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Synchronization: A General Phenomenon in an Oscillatory World

Arkady PIKOVSKY and Michael ROSENBLUM (Potsdam)

With 4 Figures

Abstract

We present a general introduction to synchronization phenomena in nonlinear systems. The notion of self-sustained oscillators is introduced, and effects of phase locking and frequency entrainment are described and illustrated with examples. Different types of synchrony in chaotic systems are also outlined.

Zusammenfassung

In diesem Artikel werden Synchronisationsphänomene in nichtlinearen Systemen diskutiert und zusammengefaßt. Die periodischen selbsterregten Schwingungen lassen sich durch die Phasendynamik charakterisieren. Durch eine Wechselwirkung bzw. äußere Kraft kann die Phase eingefangen werden, was zum Synchronisationszustand führt. Für chaotische Systeme läßt sich die Phase nicht eindeutig definieren, trotzdem ist es möglich, phasensynchronisierte Zustände experimentell zu beobachten. Die vollständige Synchronisation chaotischer Systeme wird auch beschrieben.

1. Introduction

Many natural and human-made nonlinear oscillators exhibit the ability to adjust their rhythms due to weak interaction: two lasers, being coupled, start to generate with a common frequency; cardiac pacemaker cells fire simultaneously; violinists in an orchestra play in unison. Such coordination of rhythms is a manifestation of a fundamental nonlinear phenomenon – *synchronization.* Discovered in 17th century by Christiaan HUYGENS, it was observed in physics, chemistry, biology, and even social behavior, as well as found practical applications in engineering and medicine. The notion of synchronization has been recently extended to cover the adjustment of rhythms in chaotic systems, large ensembles of oscillating units, rotating objects, continuous media, etc. In spite of essential progress in theoretical and experimental studies, synchronization remains a challenging problem of nonlinear sciences (for details and further references see PIKOVSKY et al. 2000, 2001, MOSEKILDE et al. 2002).

It is important to emphasize that synchronization is an essentially nonlinear effect. In contrast to many classical physical problems, where consideration of nonlinearity gives a correction to a linear theory, here the account of nonlinearity is crucial: the phenomenon occurs only in the so-called *self-sustained* systems.

2. Self-Sustained Oscillators

Self-sustained oscillators are models of natural oscillating objects, and these models are essentially nonlinear. Mathematically, such an oscillator is described by an autonomous (i. e., without explicit time dependence) nonlinear dynamical system. It differs both from linear oscillators (which, if a damping is present, can oscillate only due to external forcing) and from nonlinear energy conserving systems, whose dynamics essentially depends on initial state.

Dynamics of oscillators is typically described in the phase (state) space. Quite often two state variables suffice to determine unambiguously the state of the system, and we proceed here with this simplest case. As the oscillation is periodic, i. e., it repeats itself after the period T, $x(t)$ corresponds to a closed curve in the phase plane, called the *limit cycle*. The reason why we distinguish this curve from all others trajectories in the phase space is thus that it attracts phase trajectories and is therefore called an attractor of the dynamical system. The limit cycle is a simple attractor, in contrast to a *strange (chaotic) attractor*. The latter is a geometrical image of *chaotic* self-sustained oscillations.

Examples of self-sustained oscillatory systems are electronic circuits used for the generation of radio-frequency power, lasers, Belousov-Zhabotinsky and other oscillatory chemical reactions, pacemakers (sino-atrial nodes) of human hearts or artificial pacemakers that are used in cardiac pathologies, and many other natural and artificial systems. An outstanding common feature of such systems is their ability to be synchronized.

This ability of periodic self-sustained oscillators is based on the existence of a special variable, phase ϕ. Mathematically, ϕ can be introduced as the variable parameterizing the motion along the stable limit cycle in the state space of an autonomous continuous-time dynamical system. One can always choose the phase proportional to the fraction of the period, i. e., in a way that it grows uniformly in time,

$$\frac{d\phi}{dt} = \omega_0 \qquad [1]$$

where ω_0 is the natural frequency of oscillations. The phase is neutrally stable: its perturbations neither grow nor decay. (In terms of nonlinear dynamics neutral stability means that the phase is a variable that corresponds to the zero Lyapunov exponent of the dynamical system.) Thus, already an infinitely small perturbation (e. g. external periodic forcing or coupling to another system) can cause large deviations of the phase contrary to the amplitude, which is only slightly perturbed due to the transversal stability of the cycle. The main consequence of this fact is that *the phase can be very easily adjusted by an external action, and as a result the oscillator can be synchronized!*

3. Entrainment by External Force

We begin our discussion of synchronization phenomena by considering the simplest case, entrainment of a self-sustained oscillator by external periodic force. Before we describe this effect in mathematical terms, we illustrate it by an example. We will speak about biological clocks that regulate daily and seasonal rhythms of living systems – from bacteria to humans.

