Controlling oscillator coherence by delayed feedback

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We demonstrate that the coherence of a noisy or chaotic self-sustained oscillator can be efficiently controlled by the delayed feedback. We develop a theory of this effect, considering noisy systems in the Gaussian approximation. We obtain a closed equation system for the phase diffusion constant and the mean frequency of oscillation. For weak feedback and strong noise, the theory is in good agreement with the numerics. We discuss possible applications of the effect for the synchronization control.

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I. INTRODUCTION

Coherence, or constancy of oscillation frequency, is one of the main characteristics of self-sustained systems. This property determines the quality of clocks, electronic generators, lasers, etc. Quite often the improvement of the coherence is one of the major goals in the design of such oscillators. In terms of the phase dynamics, the coherence of a noisy limit cycle oscillator is quantified by the phase diffusion constant; it is proportional to the width of the spectral peak of oscillations. Many chaotic oscillators also admit phase dynamics description, and, hence, their coherence can be quantified by virtue of the phase diffusion constant as well [1].

In this paper we demonstrate that the coherence of oscillations is essentially influenced by an external delayed feedback, thus offering a possibility for its effective control. Delayed feedback is widely used to achieve a qualitative change in the dynamics, e.g., to make chaotic oscillators to operate periodically (Pyragas' control method [2]) or to suppress space-time chaos [3–5]. In our study we concentrate on the quantitative effect of a delayed feedback on the phase diffusion properties of noisy periodic and chaotic oscillators.

Investigation of effects of irregularities and noise in systems with delay is a complicated problem, because one cannot apply here such well-established tools as the Fokker-Planck equation, valid for the Markov processes. In the case of delay the process is non-Markov and therefore the problems are treated by *ad hoc* statistical methods. This has been accomplished recently for bistable oscillators [6], see also Refs. [7–9]. Below we present a theory describing the effect of a delayed feedback on noisy self-sustained oscillations. It is based on the phase approximation of the dynamics, which means that the noise and the delayed feedback are assumed to be weak. On the other hand, we consider a full nonlinear phase dynamics problem, and therefore our approach goes beyond the statistical analysis of linear stochastic delay-differential equations [10,11].

II. CONTROL OF COHERENCE: NUMERICAL RESULTS

In this section we present a numerical evidence for a possibility to control the diffusion constant by a delayed feedback. We begin by presenting the results of numerical simulation for noisy Van der Pol oscillator:

$$\ddot{x}-\mu(1-x^2)\dot{x}+\Omega_0^2x=k[\dot{x}(t-\tau)-\dot{x}(t)]+\zeta(t),$$

$$\langle\zeta(t)\zeta(t')\rangle=2d^2\delta(t-t'). \qquad (1)$$

The left-hand side represents the Van der Pol equation. In the absence of noise and delay ($k=d=0$) and for small nonlinearity μ, this model has a limit cycle solution $x_0\approx 2\cos\phi$, $\dot{x}_0\approx -2\Omega_0\sin\phi$, with a uniformly growing phase $\phi(t)\approx\Omega_0 t+\phi_0$ [12]. Under the influence of noise and in the absence of feedback ($k=0$, $d>0$), $\phi(t)$ diffuses according to $\langle[\phi(t)-\langle\phi(t)\rangle]^2\rangle\propto D_0 t$; the diffusion constant D_0 is proportional to the intensity of noise d^2 [see Eq. (4) below for an exact relation].

We expect that in the presence of feedback the diffusion constant D generally differs from D_0; this is confirmed by the numerical results, shown in Fig. 1 for $\Omega_0=1$, $d=0.1$, and $\mu=0.7$. One can see that diffusion can be suppressed or enhanced, depending on the feedback strength k and the delay time τ. The main goal of this paper is to describe this picture theoretically.

