition occurs smoothly, i.e. all the elements along the chain gradually adjust their frequencies. If the frequency mismatch is larger, clustering is observed: the oscillators build phase-synchronized groups having different mean frequencies (Fig. 13). At the borders between clusters phase slips occur; this can be considered as appearance of defects in the spatio-temporal representation. Both regular and irregular patterns of defects can be seen in Fig. 13.

## 7. Synchronization of space-time chaos

The idea of phase synchronization can be also applied to space-time chaos. For example, in the famous complex Ginzburg-Landau equation (CGLE) [16, 13, 63]

$$\partial_t a = (1 + i\omega_0)a - (1 + i\alpha)|a|^2 a + (1 + i\beta)\partial_t^2 a \ , \tag{23}$$

there are regimes where the complex amplitude $a$ rotates with some mean frequency, but these rotations are not regular: the phase deviates irregularly in space and time (this regime is called "phase turbulence"). Another regime, where the complex amplitude $a$ not always rotates but experiences space-time defects (the places where the absolute value of the amplitude vanishes whereas the phase is not defined); this regime is analogous to oscillations with ill-defined phase like in the funnel Rössler oscillator.

Let us now add periodic in time spatially homogeneous forcing of amplitude $B$ and frequency $\omega_e$. Transition into a reference frame rotating with this external forcing ($a \to A \equiv a\exp(-i\omega_e t)$) reduces Eq. (23) to

$$\partial_t A = (1 + i\nu)A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\partial_t^2 A + B \ , \tag{24}$$

where $\nu = \omega_0 - \omega_e$ is the frequency mismatch between the frequency of the external force and the frequency of small oscillations. An analysis of different regimes in the system (24) has been recently performed [14]. As one can expect, a very strong force suppresses turbulence and the spatially homogeneous periodic in time synchronous oscillations are observed, while a small force has no significant influence on the turbulent state. A nontrivial regime is observed for intermediate forcing: in some parameter range the irregular fluctuations of the phase are not completely suppressed but are bounded: the whole system oscillates "in phase" with the external force

