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Reversibility vs. synchronization in oscillator lattices

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Abstract

We consider the dynamics of a lattice of phase oscillators with a nearest-neighbor coupling. The clustering hierarchy is described for the case of linear distribution of natural frequencies. We demonstrate that for small couplings prior to the appearance of the first cluster the dynamics is quasi-Hamiltonian: the phase volume is conserved in average, and the spectrum of the Lyapunov exponents is symmetric. We explain this unexpected for a dissipative system phenomenon using the concept of reversibility. We show that for a certain coupling a smooth transition from the quasi-Hamiltonian to the dissipative dynamics occurs, which is a novel type of chaos–chaos transition.

Keywords: Reversibility; Synchronization; Phase dynamics; Oscillator lattice

1. Introduction

Coupled oscillators appear in various fields of science: in biology, electronics, chemical reactions, optics, acoustics. One of the main effects here is synchronization, resulting in the adjusting of the frequencies of the oscillators. Although Huygens first discovered this phenomenon already in 17th century, and a theory of synchronization was developed by Appleton and Van der Pol in 1920s, there are many aspects of dynamics of coupled oscillators that are a subject of a current research. In particular, synchronization transitions in lattices of coupled oscillators attracted a lot of attention. Such lattices of elements with nearest-neighbor coupling appear, e.g., in laser arrays [1], Josephson junctions [2], phase-locked loops [3,4], and even in piano strings [5]. Although particular dynamical systems describing these lattices are quite different, there are many general features that can be described already in the framework of the simplest model of coupled phase equations [6–9]. Indeed, because the phase of a self-sustained oscillator is free and the amplitude is relaxing to a particular value, small coupling influences the phases only.

In the case of many coupled oscillators, between the limiting cases of full synchronization (when all oscillators have the same frequency) and complete desynchronization (all the frequencies are different) one encounters regimes of partial synchronization. For a lattice such a state appears in the form of synchronization clusters, when neighboring or even non-neighboring oscillators form groups having the same frequency. In general, the transition from a non-synchronous to a synchronous state can be described as formation and merging of clusters. Particular features of this process depend on the coupling and on the distribution of natural frequencies. Typically, one assumes that the coupling is attracting, i.e.

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it tends to equalize the phases of interacting oscillators. More variative is the distribution of frequencies, here two types of models attracted special interest. In papers [10–13] a random distribution of natural frequencies was considered and the clustering hierarchy has been calculated. Here one can generally make only statistical predictions on the transition. In [13–15] it has been demonstrated that clusters can be observed in lattices of chaotic oscillators as well.

Most close to our present study is the paper [7], where a linear distribution of the natural frequencies in a one-dimensional lattice have been studied. It was motivated by experimental observations of formation of clusters in mammalian intestinal smooth muscle [16]. The attention of Ermentrout and Kopell [7] was mainly restricted to the final stage at large couplings, when two or few clusters merge to produce the fully synchronized state. In this paper, we consider the same problem as in [7], but we follow the transition from small to large couplings. Moreover, our main attention will be devoted to a non-trivial state at small couplings prior to the formation of the first cluster. We demonstrate that in this state the dynamics of the dissipative lattice is quasi-Hamiltonian, i.e. the phase volume is conserved (in average) and the lattice demonstrates typical for Hamiltonian systems coexistence of chaotic regimes and quasi-periodic tori. To the best of our knowledge, it is the first observation of quasi-Hamiltonicity for coupled oscillator arrays. Our explanation of this property is in the reversibility of the oscillator lattice due to high symmetry of the frequency distribution. We note that the quasi-Hamiltonian dynamics of dissipative systems due to the reversibility have been already discussed for low-dimensional laser models [17] and for Josephson junctions [18,19]. The treatment is especially difficult if the system under consideration is high dimensional. Thus, our study is more rigorous for small lattices than for large ones.

The plan of the paper is as follows. We introduce the model of a lattice of phase oscillators in Section 2. Here we present numerical results illustrating the formation of clusters. We give also the numerical evidence of the quasi-Hamiltonian dynamics: the symmetry of the Lyapunov spectrum and the vanishing of the phase volume divergence on time average. Reversibility of the system is discussed in Section 3. We show that it is related to particular symmetries of the coupling function and of the distribution of natural frequencies. To support these findings, we describe in Section 4 what happens if these symmetries are broken. The results are summarized and discussed in Section 5.