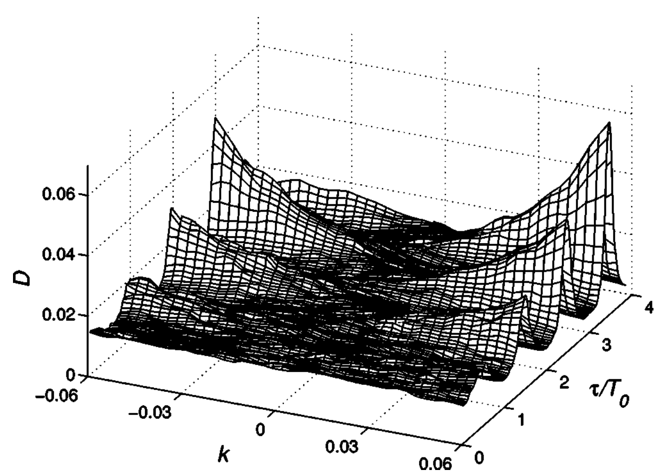

FIG. 1. Diffusion constant D for the phase of the noise-driven Van der Pol oscillator with delayed feedback (1) as the function of τ/T_0 and k; $T_0\approx 6.61$ is the oscillation period without delay.

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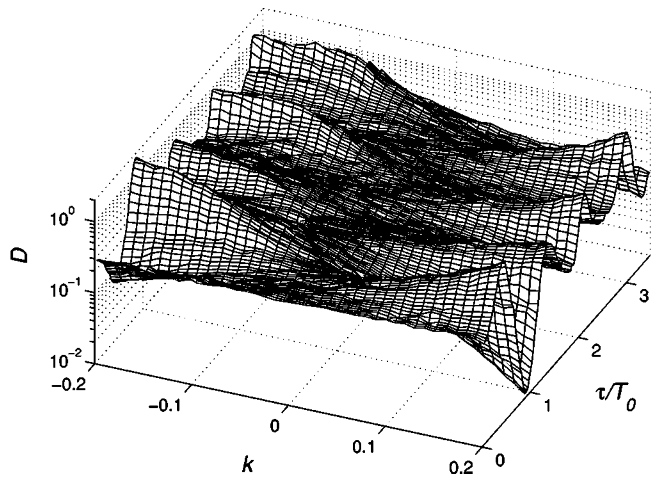

FIG. 2. Diffusion constant D for the Lorenz system (2) as the function of τ/T_0 and k. $T_0 \approx 0.69$ is the average oscillation period without delay. Note the logarithmic scale of the D axis.

Another numerical example demonstrates the effect of delayed feedback on phase diffusion in the chaotic Lorenz model:

$$\dot{x} = \sigma(y-x),$$
$$\dot{y} = rx - y - xz, \tag{2}$$
$$\dot{z} = -bz + xy + k[z(t-\tau) - z(t)],$$

where $\sigma=10$, $r=32$, and $b=8/3$. The phase of the Lorenz system is well defined if one uses a projection of the phase space on the plane $(u=\sqrt{x^2+y^2}, z)$ (see Ref. [1] and Fig. 3 below). Notice that there is no noise term in Eqs. (2): because of chaos the phase of the autonomous system grows nonuniformly, with a nonzero diffusion constant.

The dependence of the diffusion constant D of the phase on the feedback parameters k and τ is shown in Fig. 2. Qualitatively this dependence is similar to that for the Van der Pol model. However, there is an important distinction: the diffusion has a very deep minimum for positive feedback constant k and the delay time close to the mean oscillation period; here the rotation of the phase point along the trajectory of the Lorenz system becomes highly coherent.

Another representation of the effect of the delayed feedback on the coherence of the process is given by the power spectrum. Indeed, the power spectrum of an oscillatory observable has a peak at frequency Ω_0, and the width of the peak is proportional to the diffusion constant D. In Fig. 3 we show how the feedback changes the spectrum of the Lorenz system for the cases of maximal enhancement and maximal suppression of the diffusion constant. In this figure we also demonstrate that the effect is not related to the suppression of chaos: large variations of the diffusion constant (more than 10 times) are not reflected in the topology of the strange attractor; also the calculated Lyapunov exponents are very close to those without feedback. This suggests that the effect of feedback on the coherence can be described in the framework of phase approximation to the dynamics (this approximation has been used in Ref. [13] to describe phase synchronization of chaotic oscillators).