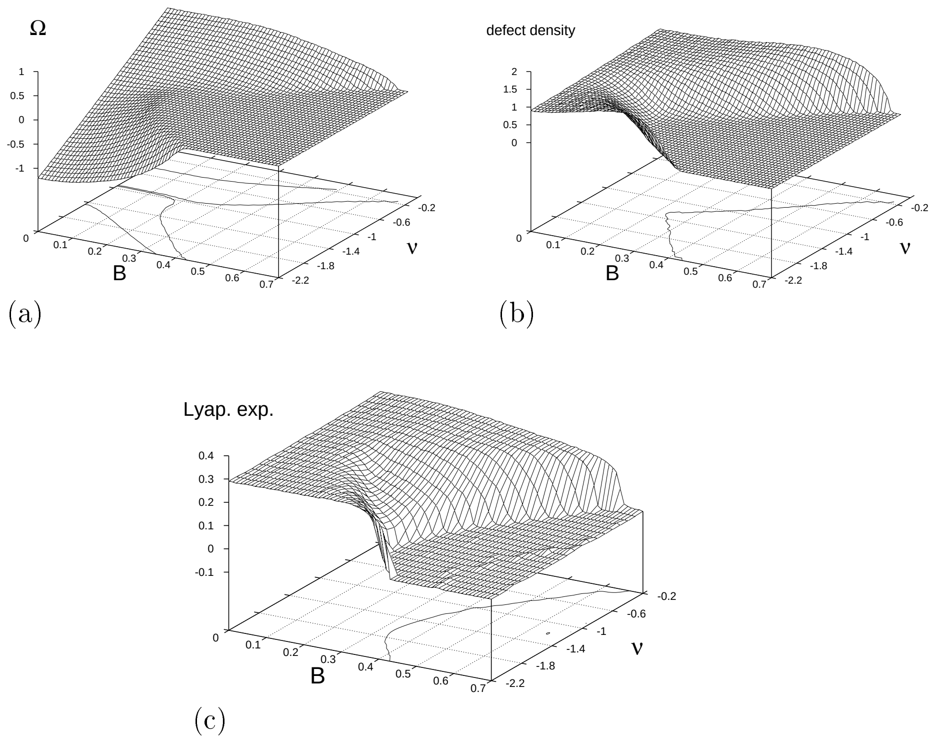


*Figure 14.* Synchronization of defect turbulence ($\alpha = -2$, $\beta = 2$): (a) Average frequency as a function of the amplitude $B$ and the frequency $\nu$ of the external force. The contour lines are drawn at levels $-0.5$, $-0.01$, $0.01$, $0.5$. (b) Density of defects (arbitrary units). The contour line shows the border of the defect-free region. (c) The largest Lyapunov exponent. The contour line shows the border of the turbulent region.

and is highly coherent, although some small chaotic variations persist. In Figs. 14,15 we show how the forcing acts on the regimes with defect and phase turbulence in the CGLE. One can see that the forcing can supress defects while not supressing completely the space-time chaos; this regime is analogous to phase synchronization of individual oscillators. In the case of phase turbulence there is a range of parameters where defects are observed, they appear due to a specific for the forced CGLE instability of special solutions – kinks – connecting two spatially homogeneous regions of complete synchronization, the phases in these regions differ by $\pm 2\pi$. These kinks can disappear through a defect, but from this defect two new kinks appear, leading to kink-breeding process (Fig. 16).

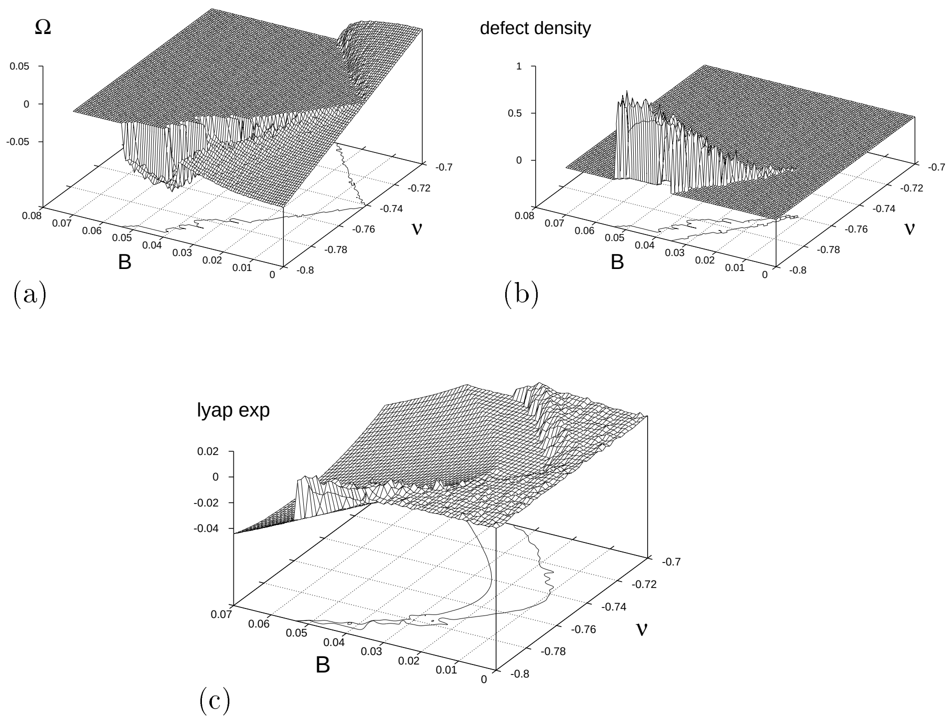


*Figure 15.* Synchronization of phase turbulence ($\alpha = -0.75$, $\beta = 2$). (a) The average frequency $\Omega$ as a function of the amplitude $B$ and the frequency $\nu$ of the external force. The contour line shows the border of the synchronization region. (b) The density of defects (arbitrary units). (c) The largest Lyapunov exponent. The contour lines show the borders between the regions of positive, zero, and negative exponents.

