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Coherence of noisy oscillators with delayed feedback

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Abstract

A theory of effect of delayed feedback on the coherence of the noisy self-sustained oscillations is developed. In the Gaussian approximation a closed system of equations is derived for the phase diffusion constant and the mean frequency. A comparison with numerics shows that the theory works well for weak feedback and strong noise.

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Coherence is one of the main characteristics of self-oscillating systems. For periodic oscillators it determines their quality as clocks, and usually the improvement of the coherence is one of the major goals in the construction of such oscillators. In terms of the phase dynamics, the coherence is quantitatively measured by the phase diffusion constant, it is proportional to the width of the spectral peak of oscillations. Many chaotic oscillators can be also represented via the phase dynamics, thus allowing one to characterize their coherence by virtue of the phase diffusion constant as well [1].

In this paper we study how the coherence of oscillations is influenced by external delayed feedback. Applying a delayed feedback is widely use to control different properties of the dynamical systems: to make chaotic oscillators operate periodically (Pyragas' control method [2]) and to suppress space–time chaos [3,4]. In our consideration below we are not considering such delay-induced qualitative changes in the

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dynamics, but focus on the quantitative effect of a delayed feedback on the phase diffusion properties.

Investigation of effects of irregularities and noise in systems with delay is not an easy task, because one cannot use here well-established methods (like the Fokker–Planck equation) valid for Markov processes. In the case of delay the process is non-Markov and ad hoc statistical methods should be developed. This has been accomplished recently for bistable oscillators [5], see also Refs. [6–8]. Below we present a theory describing the effect of a delayed feedback on noisy self-sustained oscillations. It is based on the phase approximation to the dynamics, which means that the noise and feedback are assumed to be small.

The basic model we study in this paper is the equation describing the dynamics of the phase under influence of noise and delay:

$$\dot{\phi} = \Omega_0 + \xi(t) + a\sin(\phi(t-\tau) - \phi(t)) , \tag{1}$$

where the noisy term $\xi(t)$ is assumed to be Gaussian. Eq. (1) has been used in Ref. [8] to describe evolution of the phase of an optical field in a laser with weak optical feedback. Of our main interest are the diffusion properties of the phase $\phi(t)$. Under influence of the noise, in the absence of feedback, it diffuses $\langle(\phi(t) - \langle\phi(t)\rangle)^2\rangle \propto D_0 t$ with the diffusion constant $D_0 = \int_{-\infty}^{\infty} \langle\xi(t')\xi(t'+t)\rangle\, \mathrm{d}t$. This constant determines the coherence of oscillations, as the power spectrum of an observable $x = \cos(\phi)$ has a peak at frequency Ω_0 whose width is D. The feedback changes the diffusion constant, and the main goal of our investigation is to find dependence of D on the parameters a, τ.

We start our theoretical consideration with the noise-free case, when Eq. (1) reduces to $\dot{\phi} = \Omega_0 + a\sin(\phi(t-\tau) - \phi(t))$. If we seek for a solution with a uniformly rotating phase $\phi(t) = \Omega t$, we obtain

$$\Omega + a\sin\Omega\tau = \Omega_0 . \tag{2}$$

This equation has a unique solution for any Ω_0 if $|a\tau| < 1$, otherwise multiple solutions are possible. The latter case is especially difficult and will be considered elsewhere (see numerical simulations of effect of noise on the multistable states in (1) in Ref. [8]). Below we will consider a situation with small delayed feedback only, where no multistability occurs. We will also show that noise can destroy multistability, so that in its presence the condition $|a\tau| < 1$ can be weakened.

