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Synchronization: from pendulum clocks to chaotic lasers and chemical oscillators

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Many natural and human-made nonlinear oscillators exhibit the ability to adjust their rhythms due to weak interaction: two lasers, being coupled, start to generate with a common frequency; cardiac pacemaker cells fire simultaneously; violinists in an orchestra play in unison. Such coordination of rhythms is a manifestation of a fundamental nonlinear phenomenon—synchronization. Discovered in the 17th century by Christiaan Huygens, it was observed in physics, chemistry, biology and even social behaviour, and found practical applications in engineering and medicine. The notion of synchronization has been recently extended to cover the adjustment of rhythms in chaotic systems, large ensembles of oscillating units, rotating objects, continuous media, etc. In spite of essential progress in theoretical and experimental studies, synchronization remains a challenging problem of nonlinear sciences.

1. Historical perspective

The history of synchronization goes back to the 17th century when the famous Dutch scientist Christiaan Huygens reported on his observation of synchronization of two pendulum clocks which he had invented shortly before. This invention of Huygens essentially increased the accuracy of time measurement and helped him to tackle the longitude problem. During a sea trial, he observed the phenomenon that he briefly described in his memoirs *Horologium Oscillatorium* (*The Pendulum Clock, or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks*) [1]:

> '. . . It is quite worths noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible.'

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According to a letter of Huygens to his father, the observation of synchronization was made while Huygens was sick and stayed in bed for a couple of days watching two clocks hanging on a wall. Interestingly, in describing the discovered phenomenon, Huygens wrote about '*sympathy of two clocks*' (*le phénoméne de la sympathie, sympathie des horloges*). Besides an exact description, he also gave a brilliant qualitative explanation of this effect of *mutual synchronization*; he correctly understood that the conformity of the rhythms of two clocks had been caused by an imperceptible motion of the beam. In modern terminology this would mean that the clocks were synchronized in antiphase due to *coupling* through the beam.

In the middle of the nineteenth century, in his famous treatise *The Theory of Sound*, Lord Rayleigh [2] described an interesting phenomenon of synchronization in acoustical systems:

> 'When two organ-pipes of the same pitch stand side by side, complications ensue which not unfrequently give trouble in practice. In extreme cases the pipes may almost reduce one another to silence. Even when the mutual influence is more moderate, it may still go so far as to cause the pipes to speak in absolute unison, in spite of inevitable small differences.'

Thus, Rayleigh observed not only mutual synchronization when two distinct but similar pipes begin to sound in

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unison, but also the related effect of oscillation death, when the coupling results in suppression of oscillations of interacting systems.

Being, probably, the oldest scientifically studied nonlinear effect, synchronization was understood only in the 1920s when E. V. Appleton and B. Van der Pol systematically—theoretically and experimentally—studied synchronization of triode generators. This new stage in the investigation of synchronization was related to the development of electrical and radio physics (now these fields belong to engineering). On 17 February 1920 W. H. Eccles and J. H. Vincent applied for a British Patent confirming their discovery of the synchronization property of a triode generator—a rather simple electrical device based on a vacuum tube that produces a periodically alternating electrical current. The frequency of this current oscillation is determined by the parameters of the elements of the scheme, e.g. of the capacitance. In their experiments, Eccles and Vincent coupled two generators which had slightly different frequencies and demonstrated that the coupling forced the systems to vibrate with a common frequency.

A few years later Edward Appleton and Balthasar van der Pol replicated and extended the experiments of Eccles and Vincent and made the first step in the theoretical study of this effect [3, 4]. Considering the simplest case, they showed that the frequency of a generator can be entrained, or synchronized, by a weak external signal of a slightly different frequency. These studies were of great practical importance because triode generators became the basic elements of radio communication systems. The synchronization phenomenon was used to stabilize the frequency of a powerful generator with the help of one which was weak but very precise.

To conclude the historical introduction, we cite the Dutch physician Engelbert Kaempfer [5]† who, after his voyage to Siam in 1680 wrote:

> 'The glowworms … represent another shew, which settle on some Trees, like a fiery cloud, with this surprising circumstance, that a whole swarm of these insects, having taken possession of one Tree, and spread themselves over its branches, sometimes hide their Light all at once, and a moment after make it appear again with the utmost regularity and exactness ….'

This very early observation reports on synchronization in a large population of oscillating systems. The same physical mechanism that makes the insects to keep in sync is responsible for the emergence of synchronous clapping in a large audience or onset of rhythms in neuronal populations.

We end our historical excursus in the 1920s. Since then many interesting synchronization phenomena have been observed and reported in the literature; some of them are mentioned below. More importantly, it gradually became clear that diverse effects which at first sight have nothing in common, obey some universal laws. Modern concepts also cover such objects as rotators and chaotic systems; in the latter case one distinguishes between different forms of synchronization: complete, phase, master-slave, etc. A great deal of research carried out by mathematicians, engineers, physicists and scientists from other fields, has led to the development of an understanding that, say, the conformity of the sounds of organ pipes or the songs of the snowy tree cricket is not occasional, but can be understood within a unified framework (for details and further references see [7–9]).

It is important to emphasize that synchronization is an essentially nonlinear effect. In contrast to many classical physical problems, where consideration of nonlinearity gives a correction to a linear theory, here the account of nonlinearity is crucial: the phenomenon occurs only in the so-called *self-sustained* systems.

2. Self-sustained oscillators

Self-sustained oscillators are models of natural oscillating objects, and these models are essentially nonlinear. To be not too abstract, we consider the classical object, which gave birth to synchronization theory, the pendulum clock. Let us discuss how it works. Its mechanism transforms the potential energy of the lifted weight (or compressed spring, or electrical battery) into the oscillatory motion of the pendulum. In its turn, this oscillation is transferred into the rotation of the hands on the clock's face (figure 1 (*a*)). We are not interested in the particular design of the mechanism; it is only important that it takes energy from the source in order to compensate the loss of energy due to dissipation, and in this way maintains a steady oscillation of the pendulum, which continues without any change until the supply of energy expires. The next important property is that the exact form of the oscillatory motion is entirely determined by the internal parameters of the clock and does not depend on how the pendulum was put into motion. Moreover, after being slightly perturbed, following some transient process the pendulum restores its previous internal rhythm.

These features are typical not only of clocks, but also of many oscillating objects of diverse nature (electronic generators, organ pipes etc.). In physics these oscillatory objects are denoted as self-sustained oscillators. Mathematically, such an oscillator is described by an autonomous (i.e. without explicit time dependence) nonlinear dynamical system. It differs both from linear oscillators (which, if damping is present, can oscillate only due to external forcing) and from nonlinear energy conserving systems, whose dynamics essentially depends on the initial state.

†Citation taken from [6].

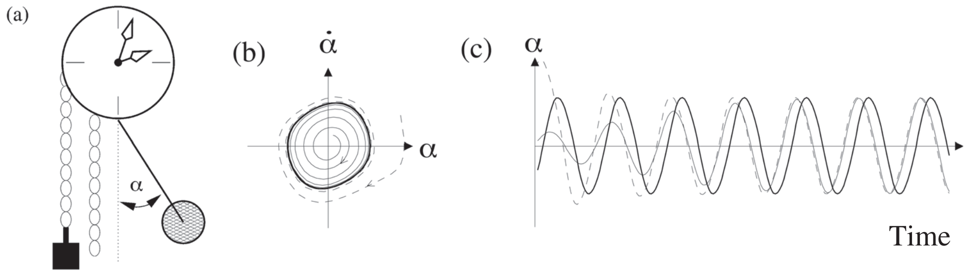

Figure 1. (*a*) An example of a self-sustained oscillator, the pendulum clock. The potential energy of the lifted weight is transformed into oscillatory motion of the pendulum and eventually into the rotation of the hands. (*b*) The state of the pendulum can be characterized by the angle α and its time derivative $\dot{\alpha}$, and the time evolution of the system can be described in the phase plane $(\alpha, \dot{\alpha})$. The closed curve (bold curve) in the phase plane attracts all the trajectories from its neighbourhood, and is therefore called the *limit cycle*. The same trajectories are shown in (*c*) as a time plot.

The dynamics of oscillators is typically described in the phase (state) space. Quite often two state variables suffice to determine unambiguously the state of the system, and we proceed here with this simplest case. For a pendulum clock, these variables can be, e.g. the angle α of the pendulum with respect to the vertical and its angular velocity $\dot{\alpha}$. Thus, the behaviour of the system can be completely described by the time evolution of a pair $(\alpha, \dot{\alpha})$. As the oscillation is periodic, i.e. it repeats itself after the period T, $x(t)$ corresponds to a closed curve in the phase plane, called the *limit cycle* (figures 1 (*b*) and (*c*)). The reason why we distinguish this curve from all others trajectories in the phase space is that it attracts phase trajectories and is therefore called an attractor of the dynamical system. The limit cycle is a simple attractor, in contrast to a *strange (chaotic) attractor*. The latter is a geometrical image of *chaotic* self-sustained oscillations.