3.1 An Example: Circadian Rhythms

In 1729 Jean-Jacques Dortous de Mairan, the French astronomer and mathematician, who was later the Secretary of the Académie Royale des Sciences in Paris, reported on his experiments with a haricot bean. He noticed that the leaves of this plant moved up and down in accordance with the change of day into night. Having made this observation, de Mairan put the plant in a dark room and found that the motion of the leaves continued even without variations in the illuminance of the environment. Since that time these and much more complicated experiments have been replicated in different laboratories, and now it is well-known that all biological systems, from rather simple to highly organized ones, have internal biological clocks that provide their "owners" with information on the change between day and night. The origin of these clocks is still a challenging problem. But it is well established that they can adjust their circadian rhythms (from *circa* = about and *dies* = day) to external signals: if the system is completely isolated from the environment and is kept under controlled constant conditions (constant illuminance, temperature, pressure, parameters of electromagnetic fields, etc.), its internal cycle can essentially differ from a 24-hour cycle. Under natural conditions, biological clocks tune their rhythms in accordance with the 24-hour period of the Earth's daily cycle.

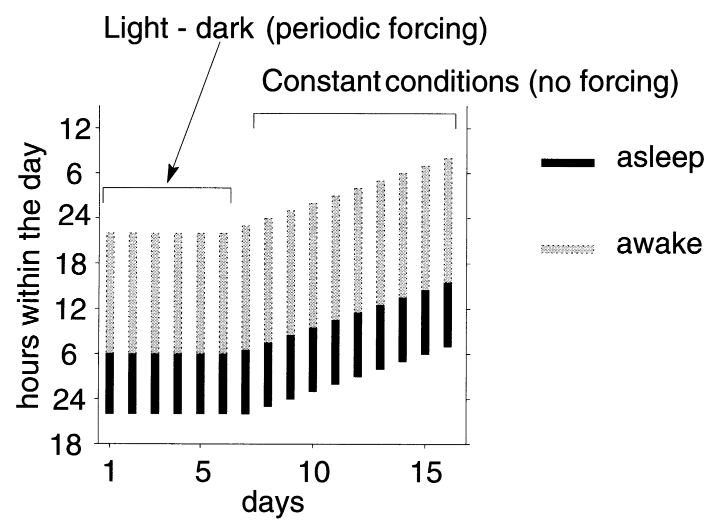

Fig. 1 Schematic diagram of the behavioral sleep-wake rhythm. This cycle (termed circadian rhythm) represents the fundamental adaptation of organisms to an environmental stimulus, the daily cycle of light and dark. Here the circadian rhythm is shown entrained for six days by the environmental light-dark cycle and autonomous for the rest of the experiment when the subject is placed under constant light conditions. The intrinsic period of the circadian oscillator is in this particular case larger than 24 hours. Correspondingly, the phase difference between the sleep-wake cycle and daily cycle increases: the internal "day" begins later and later. Such plots are typically observed in experiments with both, animals and humans (see, e. g., Aschoff et al. 1982, Czeisler et al. 1986, Moore 1999).

Experiments show that for most people the internal period of biological clocks differs from 24 h, but it is entrained by environmental signals, e. g. illuminance, having the period of the Earth's rotation (Fig. 1); see also the discussion of circadian oscillations of specific biological systems in the contributions by M. MITTAG and U. RASCHER, both in this volume. Obviously, the action here is unidirectional: the revolution of a planet cannot be influenced by mankind (yet); thus, this case constitutes an example of synchronization by an external force. In usual circumstances this force is strong enough to ensure perfect entrainment; in order to desynchronize a biological clock one can either travel to polar regions or go caving. It is interesting that although normally the period of one's activity is exactly locked to that of the Earth's rotation, the phase shift between the internal clock and the external force varies from person to person: some people say that they are "early birds" whereas others call themselves "owls".

Perturbation of the phase shift strongly violates normal activity. Every day many people perform such an experiment by rapidly changing their longitude (e. g. crossing the Atlantic) and experiencing jet lag. It can take up to several days to reestablish a proper phase relation to the force; in the language of nonlinear dynamics one can speak of different lengths of transients leading to the stable synchronous state. As other commonly known examples of synchronization by external force we mention radio-controlled clocks and cardiac pacemakers.