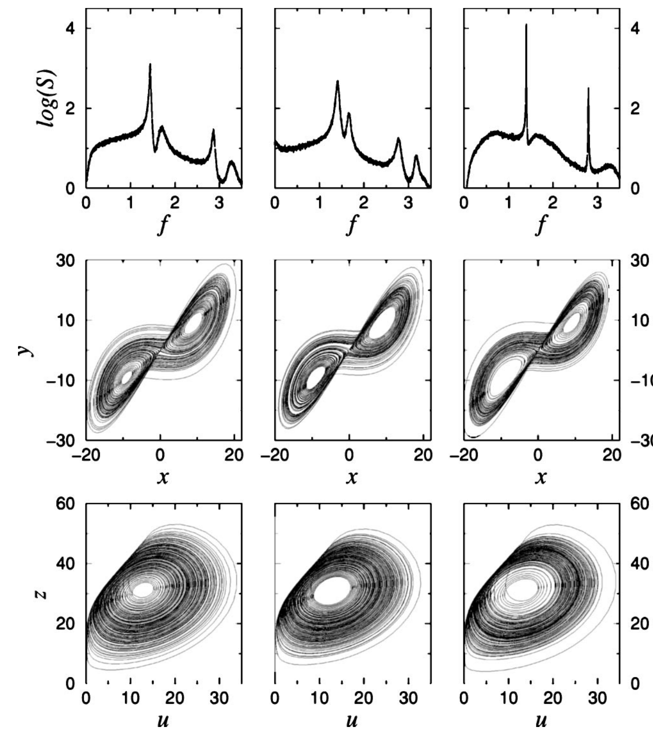

FIG. 3. Spectra $\log_{10}(S)$ of the z component of the Lorenz system and projections of the phase portrait for the system in the absence of delayed feedback (left column) and in the presence of feedback with delay $\tau=0.3$ (middle column) and $\tau=0.65$ (right column); feedback strength $k=0.2$. Note that feedback makes the spectral peak essentially more broad (enhanced diffusion, middle column) or more narrow (suppressed diffusion, right column), whereas practically no changes can be seen in the phase portraits.

One of the implications of the coherence control is a possibility to govern synchronization properties of an oscillator. Indeed, the ability of an oscillator to be entrained directly depends on the phase diffusion constant, thus improving coherence means improving of the synchronization ability [1]. We illustrate this by consideration of the phase synchronization of the Lorenz system by a periodic force $E \sin \nu t$ added to the equation for the variable z (Fig. 4). In the absence of the feedback the force is too weak to entrain the system, while the coherent oscillator demonstrates synchronization.

III. BASIC PHASE MODEL

According to a general theory (see, e.g., Ref. [16]), external force acting on a limit cycle oscillator in the first approximation affects the phase variable, but not the amplitudes, because the phase is free and can be adjusted by a very weak action, while the amplitude variables are stable and thus change only slightly. We follow this idea to derive below our basic theoretical phase model starting from Van der Pol model (1) in the case of small nonlinearity $\mu \ll 1$. For small feedback and noise we can use the perturbation theory, valid in the vicinity of the limit cycle (see, e.g., Refs. [1,16]). We rewrite Eq. (1) as a system,

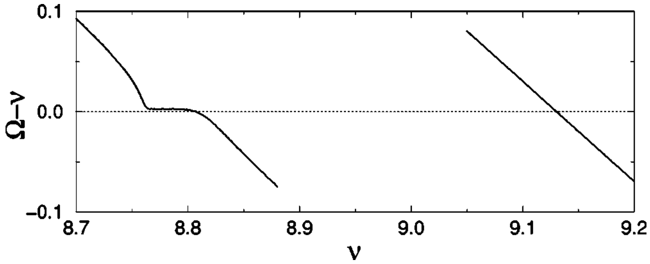

FIG. 4. Entrainment of the Lorenz system by a harmonic force with $E=2$. Right graph: without feedback the mean oscillator frequency Ω is not locked to the driving frequency ν. Left graph: the feedback with $k=0.2$, $\tau=0.65$ makes the oscillator coherent, what results in the appearance of the synchronization region $\Omega\approx\nu$ (cf. Refs. [14,15]). Note also that the mean frequency is shifted by the feedback; this effect is theoretically explained below.