Examples of self-sustained oscillatory systems are electronic circuits used for the generation of radio-frequency power, lasers, Belousov–Zhabotinsky and other oscillatory chemical reactions, pacemakers (sino-atrial nodes) of human hearts or artificial pacemakers that are used in cardiac pathologies, and many other natural and artificial systems. An outstanding common feature of such systems is their ability to be synchronized.

This ability of periodic self-sustained oscillators is based on the existence of a special variable, phase ϕ. Mathematically ϕ can be introduced as the variable parametrizing the motion along the stable limit cycle in the state space of an autonomous continuous-time dynamical system. One can always choose phase proportional to the fraction of the period, i.e. in the way that it grows uniformly with time,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0 \;, \tag{1}$$

where ω_0 is the natural frequency of oscillations. The phase is neutrally stable: its perturbations neither grow no decay. (In terms of nonlinear dynamics neutral stability means that the phase is a variable that corresponds to the zero Lyapunov exponent of the dynamical system.) Thus, already an infinitely small perturbation (e.g. external periodic forcing or coupling to another system) can cause large deviations of the phase—contrary to the amplitude, which is only slightly perturbed due to the transversal stability of the cycle. The main consequence of this fact is that *the phase can be very easily adjusted by an external action, and as a result the oscillator can be synchronized!*

3. Entrainment by external force

We begin our discussion of synchronization phenomena by considering the simplest case, entrainment of a self-sustained oscillator by external periodic force. Before we describe this effect in mathematical terms, we illustrate it by an example. We will again speak about clocks, but this time about biological clocks that regulate daily and seasonal rhythms of living systems—from bacteria to humans.

3.1. *An example: circadian rhythms*

In 1729 Jean-Jacques Dortous de Mairan, the French astronomer and mathematician, who was later the Secretary of the Académie Royale des Sciences in Paris, reported on his experiments with a haricot bean. He noticed that the leaves of this plant moved up and down in accordance with the change of day into night. Having made this observation, de Mairan put the plant in a dark room and found that the motion of the leaves continued even without variations in the illuminance of the environment. Since that time these and much more complicated experiments have been replicated in different laboratories, and now it is well known that all biological systems, from rather simple to highly organized ones, have internal biological clocks that provide their ‘owners’ with information on the change between day and night. The origin of these clocks is still a challenging problem, but it is well established that they can adjust their circadian rhythms (from *circa* = about and *dies* = day) to external signals: if the system is completely isolated from the environment and is kept under controlled constant conditions (constant illuminance, temperature, pressure, parameters of electromagnetic fields, etc.), its internal cycle can essentially differ from a 24 h cycle. Under natural conditions, biological clocks tune their rhythms in accordance with the 24 h period of the Earth's daily cycle.

Experiments show that for most people the internal period of biological clocks differs from 24 h, but it is entrained by environmental signals, e.g. illuminance, having the period of the Earth's rotation (figure 2). Obviously, the action here is unidirectional: the revolution of a planet cannot be influenced by mankind (yet); thus, this case constitutes an example of synchronization by an external force. In usual circumstances this force is strong enough to ensure perfect entrainment; in order to desynchronize a biological clock one can either travel to polar regions or go caving. It is interesting that although normally the period of one's activity is exactly locked to that of the Earth's rotation, the phase shift between the internal clock and the external force varies from person to person: some people say that they are 'early birds' whereas others call themselves 'owls'. Perturbation of the phase shift strongly violates normal activity. Every day many people perform such an experiment by rapidly changing their longitude (e.g. crossing the Atlantic) and experiencing jet lag. It can take up to several days to re-establish a proper phase relation to the force; in the language of nonlinear dynamics one can speak of different lengths of transients leading to the stable synchronous state. As other commonly known examples of synchronization by external force we mention radio-controlled clocks and cardiac pacemakers.

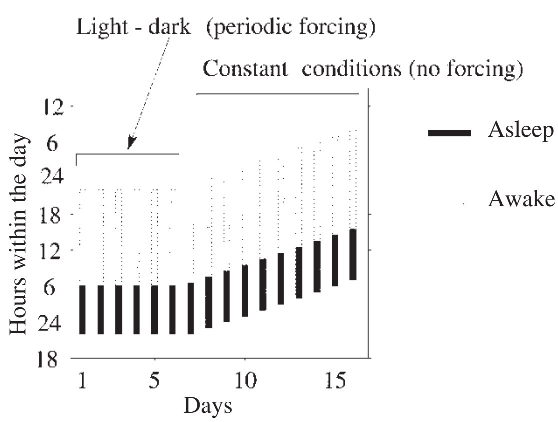

Figure 2. Schematic diagram of the behavioural sleep – wake rhythm. This cycle (termed circadian rhythm) represents the fundamental adaptation of organisms to an environmental stimulus, the daily cycle of light and dark. Here the circadian rhythm is shown entrained for five days by the environmental light – dark cycle and autonomous for the rest of the experiment when the subject is placed under constant light conditions. The intrinsic period of the circadian oscillator is in this particular case greater than 24 hours. Correspondingly, the phase difference between the sleep – wake cycle and daily cycle increases: the internal 'day' begins later and later. Such plots are typically observed in experiments with both animals and humans, see, e.g. [10 – 12].

3.2. *Phase dynamics of a forced oscillator*

For a mathematical treatment of synchronization we recall that phase of an oscillator is neutrally stable and can be adjusted by a small action, whereas the amplitude is stable. This property allows a description of the effect of small forcing/coupling within the framework of the phase approximation. Considering the simplest case of a limit cycle oscillator, driven by a periodic force with frequency ω and amplitude ε, we can write the equation for the perturbed phase dynamics in the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0 + \epsilon Q(\phi, \omega t) , \qquad (2)$$

where the coupling function Q depends on the form of the limit cycle and of the forcing. As the states with the phases ϕ_0 and $\phi_0 + 2\pi$ are physically equivalent, the function Q is 2π-periodic in both its arguments, and therefore can be represented as a double Fourier series. If the frequency of the external force is close to the natural frequency of the oscillator, $\omega \approx \omega_0$, then the series contains fast oscillating and slow varying terms, and the latter can be written as $q(\phi - \omega t)$. Introducing the difference between the phases of the oscillation and of the forcing $\psi = \phi - \omega t$ and performing an averaging over the oscillation period we get rid of the oscillating terms and obtain the following basic equation for the phase dynamics:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -(\omega - \omega_0) + \epsilon q(\psi) . \qquad (3)$$

Function q is 2π-periodic, and in the simplest case $q(\cdot) = \sin(\cdot)$ equation (3) is called the Adler equation. One can easily see that on the plane of the parameters of the external forcing (ω, ε) there exists a region $\varepsilon q_{\min} < \omega - \omega_0 < \varepsilon q_{\max}$, where equation (3) has a stable stationary solution. This solution corresponds to the conditions of phase locking (the phase ϕ just follows the phase of the force, i.e. $\phi = \omega t + constant$) and frequency entrainment (the observed frequency of the oscillator $\Omega = \langle \dot{\phi} \rangle$ exactly coincides with the forcing frequency ω; brackets $\langle\rangle$ denote time averaging).

Generally, synchronization is observed for high-order resonances $n\omega \approx m\omega_0$ as well. In this case the dynamics of the generalized phase difference $\psi = m\phi - n\omega t$, where n and m are integers, is described by an equation similar to equation (3), namely by $\mathrm{d}(\psi)/\mathrm{d}t = -(n\omega - m\omega_0) + \epsilon\tilde{q}(\psi)$. The synchronous regime then means perfect entrainment of the oscillator frequency at the rational multiple of the forcing frequency, $\Omega = (n/m)\omega$, as well as phase locking $m\phi = n\omega t + constant$. The overall picture can be shown on the (ω,ε) plane: there exists a family of triangular-shaped synchronization regions touching the ω axis at the rationals of the natural frequency $(m/n)\omega_0$; these regions are usually called Arnol'd tongues (figure 3 (*a*)). This picture is preserved for moderate forcing, although now the shape of the tongues generally differs from being exactly triangular. For a fixed amplitude of the forcing ε and varied driving frequency ω one observes different phase locking intervals where the motion is periodic, whereas in between them it is quasiperiodic. The curve Ω versus ω thus consists of horizontal plateaus at all possible rational frequency ratios; this fractal curve is called the devil's staircase (figure 3 (*b*)). A famous example of such a curve is the voltage–current plot for a Josephson junction in an ac electromagnetic field; in this context synchronization plateaus are called Shapiro steps. Note that a junction can be considered as a rotator (rotations are maintained by a dc current); this example demonstrates that synchronization properties of rotators are very close to those of oscillators.