3.2 Phase Dynamics of a Forced Oscillator

For mathematical treatment of synchronization, we recall that the phase of an oscillator is neutrally stable and can be adjusted by a small action, whereas the amplitude is stable. This property allows description of the effect of small forcing/coupling within the framework of the phase approximation. Considering the simplest case of a limit cycle oscillator, driven by a periodic force with frequency ω and amplitude ε, we can write the equation for the perturbed phase dynamics in the form

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon Q(\varphi, \omega t), \quad [2]$$

where the coupling function Q depends on the form of the limit cycle and of the forcing. As the states with the phases φ_0 and $\varphi_0 + 2\pi$ are physically equivalent, the function Q is 2π-periodic in its both arguments, and therefore can be represented as a double Fourier series. If the frequency of the external force is close to the natural frequency of the oscillator, $\omega \approx \omega_o$, then the series contains fast oscillating and slow varying terms, the latter can be written as $q(\phi - \omega t)$. Introducing the difference between the phases of the oscillation and of the forcing $\psi = \phi - \omega t$ and performing an averaging over the oscillation period we get rid of the oscillating terms and obtain the following basic equation for the phase dynamics:

$$\frac{d\psi}{dt} = -(\omega - \omega_0) + \varepsilon q(\psi). \quad [3]$$

Function q is 2π-periodic, and in the simplest case, $q(\cdot) = \sin(\cdot)$, Equation [3] *is* called the Adler equation. One can easily see that on the plane of the parameters of the external forcing (ω, ε) there exists a region $\varepsilon q_{\min} < \omega - \omega_0 < \varepsilon q_{\max}$, where Equation [3] has a stable stationary solution. This solution corresponds to the conditions of phase locking (the phase ϕ just follows the phase of the force, he. $\phi = \omega t + c$onstant) and frequency entrainment (the observed frequency of the oscillator $\Omega = \langle \varphi \rangle$ exactly coincides with the forcing frequency ω; brackets $\langle\ \rangle$ denote time averaging).

Generally, synchronization is observed for high-order resonances $n\omega \approx m\omega_0$ as well. In this case the dynamics of the generalized phase difference $\psi = m\varphi - n\omega t$, where n, m are integers, is described by an equation similar to Equation [3], namely by $d(\psi)/dt = -(n\omega - m\omega_0) + \varepsilon\tilde{q}(\psi)$. Synchronous regime then means perfect entrainment of the oscillator frequency at the rational multiple of the forcing frequency, $\Omega = \frac{n}{m}\omega$, as well as phase locking $m\varphi = n\omega t + \text{constant}$. The overall picture can be shown on the (ω, ε) plane: there exist a family of triangular-shaped synchronization regions touching the ω-axis at the rationals of the natural frequency $\frac{m}{n}\omega_0$, these regions are usually called Arnold tongues (Fig. 2*A*). This picture is preserved for moderate forcing, although now the shape of the tongues generally differs from being exactly triangular. For a fixed amplitude of the forcing ε and varied driving frequency ω one observes different phase