$$\dot{x}=\Omega_0 y,$$

$$\dot{y}=-\Omega_0 x+\mu[1-x^2)y+k[y(t-\tau)-y(t)]+\frac{1}{\Omega_0}\zeta(t),$$

and obtain according to [1,16]

$$\dot{\phi}=\Omega_0+\frac{\partial\phi}{\partial y_0}\left(k[y_0(t-\tau)-y_0(t)]+\frac{1}{\Omega_0}\zeta(t)\right),$$

where $x_0=2\cos\phi$, $y_0=-2\sin\phi$ are the limit cycle solutions related to the phase as $\phi=-\arctan(y_0/x_0)$; therefore $\partial\phi/\partial y_0=-x_0/(x_0^2+y_0^2)$. Substituting the variables x_0, y_0 on the right-hand side (rhs) by ϕ, we obtain

$$\dot{\phi}=\Omega_0+k[\sin\phi(t-\tau)-\sin\phi(t)]\cos[\phi(t)] + \frac{1}{2\Omega_0}\zeta(t)\cos(\phi). \quad (3)$$

We are mostly interested in the long-time behavior of the phase; therefore, we average the rhs over the period of oscillations. As a result, the rhs contains only the terms depending on the phase differences. Next, we use that ζ is δ correlated and independent of ϕ, so that

$$\langle\zeta(t)\zeta(t')\cos\phi(t)\cos\phi(t')\rangle\approx\langle\zeta(t)\zeta(t')\rangle \times\langle\cos\phi(t)\cos\phi(t')\rangle = d^2\delta(t-t').$$

Finally we obtain our basic phase equation

$$\dot{\phi}=\Omega_0+a\sin[\phi(t-\tau)-\phi(t)]+\xi(t), \quad (4)$$

where $a=k/2$ is the renormalized strength of the feedback and $\xi(t)$ is the effective noise satisfying $\langle\xi(t)\xi(t')\rangle =(d^2/4\Omega_0^2)\delta(t-t')$.

We emphasize that, although we derived Eq. (4) for the Van der Pol equation, a similar equation can be obtained for any limit cycle oscillator (if the assumption of weak perturbations is valid)—the only difference may be in a more complex dependence on the phase difference, containing not only one sine function but its harmonics as well. Moreover, as the phase dynamics of chaotic oscillators is qualitatively similar to the dynamics of noisy periodic oscillators (see Ref. [1]), Eq. (4) can serve as a model for chaotic oscillators in the presence of the feedback loop. In the latter case the term $\xi(t)$ reflects the irregularity of chaotic amplitudes. Note that Eq. (4) has been used in Ref. [9] to describe the evolution of the phase of an optical field in a laser with a weak optical feedback.

IV. STATISTICAL ANALYSIS OF THE PHASE MODEL

As the first step in the theoretical analysis of model (4), we separate the phase growth into the average growth and the fluctuations, according to $\phi=\Omega t+\psi$, where Ω is the unknown mean frequency and ψ is the slow phase. For the fluctuating instantaneous frequency $v(t)=\dot{\psi}$, we obtain from Eq. (4),

$$v(t)=\Omega_0-\Omega+\xi(t)-a\sin\Omega\tau\cos[\psi(t-\tau)-\psi(t)] + a\cos\Omega\tau\sin[\psi(t-\tau)-\psi(t)]. \quad (5)$$

In the following we analyze this equation using different approximations.

A. Noise-free case: Multistability in oscillation frequency

We begin our consideration with the noise-free case, $\xi=\psi=v=0$, when Eq. (5) reduces to

$$\Omega+a\sin\Omega\tau=\Omega_0. \quad (6)$$

Thus, the delayed feedback changes the frequency of the oscillator. The transcendent Eq. (6) has a unique solution for any Ω_0, if $|a\tau|<1$, and multiple solutions otherwise. The latter case is especially difficult and will be considered elsewhere. (Numerical simulation of the effect of the noise on the multistable states in Eq. (4) was performed in Ref. [9].) Below we will consider a situation with weak delayed feedback only, when no multistability occurs. We will also show that noise can destroy multistability, so that in its presence the condition $|a\tau|<1$ can be weakened [see Eq. (11) below].