Finally, we note that the phase difference in the synchronous state is not necessarily constant, but may oscillate around a constant value. Indeed, a solution $m\phi - n\omega t$ = constant was obtained from equation (2) by means of averaging, i.e. by neglecting the fast oscillating terms. If we take these terms into account, then we have to reformulate the condition of phase locking as $|m\phi - n\omega t| <$ constant. Thus, in the synchronous regime the phase difference is bounded, otherwise it grows infinitely.

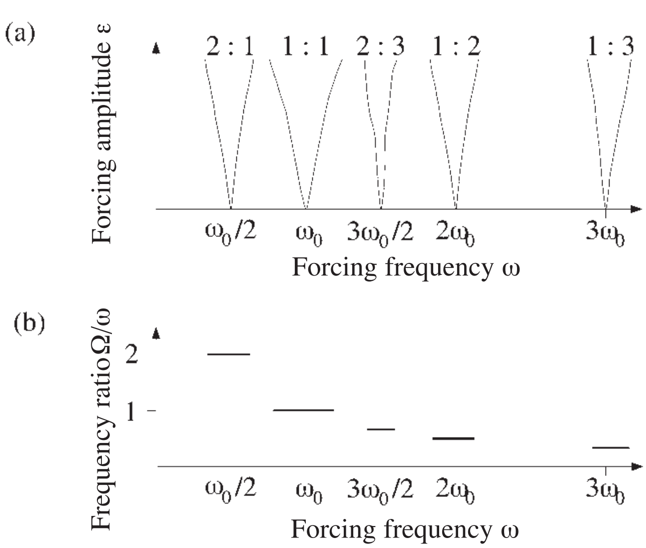

Figure 3. Family of synchronization regions, or Arnol'd tongues (schematically). The numbers on top of each tongue indicate the order of locking; e.g. 2:3 means that the relation $2\omega = 3\Omega$ is fulfilled. (*b*) The Ω/ω versus ω plot for a fixed amplitude of the force (shown by the dashed line in (*a*)) has a characteristic shape, known as the *devil's staircase*. (In this scheme the variation of the frequency ratio between the main plateaus of the staircase is not shown.)

3.3. *Synchronization versus resonance*

At this point we would like to underline the difference to another phenomenon, well known in oscillatory systems—the resonance. Resonance is the response of a system that is non-active, i.e. demonstrates no oscillations without external driving. In other words, here one cannot speak of an adjustment of intrinsic oscillations to an external force, as this force is the source of oscillations. In the case of resonance, if the force is switched off, the oscillations disappear, while self-sustained oscillations continue to exist even without forcing.

As a simple example of this difference let us consider *radio-controlled clocks* and *railway station clocks*. Radio-controlled clocks are self-oscillating, they continue to show time even if there is no radio signal from the high-precision centre. The role of the latter is only to adjust—to correct—the oscillations in order to synchronize them with the time standard. The railway station clocks receive signals from a central clock, and if these signals are absent—they stop; this is an example of resonance, not of synchronization.

Sometimes, when a system is forced very strongly and operates in a highly nonlinear regime, it is hard to distinguish between synchronization and resonance (especially if one can hardly control the forcing like for circadian rhythms); here the observed features at the resonance may be very close to those at the synchronization (e.g. one can observe the devil's staircase-like dependence on the forcing frequency). Nevertheless, the difference becomes evident if the forcing is reduced or switched off.

4. Two and more oscillators

4.1. *Phase dynamics of two coupled oscillators*

Synchronization of two coupled self-sustained oscillators can be described in a similar way. A weak interaction affects only the phases of two oscillators ϕ_1 and ϕ_2, and equation (1) generalizes to

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = \omega_1 + \varepsilon Q_1(\phi_1, \phi_2) , \quad \frac{\mathrm{d}\phi_2}{\mathrm{d}t} = \omega_2 + \varepsilon Q_2(\phi_2, \phi_1) . \quad (4)$$

For the phase difference $\psi = \phi_2 - \phi_1$ one obtains after averaging an equation of the type of (3). Synchronization now means that two non-identical oscillators start to oscillate with the same frequency (or, more generally, with rationally related frequencies). This common frequency usually lies between ω_1 and ω_2. It is worth mentioning that locking of the phases and frequencies implies no restrictions on the amplitudes, in fact the synchronized oscillators may have very different amplitudes and waveforms (e.g. oscillations may be relaxation (pulse-like) or quasiharmonic). We illustrate the effect of mutual synchronization by the classical experiment of Appleton.

4.2. *Example: Appleton's experiment*

Appleton [3] systematically studied synchronization properties of triode generators in a specially designed experiment. He investigated both synchronization by an external force, and mutual synchronization of two coupled non-identical systems. The set-up of the latter experiment is sketched in figure 4. Each generator consists of an amplifier (triode vacuum tube), an oscillatory *LC*-circuit and a feedback implemented by means of the second inductance. This coil, submitting a signal proportional to the oscillation in the *LC*-circuit to the grid, therefore connects the output and input of the amplifier.

There are several ways to couple two triode generators. For example, they can be coupled via a resistor. In his experiments, Appleton placed the coils nearby so that their magnetic fields overlapped, and, hence, the currents in the *LC*-circuits influenced each other.

The experiment was carried out with oscillators having low frequencies of ≈ 400 Hz. The frequency of one system was varied by tuning a capacitor. The effect of detuning was followed in two ways. First, the Lissajous figure was observed on the screen of the oscilloscope indicating the equality of frequency for a certain range of detuning. The phase shift between the synchronized generators was estimated from the Lissajous figures. Second, the beat frequency, i.e. the difference between the frequencies of two generators, was measured in a rather simple way: the beats were so slow that Appleton was able to count them by ear. The beat frequency ($|\Omega_1 - \Omega_2|$) is depicted in figure 5 as a function of the readings of the tuning capacitor (arbitrary units), i.e. as a function of detuning.

We conclude the discussion of mutual synchronization of two coupled systems with two remarks. (i) Similar to

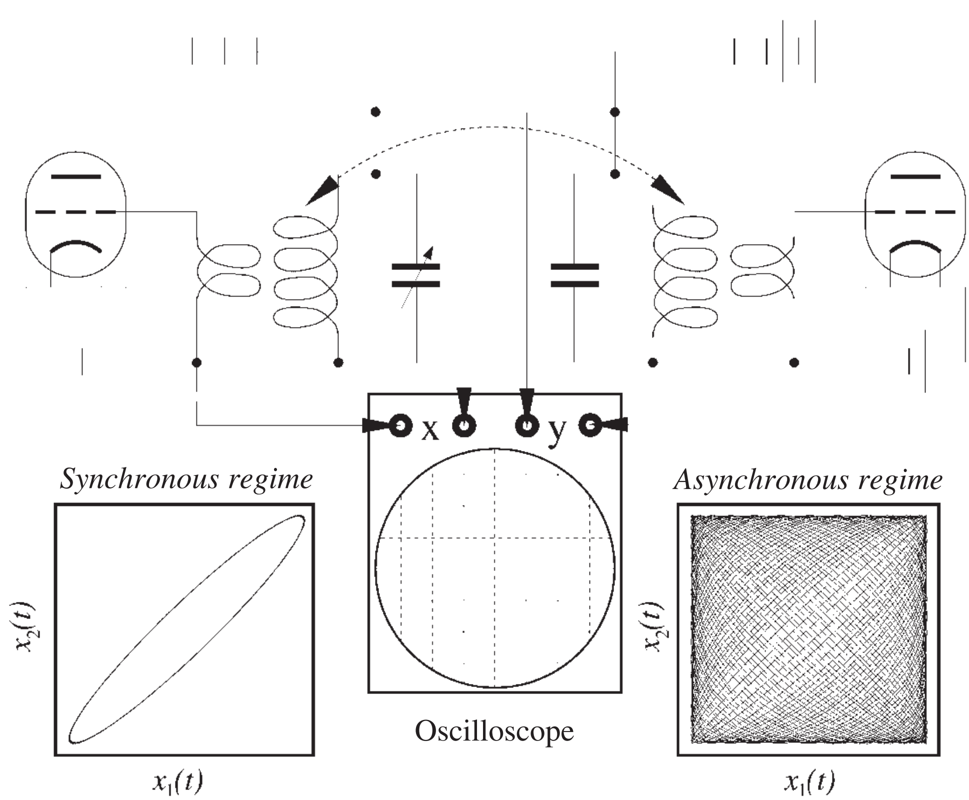

Figure 4. Set-up of the triode generator experiment by E. V. Appleton [3]. The dashed arc indicates that the coils were placed in such a way that their magnetic fields overlapped, thus coupling the generators. Synchronization can be identified from Lissajous figures $x_1(t)$ versus $x_2(t)$ observed on the oscilloscope. The left figure corresponds to a synchronous state. The periods of oscillations are identical, therefore the plot is a closed curve. The right figure corresponds to an asynchronous, quasiperiodic state. The point never returns to the same coordinates and the unclosed curve fills the region.