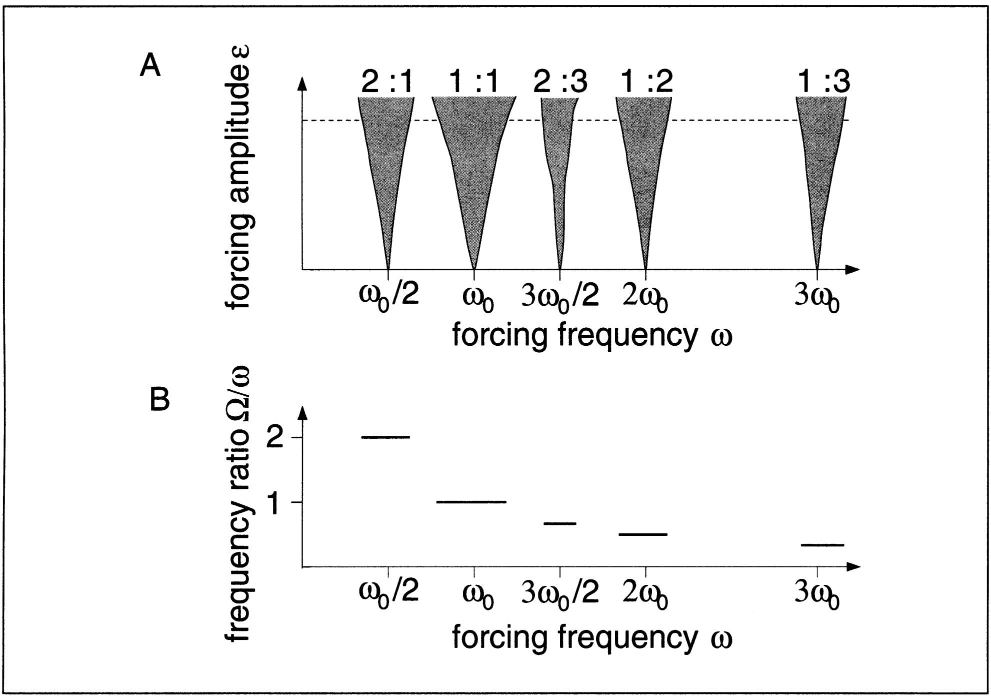

Fig. 2 (*A*) Family of synchronization regions, or Arnold tongues (schematically). The numbers on top of each tongue indicate the order of locking; e. g., 2 : 3 means that the relation $2\omega = 3\Omega$ is fulfilled. (*B*) The Ω/ω *versus* ω plot for a fixed amplitude of the force (shown by the dashed line in (*A*)) has a characteristic shape, known as the *devil's staircase.* (In this scheme the variation of the frequency ratio between the main plateaus of the staircase is not shown.)

locking intervals where the motion is periodic, whereas in between them it is quasiperiodic. The curve Ω *versus* ω thus consists of horizontal plateaus at all possible rational frequency ratios; this fractal curve is called devil's staircase (Fig. 2*B*). A famous example of such a curve is the voltage-current plot for a Josephson junction in an ac electromagnetic field; in this context synchronization plateaus are called Shapiro steps (SHAPIRO 1963). Note that a junction can be considered as a rotator (rotations are maintained by a dc current); this example demonstrates that synchronization properties of rotators are very close to those of oscillators (PIKOVSKY et al. 2001).

Finally, we note that the phase difference in the synchronous state is not necessarily constant, but may oscillate around a constant value. Indeed, a solution $m\varphi - n\omega t =$ constant was obtained from Equation [2] by means of averaging, i. e. by neglecting the fast oscillating terms. If we take this terms into account, then we have to reformulate the condition of phase locking as $|m\phi - n\omega t| <$ constant. Thus, in the synchronous regime the phase difference is bounded, otherwise it grows infinitely.

3.3 Synchronization versus Resonance

At this point we would like to underline the difference to another phenomenon, well-known for oscillatory systems – the resonance. Resonance is a response of a system that is non-active, i. e., demonstrates no oscillations without external driving. In other words, here one cannot speak of an adjustment of intrinsic oscillations to an external force, as this force is the source of the oscillations. In the case of resonance, if the force is switched off, the oscillations disappear, while self-sustained oscillations continue to exist even without forcing.

As a simple example of this difference let us consider *radio-controlled clocks* and *older railway station clocks*. Radio-controlled clocks are self-oscillating, they continue to show time even if there is no radio signal from the high-precision center. The role of the latter is only to adjust – to correct – the oscillations in order to synchronize them with the time standard. The railway station clocks receive signals from a central clock, and if these signals are absent they stop; this is an example of resonance, not of synchronization.

Sometimes, when a system is forced very strongly and operates in a highly nonlinear regime, it is hard to distinguish between synchronization and resonance (especially if one can hardly control the forcing like for circadian rhythms), here the observed features at the resonance may be very close to those at the synchronization (e. g., one can observe the devil's staircase-like dependence on the forcing frequency). Nevertheless, the difference becomes evident if the forcing is reduced or switched off.

4. Two and More Oscillators

4.1 Phase Dynamics of Two Coupled Oscillators

Synchronization of two coupled self-sustained oscillators can be described in a similar way. A weak interaction affects only the phases of two oscillators ϕ_1 and ϕ_2, and Equa-

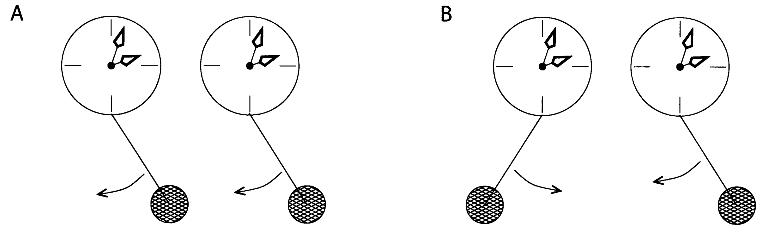

Fig. 3 Two coupled oscillators may be synchronized almost in-phase, i. e., with the phase difference $\phi_2 - \phi_1 \approx 0$ (*A*), or in anti-phase, when $\phi_2 - \phi_1 \approx \pi(B)$, in dependence on how the coupling was introduced. The discoverer of synchronization, Christiaan Huygens, observed synchronization in anti-phase. Later experiments, reported in Blekhman (1981) demonstrated that both anti-phase and in-phase synchronous regimes are possible.