B. Linear approximation

Here we assume that the fluctuations of the phase are very weak, i.e., $\psi(t)-\psi(t-\tau)\ll 2\pi$. In this first order in ψ approximation, we obtain from Eq. (5) with account of Eq. (6)

$$v(t)=\dot{\psi}=\xi(t)+a\cos\Omega\tau[\psi(t-\tau)-\psi(t)], \quad (7)$$

where Ω is a solution of Eq. (6). This linear equation can be easily solved in the Fourier domain. As a result the power spectrum of frequency fluctuations $S_v(\omega)$ can be related to the power spectrum of noise $S_\xi(\omega)$ (note that no further assumption on the noise statistics is needed):

$$S_v(\omega) = \frac{\omega^2 S_\xi(\omega)}{\omega^2 + 2\omega a \sin\omega\tau \cos\Omega\tau + 2(1-\cos\omega\tau)a^2\cos^2\Omega\tau}.$$

The diffusion constant can be obtained by considering the limit $\omega \to 0$:

$$S_v(0) = \frac{S_\xi(0)}{(1 + a\tau\cos\Omega\tau)^2}.$$

Thus, the diffusion constant $D = 2\pi S_v(0)$ is obtained in the linear approximation as

$$D = \frac{D_0}{(1 + a\tau\cos\Omega\tau)^2}, \tag{8}$$

where $D_0 = 2\pi S_\xi(0)$ is the diffusion of the "no control" oscillator.

Below we will obtain a more accurate expression for the diffusion constant; however, the simple formula (8) allows us to give a qualitative explanation of the numerical results presented in Figs. 1 and 2. As it follows from Eq. (8), the feedback term can compensate or amplify the fluctuations in the phase growth, in dependence on the sign of the product $a\cos\Omega\tau$ (for small feedback this term can be estimated as $a\cos\Omega_0\tau$), because this product appears in Eq. (7) as the effective strength of the feedback regulating the fluctuations of the phase. This explains the oscillatory dependence of the diffusion constant on the delay time τ.

C. Gaussian approximation

Our main statistical approach in the treatment of full nonlinear Eq. (4) is based on the Gaussian approximation for $\psi(t)$. We also assume the noisy term $\xi(t)$ to be Gaussian. However, contrary to the numerical simulation, where the noise is white, we consider a general spectrum of the noise. Averaging Eq. (5) for the fluctuations of the instantaneous frequency $v(t) = \dot{\psi}$ (which is also Gaussian), we come to the equation for the mean frequency Ω:

$$0 = \Omega_0 - \Omega - a\sin\Omega\tau \langle \cos[\psi(t-\tau) - \psi(t)] \rangle. \tag{9}$$

The phase difference $\eta(t) = \psi(t-\tau) - \psi(t)$ is Gaussian with zero average, hence $\langle \cos\eta \rangle = \exp[-\langle \eta^2 \rangle/2]$. The phase difference η can be represented as an integral of the instantaneous frequency:

$$\eta(t) = -\int_{t-\tau}^{t} v(s)\,ds,$$

which gives for the variance of η,

$$\langle \eta^2 \rangle = 2\int_0^\tau (\tau - s)V(s)\,ds \equiv 2R. \tag{10}$$

Here we have introduced the autocorrelation function of the instantaneous frequency,

$$V(u) = \langle v(t)v(t+u) \rangle.$$

Using the notation introduced in Eq. (10) we rewrite Eq. (9) for the average frequency as

$$\Omega = \Omega_0 - ae^{-R}\sin\Omega\tau. \tag{11}$$

We note that it is similar to Eq. (6), but contains an additional factor e^{-R}, which describes the above-mentioned partial suppression of the effect of the delayed feedback due to phase diffusion.

To obtain equations for the autocorrelation function $V(u)$ we introduce also the autocorrelation function of the noise $C(u)$ and the cross-correlation function $S(u)$, defined according to

$$C(u) = \langle \xi(t)\xi(t+u) \rangle, \quad S(u) = \langle \xi(t)v(t+u) \rangle.$$

After the averaging described in the Appendix we obtain the equations for the correlation functions

$$V(u) = S(u) - ae^{-R}\cos\Omega\tau \int_0^\tau V(s+u)\,ds, \tag{12}$$

$$S(u) = C(u) - ae^{-R}\cos\Omega\tau \int_0^\tau S(u-s)\,ds. \tag{13}$$

Together with Eq. (11) and the definition of quantity R given by Eq. (10), they constitute a closed system.