the case of periodic forcing, synchronization of order $n:m$ is also possible. Examples are synchronization of running and breathing in mammals and locking of breathing and wing beat frequencies in flying birds (see [8] for citations and further examples). (ii) Depending on the parameters of coupling, two oscillators can be locked almost in-phase or almost in anti-phase (figure 6). Moreover, varying the parameters of coupling one can observe transition between different synchronous states. As an example we mention the effect observed by J. A. S. Kelso and later studied by Haken, Kelso and co-workers (see [13, 14] for references and details). In their experiments, a subject was instructed to perform an anti-phase oscillatory movement of index fingers and gradually increase the frequency. It turned out that at higher frequency this movement becomes unstable and a rapid transition to the in-phase mode is observed.

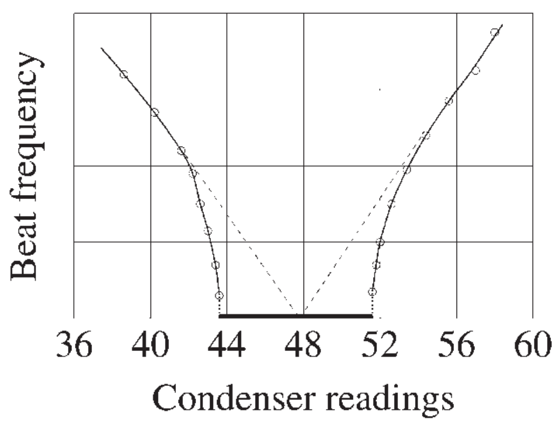

Figure 5. Results of the experiment with coupled triode generators. If no synchronization effects are taken into account, the theoretical change of the beat frequency with capacity is indicated by the dotted lines. The continuous lines are drawn through the experimental values. The synchronization region (for a fixed coupling strength) is shown by the horizontal bar. From [3].

4.3. *Synchronization in a lattice. An example: laser arrays*

In many natural situations more than two oscillating objects interact. If two oscillators can adjust their rhythms, we can expect that a large number of systems could do the same. One example has already been mentioned in section 1: a large population of flashing fireflies constitutes what we can call an ensemble of mutually coupled oscillators, and can flash in synchrony. A firefly communicates via light pulses with all other insects in the population. In this case one speaks of *global* (all-to-all) coupling. There are other situations when oscillators are ordered into chains or lattices, where each element interacts only with its several neighbours. Such structures are common for man-made systems, examples are laser arrays and series of Josephson junctions, but may also be encountered in nature. So, mammalian intestinal smooth muscle may be electrically regarded as a series of loosely coupled pacemakers having different intrinsic frequencies. Their activity triggers the muscle contraction. Experiments show that neighbouring sources often adjust their frequencies and form synchronous clusters. We first discuss synchronization effects in large spatially ordered ensembles of oscillators, and then proceed with ensembles of globally coupled elements.

The simplest example of a regular spatial structure is a chain, where each element interacts with its nearest neighbours. Generally, both the spatial ordering and the interaction is more complicated, e.g. the oscillators can interact with several neighbours. For illustration we consider experiments with a multibeam CO_2 wave guide laser consisting of 61 glass tubes in a honeycomb arrangement conducted by Antyukhov *et al.* [16], see figure 7 (*a*).

Synchronization of laser arrays is a basic tool used to create a source of high-intensity radiation. This can be achieved by coupling the lasers in a linear array so that they either interact with their nearest neighbours or with all other elements in the structure. In the experiments of Antyukhov *et al.* the lasers were coupled by means of an

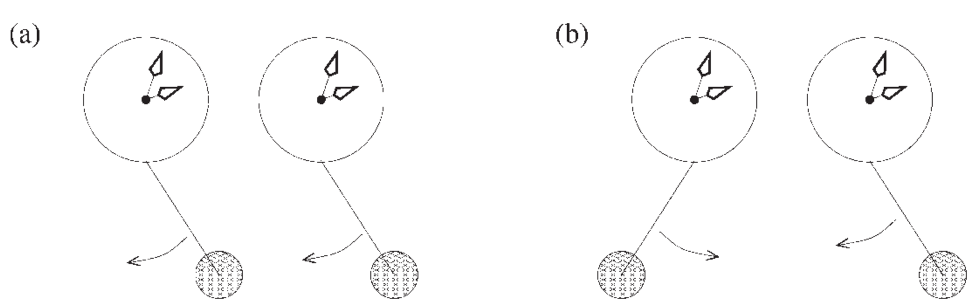

Figure 6. Two coupled oscillators depending on the way the coupling is introduced, may be synchronized almost in-phase, i.e. (*a*) with the phase difference $\phi_2-\phi_1\approx 0$, or (*b*) in anti-phase, with $\phi_2-\phi_1\approx\pi$. The discoverer of synchronization, Christiaan Huygens, observed synchronization in anti-phase. Later experiments, reported in [15] demonstrated that both anti-phase and in-phase synchronous regimes of coupled clocks are possible.

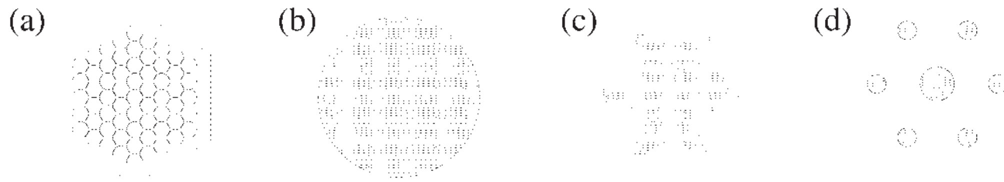

Figure 7. Synchronization in a lattice of 61 laser oscillators arranged in a honeycomb (*a*). For low coupling, the intensity in the focal spot of the array output is approximately uniform (*b*). Stronger coupling results in synchronization which manifests itself as a spatially ordered intensity distribution (*d*). The case of intermediate coupling is shown in (*c*). Schematically drawn after [16].

external coupling mirror. The results, presented in figures 7 (*c*) and (*d*), clearly indicate synchronization. Indeed, if the lasers were not synchronized then the radiation intensity in the far-field zone, as the sum of non-coherent oscillations, would be spatially uniform. The non-uniform distribution in figure 7 appears because phase locking sets in; this is a typical interference image.

4.4. *Formation of clusters. An example: electrical activity of mammalian intestine*

Suppose the oscillators have slightly different frequencies that are somehow distributed over the ensemble. What kind of collective behaviour can be expected in such a population? Certainly, if the interaction is very weak, there will be no synchronization so that all the systems will oscillate with their own frequencies. We can also imagine that sufficiently strong coupling can synchronize the whole ensemble, provided the natural frequencies are not too different; this expectation is confirmed by the above considered example. For an intermediate coupling or a broader distribution of natural frequencies of elements we can expect some partially synchronous states. Indeed, it may be that several oscillators synchronize and oscillate with a common frequency, whereas their neighbours have their own, different, frequencies. There may appear several such groups, or *clusters* of synchronized elements. We illustrate these expectations with a description of an experiment with mammalian intestine.

Intestine consists of layers of muscle fibres supporting propagation of travelling waves of electrical activity that run from the oral to aboral end. These waves trigger the waves of muscular contraction. Diamant and Bortoff [17] experimentally investigated the distribution of frequencies of electrical activity along the intestine. The majority of experiments was performed on cats, with most of the basic observations being repeated in dogs and rhesus monkeys. Each frequency determination represents the average over a 5 min period.