tion [1] generalizes to

$$\frac{d\varphi_1}{dt} = \omega_1 + \varepsilon Q_1(\varphi_1, \varphi_2) \qquad \frac{d\varphi_2}{dt} = \omega_2 + \varepsilon Q_2(\varphi_2, \varphi_1). \qquad [4]$$

For the phase difference $\psi = \phi_2 - \phi_1$ one obtains after averaging an equation of the type of Equation [3]. Synchronization now means that two non-identical oscillators start to oscillate with the same frequency (or, more generally, with rationally related frequencies). This common frequency usually lies between ω_1 and ω_2. It is worth mentioning that locking of the phases and frequencies implies no restrictions on the amplitudes, in fact the synchronized oscillators may have very different amplitudes and waveforms (e. g., oscillations may be relaxation [pulse-like] or quasiharmonic).

We conclude the discussion of mutual synchronization of two coupled systems with two remarks. (*i*) Similar to the case of periodic forcing, synchronization of the order $n:m$ is also possible. Examples are synchronization of running and breathing in mammals and locking of breathing and wing beat frequencies in flying birds (see Pikovsky et al. 2001 for citations and further examples). (*ii*) Depending on the parameters of coupling two oscillators can be locked almost in-phase or almost in anti-phase (Fig. 3). Moreover, varying the parameters of coupling one can observe transition between different synchronous states. As an example we mention the effect observed by J. A. S. Kelso and later studied by Haken, Kelso and co-workers (see Haken et al. 1985, Haken 1999 for references and details). In their experiments, a subject was instructed to perform an anti-phase oscillatory movement of index fingers and gradually increase the frequency. It turned out that at higher frequency this movement becomes unstable and a rapid transition to the in-phase mode is observed.

4.2 Globally Coupled Oscillators

Now we study synchronization phenomena in large ensembles of oscillators, where each element interacts with all others. This is usually denoted as *global,* or all-to-all coupling.

As a representative example we mention synchronous flashing in a population of fireflies. A very similar phenomenon, self-organization in a large applauding audience, has probably been experienced by every reader of this article, e. g. in a theater. Indeed, if the audience is large enough, then one can often hear a rather fast (several oscillatory periods) transition from noise to a rhythmic, nearly periodic, applause. This happens when the majority of the public applaud in unison, or synchronously.

The each-to-each interaction is also denoted as *mean field* coupling. Indeed, each firefly is influenced by the light field that is created by the whole population. Similarly, each applauding person hears the sound that is produced by all other people in the hall. Thus, we can say that each element is exposed to a force that comes to it from all others by one input from the whole ensemble. Let us denote these outputs of elements by $x_k(t)$, where $k = 1, \ldots, N$ is the index of an oscillator, and N is the number of elements in the ensemble; x can be variation of light intensity or of the acoustic field around some average value, or, generally, any other oscillating quantity. Then the force that drives each oscillator is proportional to $\sum_k x_k(t)$. It is conventional to write this proportionality as $\varepsilon N^1 \sum_k x_k(t)$, so that it includes the normalization by the number of oscillators N. The term $N^{-1}\sum_k x_k(t)$ is just an arithmetic mean of all oscillations, what explains the origin of the term "mean field coupling".

Thus, the oscillators in a globally coupled ensemble are driven by a common force. Clearly, this force can entrain many oscillators if their frequencies are close. The problem is that this force (the mean field) is not predetermined, but arises from interaction within the ensemble. This force determines whether the systems synchronize, but it itself depends on their oscillation – it is a typical example of self-organization (HAKEN 1993). To explain qualitatively the appearance of this force (or to compute it, as is done in KURAMOTO 1984, PIKOVSKY et al. 2001) one should consider this problem self-consistently.

First, assume for the moment that the mean field is zero. Then all the elements in the population oscillate independently, and their contributions to the mean field nearly cancel each other. Even if the frequencies of these oscillations are identical, but their phases are independent, the average of the outputs of all elements of the ensemble is small if compared with the amplitude of a single oscillator. (According to the law of large numbers, it tends to zero when the number of interacting oscillators tends to infinity; the fluctuations of the mean field are of the order $N^{-1/2}$.) Thus, the asynchronous, zero mean field state obeys the self-consistency condition.

Next, to demonstrate that synchronization in the population is also possible, we suppose that the mean field is non-vanishing. Then, naturally, it entrains at least some part of the population, the outputs of these entrained elements sum up coherently, and the mean field is indeed nonzero, as assumed. Which of these two states – synchronous or asynchronous – is realized, or, in other words, which one is stable, depends on the strength of interaction between each pair and on how different the elements are. The interplay between these two factors, the coupling strength and the distribution of the natural frequencies, also determines how many oscillators are synchronized, and, hence, how strong the mean field is.