To proceed it is convenient to consider the spectra according to

$$\mathcal{V}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, V(u) e^{-i\omega u},$$

and similarly for $\mathcal{S}$ and $\mathcal{C}$. Then Eqs. (12) and (13) yield

$$\mathcal{V}(\omega) = \mathcal{S}(\omega) - ae^{-R}\cos\Omega\tau \frac{e^{i\omega\tau} - 1}{i\omega}, \tag{14}$$

$$\mathcal{S}(\omega) = \mathcal{C}(\omega) - ae^{-R}\cos\Omega\tau\, \mathcal{S}(\omega) \frac{1 - e^{-i\omega\tau}}{i\omega}, \tag{15}$$

which allows us to exclude $\mathcal{S}(\omega)$ and obtain

$$\mathcal{V}(\omega) = \mathcal{C}(\omega)\left[1 + 2a\tau e^{-R}\cos\Omega\tau \frac{\sin\omega\tau}{\omega\tau} + a^2\tau^2 e^{-2R}\cos^2\Omega\tau \frac{2 - 2\cos\omega\tau}{\omega^2}\right]^{-1}. \tag{16}$$

Equation (10) in the spectral form reads

$$R = \int_{-\infty}^{\infty} \frac{1 - \cos\omega\tau}{\omega^2} \mathcal{V}(\omega)\,d\omega. \tag{17}$$

Here we have used that $\mathcal{V}(\omega)$ is an even function. System (16) and (17) is still hard to solve in the general form, due to integration in Eq. (17).

The quantity of our main interest is the diffusion constant D of the phase ψ. D is related to the spectral density of the frequency fluctuations at zero frequency: $D=2\pi\mathcal{V}(0)$. Using Eq. (16) we obtain, for this quantity,

$$D=\frac{D_0}{(1+a\tau e^{-R}\cos\Omega\tau)^2}, \tag{18}$$

where $D_0=2\pi\mathcal{C}(0)$ is the "no control" diffusion constant in the absence of the feedback. To obtain a closed system for the determination of D we further assume that the spectrum of the frequency fluctuations $\mathcal{V}(\omega)$ is very broad. One can expect this if the spectrum of noise $\mathcal{C}(\omega)$ is broad, i.e., if the noise is nearly δ correlated. More precisely, we assume that the correlation time of frequency fluctuation is much smaller than the delay time τ, so that integral (17) can be approximated as

$$R\approx\int_{-\infty}^{\infty}\frac{1-\cos\omega\tau}{\omega^2}\mathcal{V}(0)d\omega=\frac{\tau D}{2}. \tag{19}$$

As a result we obtain a closed system of equations—the main result of our analysis,

$$D=\frac{D_0}{(1+a\tau e^{-\tau D/2}\cos\Omega\tau)^2}, \tag{20}$$

$$\Omega=\Omega_0-ae^{-\tau D/2}\sin\Omega\tau, \tag{21}$$

relating the diffusion constant D in the presence of the feedback to the "no control" diffusion constant D_0 and to the parameters of the feedback τ and a, as well as to the "no control" frequency Ω_0. This is a nonlinear system of two equations for two variables D and Ω, which can be solved numerically for a given set of parameters. In the case of small noise, $D_0\tau\ll1$, we can set $e^{-\tau D/2}\approx1$ and end with Eqs. (6) and (8), obtained above in the linear approximation.

Another useful approximation is that of small feedback, then we can approximate the diffusion constant in Eq. (19) by its "no control" value, this gives

$$D=\frac{D_0}{(1+a\tau e^{-\tau D_0/2}\cos\Omega\tau)^2},\quad \Omega=\Omega_0-ae^{-\tau D_0/2}\sin\Omega\tau. \tag{22}$$

Now only the equation for Ω is implicit, while the diffusion constant depends on the parameters in an explicit way.

We compare the theoretical results given by Eqs. (20) and (21) with the direct numerical simulations in Figs. 5 and 6. In Fig. 5 we present numerical results for phase model (4). The presented case of relatively strong noise demonstrates a good correspondence with theory. Furthermore, one can see that the effect of delayed feedback decreases with τ, because of the diffusion. Physically, it can be explained as follows. The feedback either compensates or amplifies the deviations from the uniform phase growth. If the diffusion constant is large, then during a large delay time the phases $\phi(t)$ and $\phi(t-\tau)$ are practically uncorrelated; thus the feedback reduces to a random term, which neither compensates nor amplifies the fluctuations.