From the physical viewpoint, if we consider electrical activity only, the intestine can be regarded as a one-dimensional continuous medium, where each point is oscillatory. Indeed, Diamant and Bortoff [17] found that

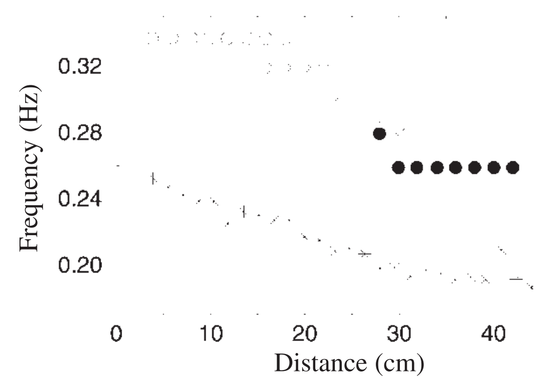

Figure 8. Synchronous clusters in a mammalian intestine. The frequency of slow electrical muscle activity plotted as a function of distance along the intestine typically shows a step-wise structure. (The distance is measured from the ligament of Treitz.) The ○, ◇ and ● symbols represent the three consecutive (at 30 min intervals) measurements of frequency along the intestine *in situ*; for each measurement the electrodes were re-positioned. The stars show the frequency of the consecutive segments of the same intestine *in vitro*. From [17].

if a section of intestine is cut into pieces, each piece is capable of maintaining nearly sinusoidal oscillations of a constant frequency. There exists an approximately linear gradient of these frequencies, so that they decrease from the oral to aboral end. Measured *in situ* and plotted as a function of the coordinate along the intestine, the frequency of electrical activity typically exhibits plateaus (figure 8). This indicates the existence of clusters of synchronous activity [18]. Within each cluster the phase shift between the oscillations increases with the spatial coordinate (in accordance with the gradient of frequencies in the pieces of intestine); neighbouring clusters are separated by regions of modulated oscillations, or beats.

4.5. *Globally coupled oscillators*

Now we study synchronization phenomena in large ensembles of oscillators, where each element interacts with all others. This is usually denoted as *global*, or all-to-all coupling. As a representative example we have already mentioned synchronous flashing in a population of fireflies.

A very similar phenomenon, self-organization in a large applauding audience, has probably been experienced by every reader of this article, e.g. in a theatre. Indeed, if the audience is large enough, then one can often hear a rather fast (several oscillatory periods) transition from noise to a rhythmic, nearly periodic, applause. This happens when the majority of the public applaud in unison, or synchronously.

The each-to-each interaction is also denoted as a *mean field* coupling. Indeed, each firefly is influenced by the light field that is created by the whole population. Similarly, each applauding person hears the sound that is produced by all the other people in the hall. Thus, we can say that all elements are exposed to a common force. This force results from the summation of outputs of all elements. Let us denote these outputs by $x_k(t)$, where $k = 1, \ldots, N$ is the index of an oscillator and N is the number of elements in the ensemble; x can be a variation of light intensity or of the acoustic field around some average value, or, generally, any other oscillating quantity. Then the force that drives each oscillator is proportional to $\sum_k x_k(t)$. It is conventional to write this proportionality as $\varepsilon N^{-1} \sum_k x_k(t)$, so that it includes the normalization by the number of oscillators N. The term $N^{-1} \sum_k x_k(t)$is just an arithmetic mean of all oscillations, which explains the origin of the term 'mean field coupling'.

Thus, the oscillators in a globally coupled ensemble are driven by a common force. Clearly, this force can entrain many oscillators if their frequencies are close. The problem is that this force (the mean field) is not predetermined, but arises from interaction within the ensemble. This force determines whether the systems synchronize, but it itself depends on their oscillation—it is a typical example of self-organization [19]. To explain qualitatively the appearance of this force (or to compute it, as is done in [8, 20]) one should consider this problem self-consistently.

First, assume for the moment that the mean field is zero. Then all the elements in the population oscillate independently, and their contributions to the mean field nearly cancel each other. Even if the frequencies of these oscillations are identical, but their phases are independent, the average of the outputs of all elements of the ensemble is small if compared with the amplitude of a single oscillator. (According to the law of large numbers, it tends to zero when the number of interacting oscillators tends to infinity; the fluctuations of the mean field are of the order $N^{-1/2}$.) Thus, the asynchronous, zero mean field state obeys the self-consistency condition.

Next, to demonstrate that synchronization in the population is also possible, we suppose that the mean field is non-vanishing. Then, naturally, it entrains at least some part of the population, the outputs of these entrained elements sum up coherently, and the mean field is indeed non-zero, as assumed. Which of these two states—synchronous or asynchronous—is realized, or, in other words, which one is stable, depends on the strength of interaction between each pair and on how different the elements are. The interplay between these two factors, the coupling strength and the distribution of the natural frequencies, also determines how many oscillators are synchronized, and, hence, how strong the mean field is.

We discuss now how the synchronization transition occurs, taking the applause in an audience as an example (experimental study of synchronous clapping is reported in [21]). Initially, each person claps with an individual frequency, and the sound they all produce is noisy†. As long as this sound is weak, and contains no characteristic frequency, it does not essentially affect the ensemble. Each oscillator has its own frequency ω_k, each person applauds and each firefly flashes with its individual rate, but there always exists some value of it that is preferred by the majority. Definitely, some elements behave in a very individualistic manner, but the main part of the population tends to be 'like the neighbour'. So, the frequencies ω_k are distributed over some range, and this distribution has a maximum around the most probable frequency. Therefore, there are always at least two oscillators that have very close frequencies and, hence, easily synchronize. As a result, the contribution to the mean field at the frequency of these synchronous oscillations increases. This increased component of the driving force naturally entrains other elements that have close frequencies, this leads to the growth of the synchronized cluster and to a further increase of the component of the mean field at a certain frequency. This process develops (quickly for relaxation oscillators, relatively slow for quasilinear ones), and eventually almost all elements join the majority and oscillate in synchrony, and their common output—the mean field—is not noisy any more, but rhythmic.

The physical mechanism we described is known as the Kuramoto self-synchronization transition [22]. The scenario of this transition does not depend on the origin of the oscillators (biological, electronic, etc.) or on the origin of interaction. In the above presented examples the coupling occurred via an optical or acoustic field. Global coupling of electronic systems can be implemented via a common load; in this case the voltage applied to individual systems depends on the sum of the currents of all elements. (As an example we mention an array of Josephson junctions.) Chemical oscillators can be coupled via a common medium, where concentration of a reagent depends on the reaction in each oscillator and, on the other hand,

†Naturally, the common (mean) acoustic field is non-zero, because each individual oscillation is always positive; the intensity of the sound cannot be negative, it oscillates between zero and some maximal value. Correspondingly, the sum of these oscillations contains some rather large constant component, and it is the deviation from this constant that we consider as the oscillation of the mean field and which is small. Therefore, the applause is perceived as some noise of almost constant intensity.

influences these reactions (figure 9 (*a*)). The Kuramoto transition can be treated as a non-equilibrium phase transition, the mean oscillating field serving as an order parameter (figure 9 (*b*)).

The scenarios of the Kuramoto transition may also be more complicated, e.g. if the distribution of the individual frequencies ω_k has several maxima. Then several synchronous clusters can be formed; they can eventually merge or coexist. Clustering can also happen if, say, the strength of interaction of an element of the population with its nearest (in space) neighbours is larger than with those that are far away.

4.6. *An example: synchronization of glycolytic oscillations in a population of yeast cells*

Suppose that some reagent is produced at a chemical reaction and that the reaction rate depends on the concentration of that reagent. If the medium is constantly stirred, then the concentration is spatially homogeneous and is determined by all oscillators (figure 9 (*a*)). Hence, it can be considered as the mean field.

Under certain conditions sustained glycolitic oscillations can be observed in a suspension of yeast cells in a stirred cuvette (see [23] and references therein). The oscillations can be followed by measuring the fluorescence of one of the metabolites, namely nicotinamide adenine dinucleotide (NADH). Richard *et al.* [23] considered two alternative hypotheses on the origin of macroscopic NADH oscillation. First, one can assume that it arises due to the summation of simultaneously induced oscillations of individual cells. Indeed, the glycolytic oscillation is induced by adding glucose to the starved cell culture. As the cells are not too different and begin to oscillate at the same instant of time, one can expect that at least for some time the cells remain approximately in phase. The alternative hypothesis is synchronization of chemical oscillators globally coupled via the common medium. Richard *et al.* [23] confirmed the second alternative by performing the following experiment. They initiated glycolytic oscillations in two populations of cells, so that the phase shift between them was about π, and then mixed these two populations together. If the cells were oscillating independently, the oscillations would cancel each other. If the cells are coupled, one expects synchronization to occur in the mixture of two (previously synchronous) populations. This latter effect was indeed observed in the experiments: immediately after mixing there was no oscillation of NADH, but it re-appeared after approximately 3 min (the characteristic oscillation period is about 40 s). Next, Richard *et al.* [23] demonstrated that the extracellular free acetaldehyde concentration oscillates at the frequency of intracellular glycolytic oscillations. They concluded that this chemical plays the role of the communicator between the cells, or what we call the mean field. This conclusion is confirmed by two facts. First, the yeast cells respond to acetaldehyde pulses. Added during oscillations, acetaldehyde induces a phase shift that depends on the concentration of the addition (i.e. strength of the pulse) and its phase. Second, the acetaldehyde is secreted by oscillating cells.