Our main statistical approach in studying Eq. (1) is based on the Gaussian approximation for $\phi(t)$. First, we separate the average rotation and the fluctuations according to $\phi = \Omega t + \psi$. For the fluctuating instant frequency (which is also Gaussian) $v(t) = \dot{\psi}$ we get from (1)

$$v(t) = \Omega_0 - \Omega + \xi(t) - a\sin\Omega\tau\cos\eta + a\cos\Omega\tau\sin\eta . \tag{3}$$

The equation for the mean frequency Ω results from the averaging of (3): $0 = \Omega_0 - \Omega - a\sin\Omega\tau\langle\cos\eta\rangle$. The phase difference $\eta = \psi(t-\tau) - \psi(t) = -\int_{t-\tau}^{t} v(s)\,\mathrm{d}s$ is Gaussian with zero average, thus

$$\langle\cos\eta\rangle = \exp\left[-\frac{\langle\eta^2\rangle}{2}\right] \qquad \langle\eta^2\rangle = 2\int_0^{\tau}(\tau - s)V(s)\,\mathrm{d}s \equiv 2R . \tag{4}$$

Here we have introduced the autocorrelation function of the instant frequency $V(s) = \langle v(t)v(t+s)\rangle$. In the introduced notations the equation for the average frequency can be rewritten as

$$\Omega = \Omega_0 - a\mathrm{e}^{-R}\sin\Omega\tau\ . \tag{5}$$

One can note that it is analogous to Eq. (2), but with an additional factor e^{-R}, which describes a diminishing of the influence of the delayed feedback due to phase diffusion.

To obtain equations for the autocorrelation function $V(s)$ we introduce also the autocorrelation function of noise C and the cross-correlation function S, defined according to $C(s) = \langle\xi(t)\xi(t+s)\rangle, S(s) = \langle\xi(t)v(t+s)\rangle$. Equations for V and S are obtained via multiplying Eq. (3) with $v(t+u)$ and $\xi(t+u)$ and averaging

$$\langle v(t)v(t+u)\rangle = \langle\xi(t)v(t+u)\rangle - a\sin\Omega\tau\left\langle v(t+u)\cos\left(\int_{t-\tau}^{t} v(s)\,\mathrm{d}s\right)\right\rangle$$
$$-a\cos\Omega\tau\left\langle v(t+u)\sin\left(\int_{t-\tau}^{t} v(s)\,\mathrm{d}s\right)\right\rangle$$

and similar for $\langle v(t)\xi(t+u)\rangle$. To accomplish the averaging we use the Furutsu–Novikov formula, valid for having zero averages Gaussian variables $x, y : \langle xF(y)\rangle = \langle xy\rangle\langle F'(y)\rangle$. In application to our averages this means that all averages of the form $\langle x\cos y\rangle$ vanish and the other give

$$\left\langle v(t+u)\sin\left(\int_{t-\tau}^{t} v(s)\,\mathrm{d}s\right)\right\rangle = \mathrm{e}^{-R}\int_{-\tau}^{0} V(s-u)\,\mathrm{d}s,$$
$$\left\langle \xi(t+u)\sin\left(\int_{t-\tau}^{t} v(s)\,\mathrm{d}s\right)\right\rangle = \mathrm{e}^{-R}\int_{-\tau}^{0} S(s-u)\,\mathrm{d}s\ .$$

As a result we obtain

$$V(u) = S(u) - a\mathrm{e}^{-R}\cos\Omega\tau\int_{0}^{\tau} V(s+u)\,\mathrm{d}s\ , \tag{6}$$

$$S(u) = C(u) - a\mathrm{e}^{-R}\cos\Omega\tau\int_{0}^{\tau} S(u-s)\,\mathrm{d}s\ . \tag{7}$$

Together with Eq. (5) and the definition of quantity R (4) they constitute a closed system.

To proceed further it is convenient to consider the spectra, according to $\mathscr{V}(\omega) = (1/2\pi)\int_{-\infty}^{\infty}\mathrm{d}uV(u)\mathrm{e}^{-\mathrm{i}\omega u}$ and similar expressions for $\mathscr{S},\mathscr{C}$. Then Eqs. (6) and (7) yield

$$\mathscr{V}(\omega) = \mathscr{S}(\omega) - a\mathrm{e}^{-R}\cos\Omega\tau\frac{\mathrm{e}^{\mathrm{i}\omega\tau}-1}{\mathrm{i}\omega},$$

$$\mathscr{S}(\omega) = \mathscr{C}(\omega) - a\mathrm{e}^{-R}\cos\Omega\tau\mathscr{S}(\omega)\frac{1-\mathrm{e}^{-\mathrm{i}\omega\tau}}{\mathrm{i}\omega}\ ,$$