## 8. Detecting synchronization in data

The analysis of relation between the phases of two systems, naturally arising in the context of synchronization, can be used to approach a general problem in time series analysis. Indeed, bivariate data are often encountered in the study of real systems, and the usual aim of the analysis of such data is to find out whether two signals are dependent or not. As experimental data are very often non-stationary, the traditional techniques, such as cross–spectrum and cross–correlation analysis [44], or non–linear characteristics like generalized mutual information [53] or maximal correlation [71] have their limitations. From the other side, sometimes it is reasonable to assume that the observed signals originate from two weakly interacting systems. The presence of this interaction can be found by means of the analysis of *instantaneous* phases of these signals. These phases can be unambigu-

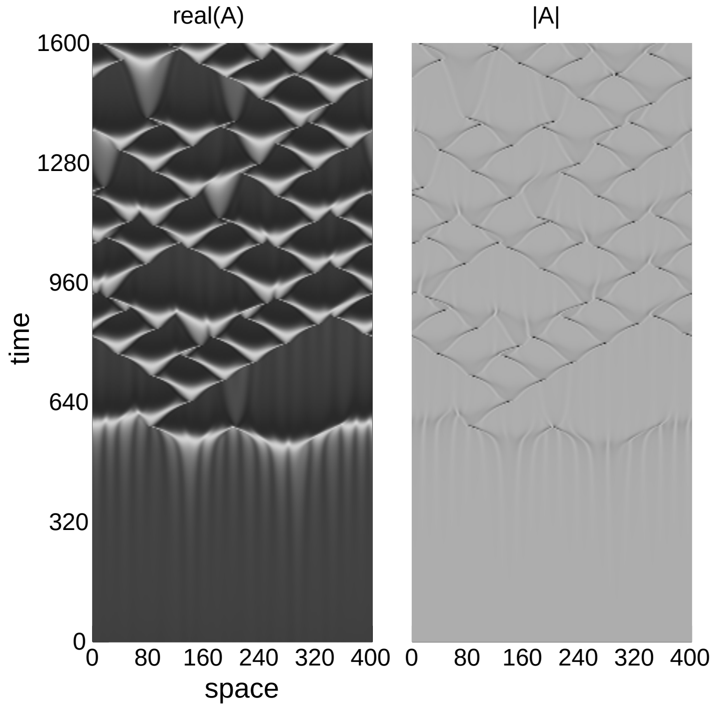


*Figure 16.* Kink-breeding process in the phase turbulence for $\alpha = -0.75$, $\beta = 2$, $\nu = -0.8$, and $B = 0.041$. Initial conditions are small fluctuations around the unstable synchronized plane wave. Grey scale space-time plots of $Re(A)$ and $|A|$ reveal the occurrence of defects (black spots in the right panel) corresponding to zeros of $|A|$.

ously obtained with the help of the analytic signal concept based on the Hilbert transform (for an introduction see [44, 50]). It goes as follows: for an arbitrary scalar signal $s(t)$ one can construct a complex function of time (analytic signal) $\zeta(t) = s(t) + i\tilde{s}(t) = A(t)e^{i\phi_H(t)}$ where $\tilde{s}(t)$ is the Hilbert transform of $s(t)$,

$$\tilde{s}(t) = \pi^{-1}\text{P.V.} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau \ , \tag{25}$$

and $A(t)$ and $\phi_H(t)$ are the instantaneous amplitude and phase (P.V. means that the integral is taken in the sense of the Cauchy principal value). Alternatively, one can use a wavelet transform with a complex Gabor wavelet, see [54].

As has been recently shown in [56, 50], the phase estimated by this method from an appropriately chosen oscillatory observable practically coincides with the phase of an oscillator computed according to one of the definitions given in Sec. 3. Therefore, the analysis of the relationship between these Hilbert phases appears to be an appropriate tool to detect synchronous epochs from experimental data and to check for a weak interaction between systems under study. It is very important that the Hilbert transform does not require stationarity of the data, so we can trace synchronization transitions even from nonstationary data.