5. Chaotic systems

Nowadays it is well known that self-sustained oscillators, e.g. nonlinear electronic devices, can generate rather complex, *chaotic* signals. Most oscillating natural systems also exhibit rather complex behaviour. Recent studies have revealed that such systems, being coupled, are also capable to undergo synchronization. Certainly, in this case we have to specify this notion more precisely, because it is not obvious, how to characterize the rhythm of a chaotic oscillator. It is helpful that sometimes chaotic waveforms are rather simple, so that a signal is 'almost periodic'; we can consider it as consisting of similar cycles with varying amplitude and period (which can be roughly defined as the time interval between the adjacent maxima). Taking a large

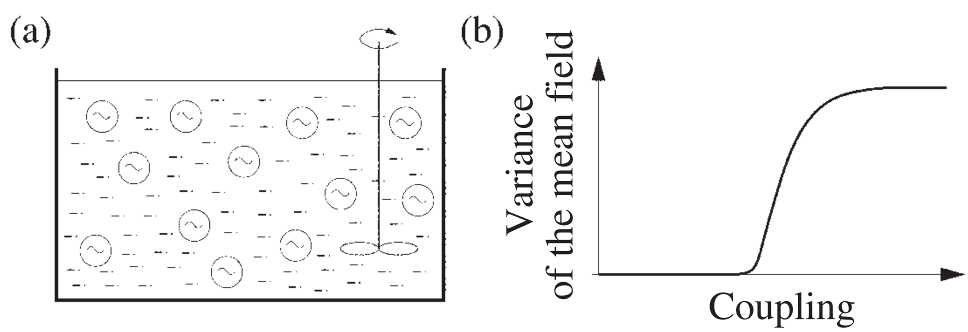

Figure 9. (*a*) Chemical oscillators in a stirred tank are globally coupled via a common medium. This is an example of mean field coupling, with the concentration of a certain reagent playing the role of the mean field. (*b*) Variance of the mean field depends on the coupling between each pair of oscillators and can be considered as an order parameter of the synchronization transition: if the coupling is below some critical value then the mean field is nearly zero; if the coupling exceeds some threshold then a macroscopic mean field appears in the population due to self-synchronization of its elements.

time interval τ we can count the number of cycles within this interval N_τ, compute the *mean frequency*

$$\omega_0 = \lim_{\tau \to \infty} 2\pi \frac{N_\tau}{\tau} , \quad (5)$$

and take it for characterization of the chaotic oscillatory process.

With the help of the mean frequencies we can describe the collective behaviour of interacting chaotic systems in the same way as we did for periodic oscillators. If the coupling is large enough (e.g. in the case of resistively coupled electronic circuits it means that the resistor should be sufficiently small), the mean frequencies of two oscillators become equal, and one can obtain a synchronization region, exactly as in the case for periodic systems. It is important that coincidence of mean frequencies does not imply that the signals coincide as well. It turns out that weak coupling does not affect the chaotic nature of both oscillators; the amplitudes remain irregular and uncorrelated, whereas the frequencies are adjusted in a fashion that allows us to speak of the phase shift between the signals. This regime is denoted as *phase synchronization* of chaotic systems.

Very strong coupling tends to make the states of both oscillators identical. It influences not only the mean frequencies but also the chaotic amplitudes. As a result, the signals coincide (or nearly coincide) and the regime of *complete synchronization* sets in. Also known are the so-called generalized and master–slave synchronizations (see, e.g. [8] and references therein); these effects are related to the complete synchronization of chaos.

5.1. *Phase synchronization. An example: electrochemical system*

Phase synchronization of chaotic systems is mostly close to the classical locking phenomena. It is based on the observation that many chaotic self-sustained oscillators admit determination of the instantaneous phase and the corresponding mean frequency. Below we illustrate this with an electrochemical oscillator, experimentally investigated by Kiss and Hudson [24]. An autonomous system demonstrates chaotic dynamics; its three-dimensional representation in delay coordinates is shown in figure 10 (*a*). The strange attractor looks like a smeared limit cycle; this allows one to introduce the phase as a variable that gains 2π with each rotation of the phase space trajectory.

There also exists an alternative way to introduce the phase of the observed chaotic signal $x(t)$, namely by virtue of the Hilbert transform. With this technique one constructs from an oscillatory observable $x(t)$ a complex *analytic signal*

$$\zeta(t) = s(t) + \mathrm{i}s_{\mathrm{H}}(t) = A(t) \exp[\mathrm{i}\phi(t)] . \quad (6)$$

Here the function $s_{\mathrm{H}}(t)$ is the Hilbert transform of $s(t)$

$$s_{\mathrm{H}}(t) = \pi^{-1}\mathrm{PV} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau}\, \mathrm{d}\tau , \quad (7)$$

and PV means that the integral is taken in the sense of the Cauchy principal value. The instantaneous phase $\phi(t)$ (and, if needed, instantaneous amplitude $A(t)$) of the signal $s(t)$ can be thus uniquely defined from (6). A harmonic oscillation $s(t) = A \cos \omega t$ is often represented in the complex form as $A \cos \omega t + \mathrm{i}A \sin \omega t$. It means that the real oscillation is complemented by the imaginary part which is delayed in phase by $\pi/2$, that is related to $s(t)$ by the Hilbert transform. The analytic signal is the direct and natural extension of this technique, as the Hilbert transform performs the $-\pi/2$ phase shift for every spectral component of $s(t)$. The Hilbert transform can be easily implemented numerically (see [8] for citations and practical hints) and therefore is effectively used in experimental studies of synchronization.

The Hilbert transform can be considered as an alternative way to construct a two-dimensional projection of the trajectory. Quite often such a projection has a rather simple structure, with trajectories rotating around some centre; for the electrochemical oscillator this projection is depicted in figure 10 (*c*). The instantaneous phase obtained as an angle on the 'signal–Hilbert transform' plane is shown in figure 10 (*d*). The evolution of this phase is rather similar to that of the phase of the periodic oscillator (1), the only difference being that it grows non-uniformly. Indeed, due to chaos one can hardly expect that the periods of rotation along different loops of the chaotic trajectory would be exactly equal: in general they depend on the amplitude, and the latter is chaotic. Due to this one can characterize the phase dynamics of a chaotic oscillator as a composition of uniform growth with the average frequency ω_0 (see equation (5)) and of a random walk. (This feature makes the synchronization properties of chaotic systems close to the properties of noisy periodic oscillators.) The average frequency corresponds to the peak in the spectrum of the chaotic signal (see figure 10 (*b*); for the electrochemical oscillator under consideration it is 1.325 Hz). The intensity of the random walk corresponds to the width of this peak, or, in other words, characterizes the suitability of the oscillator to serve as a clock.

Having introduced the phase and the frequency for chaotic oscillators we can characterize their synchronization. Now it becomes rather obvious that the effects of phase locking and frequency entrainment, known for periodic self-sustained oscillators, can be observed for chaotic systems as well.