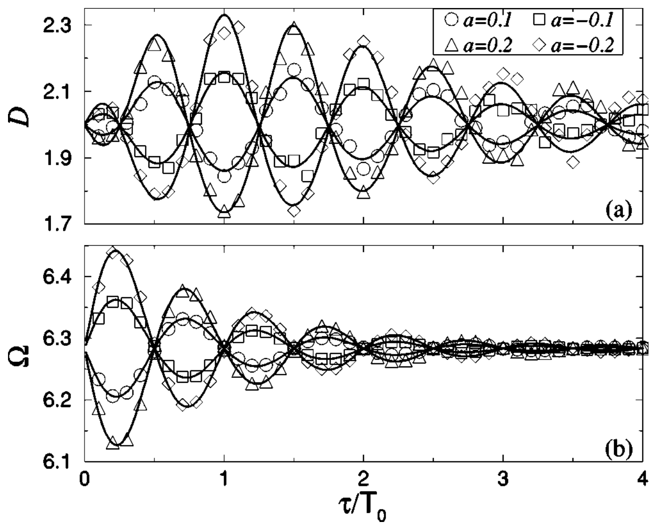

FIG. 5. Diffusion constant D (a) and mean frequency Ω (b) as functions of delay τ for model (4) with $\langle\xi(t)\xi(t+t')\rangle=2\delta(t')$ and $\Omega_0=2\pi$, and different values of feedback strength. Symbols present the results of the direct numerical simulation of model (4); solid lines show theoretical results according to Eqs. (20) and (21).

Figure 6 demonstrates the results for the Van der Pol model (1). The only parameter we have fitted here is the "no control" frequency $\Omega_0\approx0.95$. Here the correspondence with theory is good for small τ, but fails for large τ. The reason is that in this case the effective noise is small and therefore the feedback control is effective even for large delays. However, for large $a\tau$ Eq. (21) exhibits multistability, which results in an enhancement of the diffusion; here neither the linear approximation for small noise [Eqs. (6) and (8)] nor the Gaussian approximation used in derivation of Eqs. (20) and (21) is valid.

V. CONCLUSION

In summary, we have presented the effect of the coherence control by means of the delayed feedback. The control

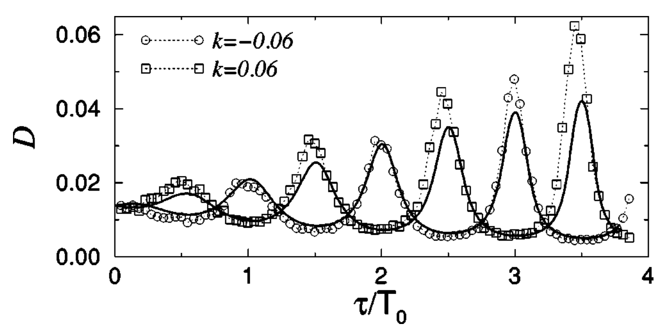

FIG. 6. Diffusion constant D of the Van der Pol model with delayed feedback [parameters are the same as in Fig. (1)]. Symbols present the results of the direct numerical simulation; solid lines show the corresponding theoretical results according to Eqs. (20) and (21). The delay time is normalized by the average period $T_0=2\pi/0.95$.

is possible for noisy limit cycle oscillators as well as for chaotic systems, admitting computation of the phase. Next, we have developed a statistical theory of phase diffusion under the influence of a delayed feedback. Using the Gaussian approximation, we have derived a closed system of equations for the diffusion constant and the mean frequency for the case of short-time correlations of the instantaneous frequency. The theory works if the feedback is not very strong, or if the noise is strong enough to suppress multistability in mean frequency. An opposite situation, where effects of multistability are dominant, will be considered elsewhere.

We would like to mention that formally the equations describing the control are the same as in the Pyragas method of chaos control. However, in our case the delay time τ is not necessarily equal to the period of some unstable limit cycle, embedded in chaos. Moreover, we consider the situation when the feedback is so small that no stabilization of periodic orbits occur. For the Lorenz system, e.g., such a stabilization by the simplest Pyragas method is anyhow not possible due to a special symmetry of the system. The main difference to the Pyragas approach is that we do not intend to suppress chaos, but to control uniformity—coherence—of phase growth in a chaotic system.

Note also that our method differs from other possibilities to control the diffusion properties of the phase. For example, synchronization of oscillations by a periodic external force reduces or even completely suppresses the diffusion (the relevant model is the noisy Adler equation [1], or, equivalently, an equation of motion of an overdamped noise-driven particle in a periodic potential, see Ref. [17] for calculation of the diffusion for the latter problem). In our method no periodic force is needed and the system remains autonomous, preserving full symmetry with respect to time shifts. In other words, the power spectrum of the delay-controlled oscillations does not contain δ peaks but is continuous.