We discuss now how the synchronization transition occurs, taking the applause in an audience as an example (experimental study of synchronous clapping is reported in NÉDA et al. 2000). Initially, each person claps with an individual frequency, and the sound they all produce is noisy.[1] As long as this sound is weak, and contains no characteristic frequency, it does not essentially affect the ensemble. Each oscillator has its own frequency ω_k, each person applauds and each firefly flashes with its individual rate, but there always exists some value of it that is preferred by the majority. Definitely, some elements behave in a very individualistic manner, but the main part of the population tends to be "like the neighbor". So, the frequencies ω_k are distributed over some range, and this distribution has a maximum around the most probable frequency. Therefore, there are always at least two oscillators that have very close frequencies and, hence, easily synchronize. As a result, the contribution to the mean field at the frequency of these synchronous oscillations increases. This increased component of the driving force naturally entrains other elements that have close frequencies, this leads to the growth of the synchronized cluster and to a further increase of the component of the mean field at a certain frequency. This process develops (quickly for relaxation oscillators, relatively slow for quasilinear ones), and eventually almost all elements join the majority and oscillate in synchrony, and their common output – the mean field – is not noisy any more, but rhythmic. Of course, the synchronization process as described above is not unconscious but depends on the intentions of the audience: in some countries a non-synchronous clapping is preferred, then even a lasting clapping does not synchronize.

The physical mechanism we described is known as the Kuramoto self-synchronization transition (Kuramoto 1975). The scenario of this transition does not depend on the origin of the oscillators (biological, electronic, etc.) or on the origin of interaction. In the above presented examples the coupling occurred *via* an optical or acoustic field. Global coupling of electronic systems can be implemented *via* a common load; in this case the voltage applied to individual systems depends on the sum of the currents of all elements. (As an example we mention an array of Josephson junctions.) Chemical oscillators can be coupled *via* a common medium, where concentration of a reagent depends on the reaction in each oscillator and, on the other hand, influences these reactions. The Kuramoto transition can be treated as a non-equilibrium phase transition, the mean oscillating field serving as an order parameter.

The scenarios of the Kuramoto transition may be also more complicated, e. g., if the distribution of the individual frequencies ω_k has several maxima. Then several synchronous clusters can be formed – they can eventually merge or coexist. Clustering can also happen if, say, the strength of interaction of an element of the population with its nearest (in space) neighbors is larger than with those that are far away.

5. Chaotic Systems

Nowadays it is well-known that self-sustained oscillators, e. g., nonlinear electronic devices, can generate rather complex, *chaotic* signals. Most oscillating natural systems also

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1 Naturally, the common (mean) acoustic field is nonzero, because each individual oscillation is always positive; the intensity of the sound cannot be negative, it oscillates between zero and some maximal value. Correspondingly, the sum of these oscillations contains some rather large constant component, and it is the deviation from this constant that we consider as the oscillation of the mean field and that is small. Therefore, the applause is perceived as some noise of almost constant intensity.

exhibit rather complex behavior. Recent studies have revealed that such systems, being coupled, are also capable to undergo synchronization. Certainly, in this case we have to specify this notion more precisely, because it is not obvious, how to characterize the rhythm of a chaotic oscillator. It is helpful that sometimes chaotic waveforms are rather simple, so that a signal is "almost periodic"; we can consider it as consisting of similar cycles with varying amplitude and period (which can be roughly defined as the time interval between the adjacent maxima). Taking a large time interval N_τ we can count the number of cycles within this interval N, compute the *mean frequency* and take it for characterization of the chaotic oscillatory process.

$$\omega_0 = \lim_{\tau \to \infty} 2\pi \frac{N_\tau}{\tau}. \qquad [5]$$

With the help of the mean frequencies we can describe the collective behavior of interacting chaotic systems in the same way as we did it for periodic oscillators. If the coupling is large enough (e. g., in the case of resistively coupled electronic circuits it means that the resistor should be sufficiently small), the mean frequencies of two oscillators become equal, and one can obtain the synchronization region, exactly as in the case of periodic systems. It is important that coincidence of mean frequencies does not imply that the signals coincide as well. It turns out that weak coupling does not affect the chaotic nature of both oscillators; the amplitudes remain irregular and uncorrelated; whereas the frequencies are adjusted in a fashion that allows us to speak of the phase shift between the signals. This regime is denoted as *phase synchronization* of chaotic systems.

Very strong coupling tends to make the states of both oscillators identical. It influences not only the mean frequencies but also the chaotic amplitudes. As a result, the signals coincide (or nearly coincide) and the regime of *complete synchronization* sets in. Known are also the so-called generalized and master slave synchronization (see, e. g. PIKOVSKY et al. 2001 and references therein); these effects are related to the complete synchronization of chaos.