The simplest case is the phase locking by an external periodic signal. When the electrochemical oscillator (figure 10) is driven by a signal with a frequency Ω close to ω_0, the

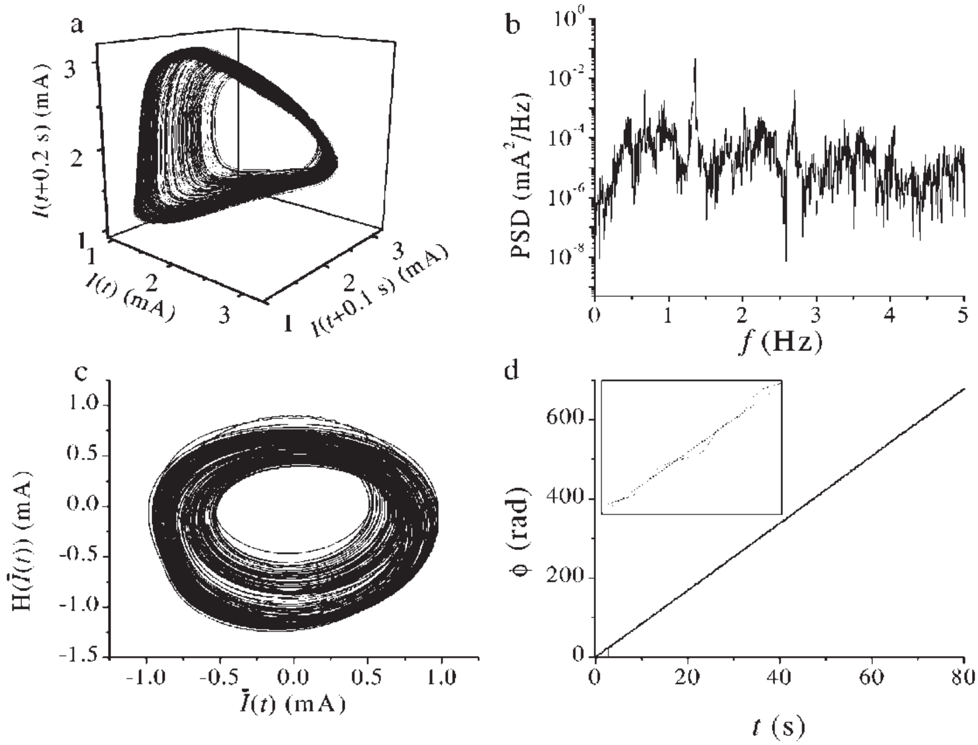

Figure 10. A chaotic electrochemical oscillator, its phase and frequency. Panels (*a*) and (*c*) show two versions of the phase portrait. In (a) the delayed coordinates are used, which is a standard way for the phase space reconstruction. For the determination of the instantaneous phase of the chaotic system the projection on the 'signal – Hilbert transform' plane (*c*) is more suitable. The phase defined as an angle on the plane (*c*) is shown in panel (*d*). It grows nearly uniformly, with the frequency $\omega_0 = 1.325$ Hz, but more detailed examination reveals non-uniformity in the growth due to chaos (inset in panel (*c*)). The average frequency can also be extracted from the power spectral density (PSD) of the signal (panel (*b*)). From [24].

forcing affects the evolution of the phase, and the observed (mean) frequency ω becomes adjusted to the external one. The results of the experiment for different amplitudes of the forcing (figure 11) allow one to define the synchronization region, where the frequency of the system is completely entrained by the external force, see figure 12. This region is a complete analogue of the synchronization regions (Arnol'd tongues) for periodic oscillators.

It is important to emphasize that the chaos itself is not suppressed by the external force. What happens is not a disappearance of chaos, but an adjustment of the mean oscillation frequency. Chaos may be destroyed by a strong force, but a small forcing affects only the phase, entraining the frequency of its rotation.

Mutual phase synchronization of chaotic oscillators is also quite similar to the classical case. To demonstrate this, one can couple two chaotic electrochemical oscillators. The quantities to be observed are the oscillator phases: for each oscillator one has to extract these phases from the portraits such as shown in figures 10 (*a*) and (*c*). Then the calculation of the phase difference and observed frequencies Ω_1, Ω_2 characterizes the entrainment. For a large enough coupling and for small mismatch of natural frequencies one observes that frequencies become equal, $\Omega_1 = \Omega_2$, like in the experiments with periodic oscillators described in section 4.2.

Furthermore, synchronization transition in a population of chaotic oscillators can be observed as well. The mechanism here is the same as described in section 4.5 above: due to interaction, some oscillators become entrained and start to oscillate with the same frequency, the fields of these oscillators sum coherently, and the resulting mean field maintains synchrony. Considering chaotic oscillators we additionally have to take into account that only the phases of the oscillators are adjusted, whereas the individual oscillations remain chaotic. Thus, the mean field arises due to contributions of mutually entrained oscillators with nearly equal phases, but with different chaotic amplitudes. Summation of these contributions leads to a periodic field with some average amplitude—chaos is 'washed out' due to the averaging over the ensemble. As a result, the synchronization transition in an ensemble of coupled chaotic oscillators manifests itself as the appearance of a periodic macroscopic mean field, while each individual oscillator remains chaotic. Such synchronization transition has been observed in experiments with an

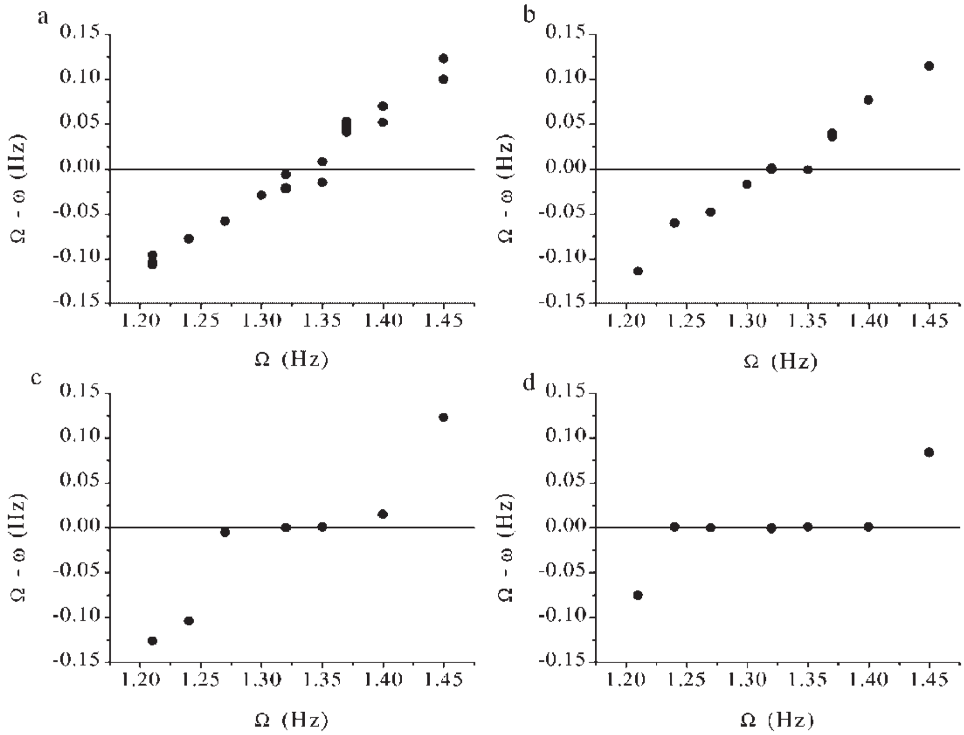

Figure 11. The difference between the observed frequency ω of the chaotic oscillator and that of the external force, for different forcing amplitudes ((*a*) 0 mV, (*b*) 6.6 mV, (*c*) 13.2 mV, (*d*) 16.5 mV). For large amplitudes the synchronization region, where $\omega = \Omega$, is large (compare with the results of the classical experiment of Appleton, figure 5). From [24].

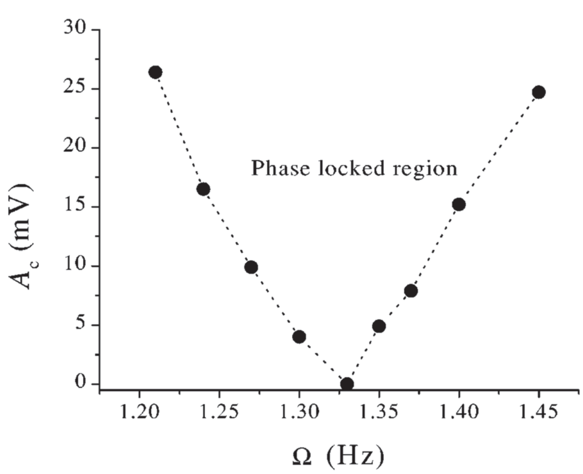

Figure 12. The synchronization region for the periodically driven chaotic oscillator on the 'frequency of the driving–amplitude of the driving' plane. From [24].

ensemble of 64 electrochemical chaotic oscillators. Figure 13 (*a*) shows the chaotic oscillations of two elements. Figure 13 (*b*) depicts the dependence of the amplitude of the mean field on the coupling constant K. The mean field starts to grow at $K \approx 0.05$; this is the synchronization threshold. For smaller couplings the mean field does not vanish, but its amplitude has values ≈ 0.2. This happens due to the finite size effect: if the oscillators are completely uncorrelated, the mean field, as the sum of N statistically independent contributions, has the amplitude of order $N^{-1/2}$: the fluctuations in a relatively small ensemble do not vanish.