We recall again the above mentioned similarity of phase dynamics in noisy and chaotic oscillators (see Sect. 3.3). A very important consequence of this fact is that, using the synchronization approach to data analysis, we can avoid the hardly solvable dilemma "noise vs chaos": irrespectively of the origin of the observed signals, the approach and techniques of the analysis are unique. Quantification of synchronization from noisy data is considered in [67].

Application of these ideas allowed us to find phase locking in the data characterizing mechanisms of posture control in humans while quiet standing [60, 57]. Namely, the small deviations of the body center of gravity in anterior–posterior and lateral directions were analyzed. In healthy subjects, the regulation of posture in these two directions can be considered as independent processes, and the occurrence of some interrelation possibly indicates a pathology. It is noteworthy that in several records conventional methods of time series analysis, i.e. the cross–spectrum analysis and the generalized mutual information failed to detect any significant dependence between the signals, whereas calculation of the instantaneous phases clearly showed phase locking.

Complex synchronous patterns have been found recently in the analysis of interaction of human cardiovascular and respiratory systems [62]. This finding possibly indicates the existence of a previously unknown type of neural coupling between these systems.

Analysis of synchronization between brain and muscle activity of a Parkinsonian patient [67] is relevant for a fundamental problem of neuroscience: can one consider the synchronization between different areas of the motor cortex as a necessary condition for establishing of the coordinated muscle activity? It was shown [67] that the temporal evolution of the coordinated pathologic tremor activity directly reflects the evolution of the strength of synchronization within a neural network involving cortical motor areas. Additionally, the brain areas with the tremor-related activity were localized from noninvasive measurements.

## 9. Conclusions

The main idea of this paper is to demonstrate that synchronization phenomena in periodic, noisy and chaotic oscillators can be understood within a unified framework. This is achieved by extending the notion of phase to the case of continuous-time chaotic systems. Because the phase is introduced as a variable corresponding to the zero Lyapunov exponent, this notion should be applicable to any autonomous chaotic oscillator. Although we are not able to propose a unique and rigorous approach to determine the phase, we have shown that it can be introduced in a reasonable and consistent way for basic models of chaotic dynamics. Moreover, we have shown that even in the case when the phases are not well-defined, i.e. they cannot be unambiguously computed explicitly, the presence of phase synchronization can be demonstrated indirectly by observations of the mean field and the spectrum, i.e. independently of any particular definition of the phase.

In a rather general framework, any type of synchronization can be considered as appearance of some additional order inside the dynamics. For chaotic systems, e.g., the complete synchronization means that the dynamics in the phase space is restricted to a symmetrical submanifold. Thus, from the point of view of topological properties of chaos, the synchronization transition usually means the simplification of the structure of the strange attractor. In discussing the topological properties of phase synchronization, we have shown that the transition to phase synchronization corresponds to splitting of the complex invariant chaotic set into distinctive attractor and repeller. Analogously to the complete synchronization, which appears through the pitchfork bifurcation of the strange attractor, one can say that the phase synchronization appears through tangent bifurcation of strange sets.

Because of the similarity in the phase dynamics, one may expect that many, if not all, synchronization features known for periodic oscillators can be observed for chaotic systems as well. Indeed, here we have described effects of phase and frequency entrainment by periodic external driving, both for simple and space-distributed chaotic systems. Further, we have described synchronization due to interaction of two chaotic oscillators, as well as self-synchronization in globally coupled large ensembles.

As an application of the developed framework we have discussed a problem in data analysis, namely detection of weak interaction between systems from bivariate data. The three described examples of the analysis of physiological data demonstrate a possibility to detect and characterize synchronization even from nonstationary and noisy data.

Finally, we would like to stress that contrary to other types of chaotic synchronization, the phase synchronization phenomena can happen already for very weak coupling, which offers an easy way of chaos regulation.

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