5.1 Phase Synchronization. An Example: Electrochemical System

Phase synchronization of chaotic systems is mostly close to the classical locking phenomena. It is based on the observation that many chaotic self-sustained oscillators admit determination of the instantaneous phase and the corresponding mean frequency. Below we illustrate this with an electrochemical oscillator, experimentally investigated by KISS and HUDSON (KISS et al. 2001). An autonomous system demonstrates chaotic dynamics. The strange attractor looks like a smeared limit cycle; this allows one to introduce the phase as a variable that gains 2π with each rotation of the phase space trajectory, and to calculate frequency according to Equation [5].

Having introduced the phase and the frequency for chaotic oscillators we can characterize their synchronization. Now it becomes rather obvious that the effects of phase locking and frequency entrainment, known for periodic self-sustained oscillators, can be observed for chaotic systems as well. The simplest ease is the phase locking by an external periodic signal. When the electrochemical oscillator is driven by a signal with a frequency Ω close to ω_0, the forcing affects the evolution of the phase, and the ob-

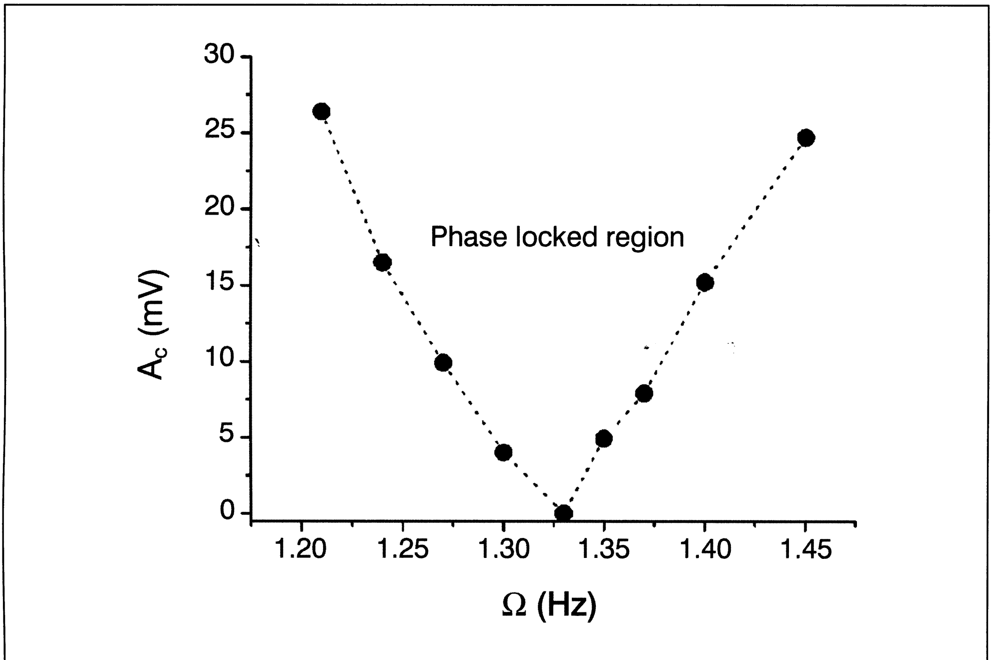

Fig. 4 The synchronization region for the periodically driven chaotic oscillator on the plane "amplitude of the driving – frequency of the driving" (Kiss et al. 2001).

served (mean) frequency ω becomes adjusted to the external one. The results of the experiment for different amplitudes of the forcing allow one to define the synchronization region, where the frequency of the system is completely entrained by the external force, see Figure 4. This region is a complete analog of synchronization regions (Arnold tongues) for periodic oscillators.

It is important to emphasize that the chaos itself is not suppressed by the external force. What happens is not a disappearance of chaos, but an adjustment of the mean oscillation frequency. Chaos may be destroyed by a strong force, but a small forcing affects only the phase, entraining the frequency of its rotation.

Mutual phase synchronization of chaotic oscillators is also quite similar to the classical case. To demonstrate this, one can couple two chaotic electrochemical oscillators. Then the calculation of the observed frequencies Ω_1, Ω_2 characterizes the entrainment. For large enough coupling and for small mismatch of natural frequencies one observes that frequencies becomes equal, $\Omega_1 = \Omega_2$, like in the experiments with periodic oscillators.