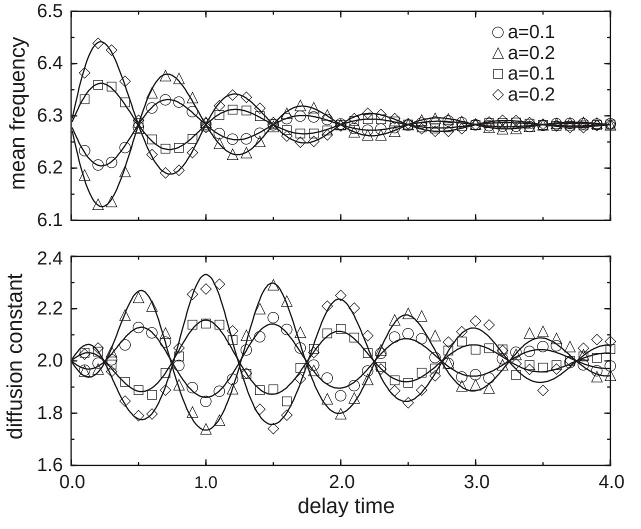

Fig. 1. Diffusion constant and mean frequency as functions of delay τ for $\sigma^2 = 1$ and $\Omega_0 = 2\pi$, and different values of feedback. Symbols: direct numerical simulation of model (1); solid lines: theory (8).

what allows us to exclude $\mathscr{S}(\omega)$ to get

$$\mathscr{V}(\omega) = \mathscr{C}(\omega)\left[1 + 2a\tau e^{-R}\cos\Omega\tau\frac{\sin\omega\tau}{\omega\tau} + a^2\tau^2 e^{-2R}\cos^2\Omega\tau\frac{2-2\cos\omega\tau}{\omega^2}\right]^{-1} .$$

Eq. (5) in the spectral form reads (here we use that $\mathscr{V}(\omega)$ is an even function) $R = \int_{-\infty}^{\infty}(1 - \cos\omega\tau)\omega^{-2}\mathscr{V}(\omega)\,d\omega$. This system is still hard to solve in a general form, due to integration in the expression for R.

The quantity of main interest for us is the diffusion constant of the phase ψ, it is related to the spectral density of the frequency fluctuations at zero frequency: $D = 2\pi\mathscr{V}(0)$. For this quantity we obtain: $D = D_0(1 + a\tau e^{-R}\cos\Omega\tau)^{-2}$, where $D_0 = 2\pi\mathscr{C}(0)$ is the diffusion constant in the absence of the feedback. To obtain a closed system for determining D we assume further that the spectrum of the frequency fluctuations $\mathscr{V}(\omega)$ is very broad. One can expect this for broad spectrum of noise $\mathscr{C}(\omega)$, i.e. if the noise is nearly delta-correlated. More precisely, we need to assume that the correlation time of frequency is much smaller than the delay time τ, so that the integral can be approximated as $R \approx \int_{-\infty}^{\infty}(1 - \cos\omega\tau)\omega^{-2}\mathscr{V}(0)\,d\omega = \tau D/2$. As a result we obtain a closed system of equations—the main result of our analysis—

$$D = \frac{D_0}{(1 + a\tau e^{-\tau D/2}\cos\Omega\tau)^2}, \qquad \Omega = \Omega_0 - a e^{-\tau D/2}\sin\Omega\tau , \tag{8}$$

relating the diffusion constant D in the presence of the feedback to the "bare" diffusion constant D_0 and to the parameters of the feedback τ and a, as well to the "bare" frequency Ω_0. This is a nonlinear system of two equations for two variables D and Ω, which can be solved numerically for a given set of parameters.

In Fig. 1 we compare the results of direct numerical simulation of model (1) with theoretical predictions (8). In this case of relatively strong noise the correspondence is good.

Summarizing, we have developed a statistical theory of phase diffusion under influence of delayed feedback. In the Gaussian approximation a closed system of equations for the diffusion constant and the mean frequency have been derived for the case of short correlations of the instant frequency. The theory works if the feedback is not very strong, or if the noise is strong enough to suppress multistability of mean frequency. An opposite situation, where effects of multistability are dominant, will be considered elsewhere.

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