A direction of the future development of this work is aimed at detailed understanding of the particular features of the control of chaotic systems. Indeed, in this case our theory provides only qualitative explanation of the effect. This limitation of the theory is related to the statistical properties of the effective noise in a chaotic system that definitely cannot be considered as weak or Gaussian. (We remind that effective noise here describes the effect of irregular, although deterministic, amplitudes, on the phase dynamics.) Particularly, it is known that for the Lorenz system this noise is not symmetric and possesses nontrivial correlation properties [14,15]. Our preliminary numerical investigations show that the feedback significantly affects these correlations. We illustrate this in Fig. 7, where we present the autocorrelation function of the Poincaré return times in the Lorenz system. It is seen that for the case of feedback with $\tau = 0.65 \approx T_0$, the successive return times become essentially anticorrelated, which apparently accounts for unusually high (by factor ≈ 30) suppression of the phase diffusion. We have demonstrated that this effect is of particular importance for the control of synchronization. In fact, the delayed feedback has a twofold effect on synchronization properties. On one hand, the feedback shifts the oscillation frequency, thus giving a possibility to facilitate or impede the entrainment (this effect is important for periodic oscillators as well). On the other hand, synchronization can be suppressed or enhanced by the regulation of the coherence.

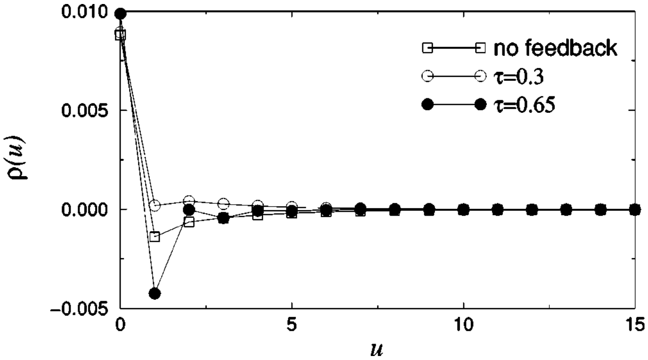

FIG. 7. Correlation functions $\rho(u)$ for the sequences of the Poincaré return times in the Lorenz system, in the absence and in the presence of the delayed feedback with $k = 0.2$. Note that variances of the return times, given by $\rho(0)$, are practically unchanged, whereas the anticorrelation between two successive intervals is either decreased (for $\tau = 0.3$) or increased (for $\tau = 0.65$).

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APPENDIX

Equations for V and S are obtained by multiplying Eq. (5) with $v(t+u)$ and $\xi(t+u)$ and averaging

$$V(u) = \langle v(t)v(t+u)\rangle = \langle \xi(t)v(t+u)\rangle - a\sin\Omega\tau\left\langle v(t+u)\cos\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle - a\cos\Omega\tau\left\langle v(t+u)\sin\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle,$$

$$S(u) = \langle v(t)\xi(t+u)\rangle = \langle \xi(t)\xi(t+u)\rangle - a\sin\Omega\tau\left\langle \xi(t+u)\cos\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle - a\cos\Omega\tau\left\langle \xi(t+u)\sin\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle.$$

To accomplish the averaging we use the Furutsu-Novikov formula [18,19], valid for zero-mean Gaussian variables x,y:

$$\langle xF(y)\rangle = \langle F'(y)\rangle\langle xy\rangle.$$

For the case under consideration this means that averages of all terms having the form $\langle x\cos y\rangle$ vanish, while other terms of type $\langle x\sin y\rangle$ yield

$$\left\langle v(t+u)\sin\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle = \left\langle \cos\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle \left\langle v(t+u)\int_{t-\tau}^{t} v(s)ds\right\rangle = e^{-R}\int_{-\tau}^{0} V(s-u)ds,$$

$$\left\langle \xi(t+u)\sin\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle = \left\langle \cos\left(\int_{t-\tau}^{t} v(s)ds\right)\right\rangle \left\langle \xi(t+u)\int_{t-\tau}^{t} v(s)ds\right\rangle = e^{-R}\int_{-\tau}^{0} S(s-u)ds.$$

This leads to Eqs. (12) and (13).

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