5.2. *Complete synchronization. Example: coupled lasers*

Strong mutual coupling of chaotic oscillators leads to their *complete synchronization* when two or more chaotic systems have exactly the same states, and these identical states vary irregularly in time. Converse to phase synchronization, it can be observed in any chaotic system, not necessarily autonomous, and in particular in periodically driven oscillators or in discrete-time systems (maps). In fact, this phenomenon is not close to the classical synchronization of periodic oscillations, as here we do not have adjustment of rhythms. Instead, complete synchronization means suppression of differences in coupled *identical* systems. Therefore, this effect cannot be described as entrainment or locking; it is closer to the onset of symmetry. Maybe another word instead of 'synchronization' would better serve for underlining this difference; we will follow the nowadays accepted terminology, using the adjective 'complete' to avoid ambiguity.

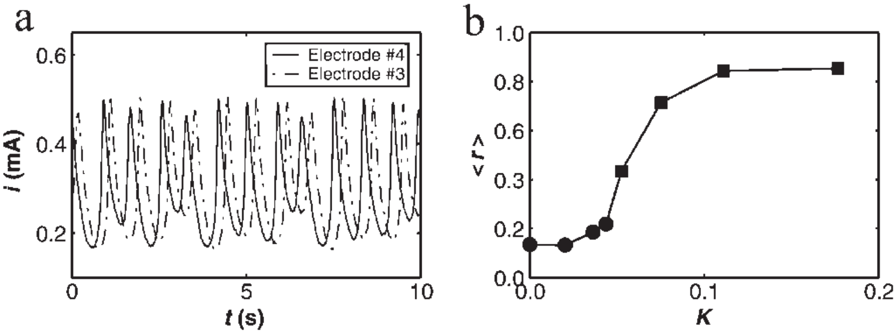

Figure 13. Illustration of the synchronization transition in an ensemble of globally coupled chaotic electrochemical oscillators. In panel (*a*) two time series of two elements are shown. In panel (*b*) the dependence of the mean field on the coupling *K* demonstrates the transition at $K \approx 0.05$. From [25].

The main precondition for complete synchronization is that the interacting systems are identical, i.e. they are described by exactly the same equations of motion. This identity implies that if the initial states of these systems are equal, then during the evolution they remain equal at all times. However, in practice this coincidence of states will be realized only if such a regime is stable, i.e. if it is restored after a small violation. This imposes a condition on the strength of the coupling between the systems.

To be more concrete in our discussion, let us consider a coupled system of the type

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \varepsilon(\mathbf{y} - \mathbf{x}) \,, \quad \frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}) + \varepsilon(\mathbf{x} - \mathbf{y}) \,. \qquad (8)$$

Here $\mathbf{x}$ and $\mathbf{y}$ are two identical systems, described by the same equations $\mathbf{F}$, and we will assume that the solutions are chaotic. ε is the coupling parameter and the corresponding terms on the right-hand side describe a so-called diffusive coupling, which tends to equalize the states of two systems (this can be easily seen if one sets $\mathbf{F} = 0$, then the difference $\mathbf{y} - \mathbf{x}$ decreases in time with the rate 2ε).

While the coupling tends to equalize the states of two systems, another mechanism prevents this. This mechanism is the inherent for chaos sensitive dependence to initial conditions. Suppose that $\varepsilon = 0$, then we have two uncoupled identical systems; they can be regarded as two realizations of one system with different initial conditions. Because chaotic motions sensitively depend on initial conditions (this phenomenon is often called the 'Butterfly effect'), the values $\mathbf{y}(t)$ and $\mathbf{x}(t)$ will differ significantly after some time, even if $\mathbf{y}(0) \approx \mathbf{x}(0)$.

Summarizing, we see two counterplaying tendencies in the diffusive interaction of two identical chaotic systems: intrinsic chaotic instability tends to make the states of the systems different, while coupling tends to equalize them. As a result, there exists a critical value of coupling ε_c, such that for stronger coupling a completely synchronized state $\mathbf{y}(t) = \mathbf{x}(t)$ sets in. For this regime the coupling term in (8) vanishes, and, hence, each of the systems vary chaotically with time as if they were uncoupled. Thus, the complete synchronization is a threshold phenomenon: it occurs only when the coupling exceeds some critical level, proportional to the largest Lyapunov exponent of the individual system. Below the threshold, the states of two chaotic systems are different but close to each other.

We illustrate the theoretical consideration by the results of Roy and Thornburg [26], who observed synchronization of chaotic intensity fluctuations of two Nd:YAG lasers with modulated pump beams. The coupling was implemented by overlapping the intracavity laser fields and varied during the experiment. For strong coupling, the intensities became identical, although they continued to vary in time chaotically (figure 14).

6. Conclusions and outlook

In spite of the long history, theory of synchronization remains a rapidly developing branch of nonlinear science. Among the ongoing directions, not discussed in this article, we mention synchronization in spatially-distributed systems and synchronization-like phenomena in stochastic and excitable systems [27].

Recent theoretical development has been strongly influenced by interdisciplinary studies, especially by widely growing applications to biological and medical problems. It turns out that synchronization is very frequently encountered in live systems [8, 28]. In particular, it is believed that the mechanism of the Kuramoto transition plays an important role in dynamics of neural ensembles and is responsible for the emergence of such severe pathologies as epilepsies and Parkinson's disease. A popular paradigmatic model, analysed in this context, is a system of pulse-coupled integrate-and-fire oscillators, see e.g. [29]. Another direction of research is related to attempts to desynchronize undesirable, pathological collective rhythms and to develop

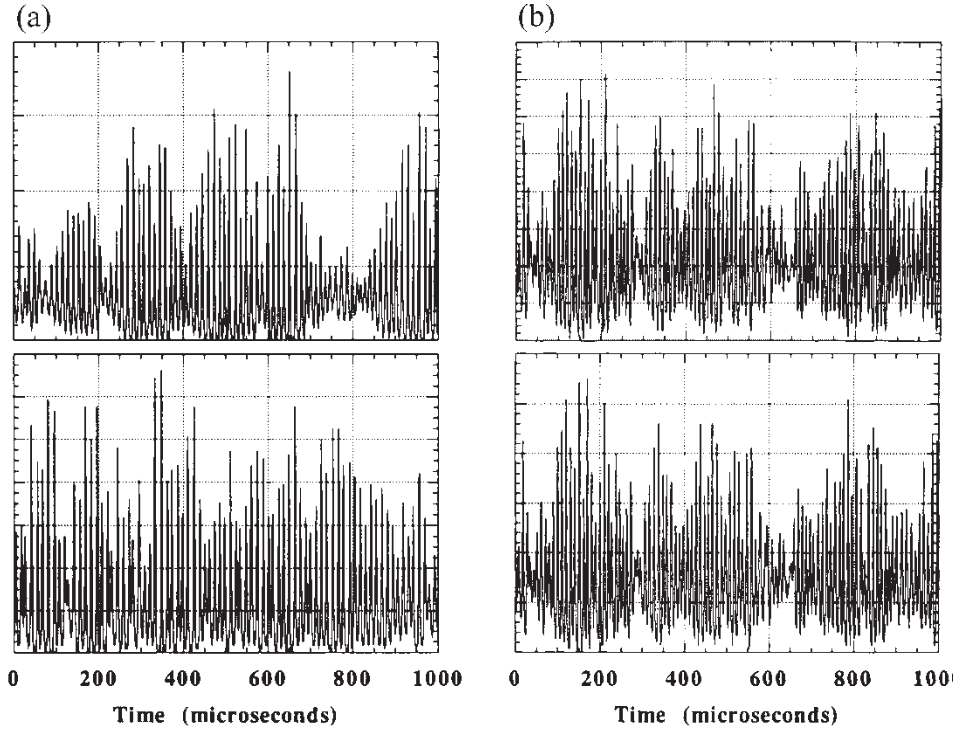

Figure 14. Complete synchronization in coupled lasers. (*a*) Under-threshold coupling: intensities (arbitrary units) of uncoupled lasers fluctuate chaotically, and the time course is different, although both lasers experience the same pump modulation. (*b*) Under strong coupling, beyond the synchronization threshold, both lasers remain chaotic, but now the oscillations are nearly identical, i.e. complete synchronization sets in. From [26].

in this way a therapeutic tool [30]. Finally we mention that ideas from the synchronization theory are used in the analysis of multivariate experimental data. The goal of such an analysis is to detect weak interaction between oscillatory systems, e.g. to reveal a coordination between respiratory and cardiac rhythms in humans [31] or localize the source of pathological brain activity in Parkinson's disease [32, 33].

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