5.2 Complete Synchronization

Strong mutual coupling of chaotic oscillators leads to their *Complete Synchronization* when two or more chaotic systems have exactly the same states, and these identical states vary irregularly in time. Contrary to phase synchronization, it can be observed in any chaotic system, not necessarily autonomous, in particular in periodically driven oscillators or in discrete time systems (maps). In fact, this phenomenon is not close to the classical synchronization of periodic oscillations, as here we do not have adjustment of rhythms. Instead, complete synchronization means suppression of differences in coupled *identical* systems. Therefore, this effect cannot be described as entrainment or locking; it is closer to the onset of symmetry. Maybe another word instead of "synchronization" would better serve for underlining this difference; we will follow the nowadays accepted terminology, using the adjective "complete" to avoid ambiguity.

The main precondition for complete synchronization is that the interacting systems are identical, i. e., they are described by exactly the same equations of motion. This identity implies that if the initial states of these systems are equal, then during the evolution they remain equal at all times. However, in practice this coincidence of states will be realized only if such regime is stable, i. e., if it is restored after a small violation. This imposes a condition on the strength of the coupling between the systems.

To be more concrete in our discussion, let us consider the coupled system of the type

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}) + \varepsilon(\vec{y} - \vec{x}) \qquad \frac{d\vec{y}}{dt} = \vec{F}(\vec{y}) + \varepsilon(\vec{x} - \vec{y}). \qquad [6]$$

Here $\vec{x}$ and $\vec{y}$ are two identical systems, described by the same equations $\vec{F}$, and we will assume that the solutions are chaotic. ε is the coupling parameter, the corresponding terms on the right hand sides describe a so-called diffusive coupling, which tends to equalize the states of two systems (this can be easily seen if one sets $\vec{F} = 0$, then the difference $\vec{y} - \vec{x}$ decreases in time with the rate 2ε).

While the coupling tends to equalize the states of two systems, another mechanism prevents this. This mechanism is the sensitive dependence to initial conditions, inherent for chaos. Suppose that $\varepsilon = 0$, then we have two uncoupled identical systems; they can be regarded as two realizations of one system with different initial conditions. Because chaotic motions sensitively depend on initial conditions (this phenomenon is often called the "Butterfly Effect"), the values $\vec{y}(t)$ and $\vec{x}(t)$ will differ significantly after some time, even if $\vec{y}(0) \approx \vec{x}(0)$.

Summarizing, we see two counter playing tendencies in the diffusive interaction of two identical chaotic systems: intrinsic chaotic instability tends to make the states of the systems different, while coupling tends to equalize them. As a result, there exists a critical value of coupling such that for stronger coupling a completely synchronized state $\vec{y}(t) = \vec{x}(t)$ sets on. At this regime the coupling term in [6] vanishes, and, hence, each of the systems varies chaotically in time as if they were uncoupled. Thus, the complete synchronization is a threshold phenomenon: it occurs only when the coupling exceeds some critical level, proportional to the largest Lyapunov exponent of the individual system. Below the threshold, the states of two chaotic systems are different but close to each other.

6. Conclusions and Outlook

In spite of the long history, theory of synchronization remains a rapidly developing branch of nonlinear science. Among the ongoing directions, not discussed in this article, we mention synchronization in spatially-distributed systems and synchronization-like phenomena in stochastic and excitable systems (ANISHCHENKO et al. 2002).

Recent theoretical development was strongly influenced by interdisciplinary studies, especially by widely growing applications to biological and medical problems. It turned out that synchronization is very frequently encountered in live systems (GLASS 2001, PIKOVSKY et al. 2001). In particular, it is believed that the mechanism of the Kuramoto transition plays an important role in dynamics of neural ensembles and is responsible for the emergence of such severe pathologies as epilepsies and Parkinson disease. A popular paradigmatic model, analyzed in this context, is a system of pulse-coupled integrate-and-fire oscillators (see e. g. MIROLLO and STROGATZ 1990). Another direction of research is related to attempts to desynchronize undesirable, pathological collective rhythms and to develop in this way a therapeutic tool (TASS 1999).

Finally we mention that ideas from the synchronization theory are used in analysis of multivariate experimental data. The goal of such an analysis is to detect weak interaction between oscillatory systems, e. g. to reveal a coordination between respiratory and cardiac rhythms in humans (SCHÄFER et al. 1998) or localize the source of pathological brain activity in Parkinson disease (TASS et al. 1998, 2003).

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Prof. Dr. Arkady PIKOVSKY
Dr. Michael ROSENBLUM
Department of Physics
University of Potsdam,
Am Neuen Palais 10
14415 Potsdam
Germany
URL:www.stat.physik.uni-potsdam.de/-pikovsky,
www.agnld.uni-potsdam.de/-mros
Phone: ++49 (0) 33 19 77 14 72
Fax: ++49 (0) 33 19 77 17 67
E-Mail: pikovsky@stat.physik.uni-potsdam.de
mros@agnld.uni-potsdam.de