2. The phase lattice model and its properties

2.1. Basic model

In this section we formulate the basic model and describe the results of its numerical simulation. We describe an individual oscillator with a phase variable φ_k, and assume that being uncoupled it rotates with a constant natural frequency ω_k. The coupling of nearest neighbors is implemented via a coupling function f that depends on the phase differences. As a result, we obtain a set of ODEs (cf. [3,6,7])

$$\dot{\varphi}_k = \omega_k + \varepsilon f(\varphi_{k-1} - \varphi_k) + \varepsilon f(\varphi_{k+1} - \varphi_k),\quad k = 1, \dots, N. \tag{1}$$

Here ε is the coupling constant, it is assumed to be the same for all oscillator pairs. It is natural to assume that the coupling vanishes when the phases of the oscillators are equal, i.e. $f(0) = 0$. Then the boundary conditions $\varphi_0 = \varphi_1$, $\varphi_{N+1} = \varphi_N$ ensure the correct equations for $k = 1, N$.

Clearly, because solely the phase differences are entering Eq. (1), a closed system can be written for these differences only, reducing the number of variables by 1. This reduction takes the simplest form if the function f is an odd one, and this is always assumed hereafter. In particular, below we will mainly consider the simplest case $f(\varphi) = \sin\varphi$, but for the moment we would like to write the equations in a more general form

$$\dot{\psi}_k = \Delta_k + \varepsilon f(\psi_{k-1}) + \varepsilon f(\psi_{k+1}) - 2\varepsilon f(\psi_k),\quad k = 1, \dots, n, \tag{2}$$

where $\psi_k = \varphi_{k+1} - \varphi_k$, $\Delta_k = \omega_{k+1} - \omega_k$, and $n = N - 1$. The boundary conditions for system (2) are $\psi_0 = \psi_{n+1} = 0$.

2.2. Small and large couplings

Before proceeding to numerical simulations, we outline some general properties of system (2) (cf. [3,7]). Mostly simple are the limiting cases of small and large couplings.

For $\varepsilon = 0$, Eq. (2) have a simple n-frequency solution. If the frequency differences Δ_k are incommensurate, this solution is quasi-periodic and can be represented as an ergodic motion on an n-dimensional torus. The natural invariant measure on this torus is uniform. According to the KAM-theory-type arguments, if the frequencies are far from resonances, the quasi-periodic motion is observed for small coupling $\varepsilon \ll 1$ as well.

For large ε a fully synchronized state is observed. In terms of the phase differences ψ_k this corresponds to a stable fixed point in (2). To see this, it is enough to mention that the equation for the stationary state $\dot{\psi}_k = 0$

$$A_{km} f(\psi_m) = -\varepsilon^{-1}\Delta_k, \tag{3}$$

is a linear system for the unknown variables $f(\psi_m)$. The matrix A is tridiagonal ($A_{k,k} = -2$, $A_{k,k+1} = A_{k-1,k} = 1$) and can be inverted ($A^{-1}_{km} = -k(N-m)/N$, with $k \le m$ and $A^{-1}_{km} = A^{-1}_{mk}$). Denoting $\alpha_m = -A^{-1}_{km}\Delta_k$, we obtain a system

$$f(\psi_m) = \varepsilon^{-1}\alpha_m. \tag{4}$$

If $f(\cdot)$ is bounded $f_{\min} \le f \le f_{\max}$, then Eq. (4) can be resolved if for all m

$$\varepsilon f_{\min} < \alpha_m < \varepsilon f_{\max}. \tag{5}$$

The 2π-periodic function f has at least two branches, so there are at least 2^n different fixed points. As have been shown in [7], only one of them is stable corresponding to the choice of the branch with $f' > 0$ for all variables. Thus, a stable phase-locked solution exists for large enough couplings. From the consideration above it is also clear, how it loses its stability. This happens when for some m^* the solutions of (4) seize to exist via a saddle-node bifurcation. Typically, beyond this transition the variable ψ_{m^*} rotates while other phase differences remain bounded. This corresponds to the splitting of the lattice (1) in two clusters $k \le m^*$ and $k > m^*$.

The limiting situations described above suggest that there exists a hierarchy of transitions from the completely phase-locked state at large ε to the quasi-periodic state at small ε. A scenario depends on the frequencies ω_k. In this paper we focus on a particular case of linearly distributed natural frequencies in the lattice. As have been discussed in [7], it corresponds to a real experimental situation of mammalian intestinal smooth muscle [16]. Furthermore, we will mostly consider the simplest case of coupling function $f(\varphi) = \sin\varphi$.

2.3. Synchronization properties: clustering hierarchy

In the rest of this section we consider the particular case of a linear distribution of natural frequencies in (1). This means that all frequency differences Δ_k in (2) are equal. Rescaling the time we can set these differences to unity, thus the resulting system has only one parameter—the coupling constant ε.

Of course, this specific choice $\Delta_k = 1$ introduces some new symmetries into the system; below we will see that they strongly influence the dynamics. Furthermore, we use in this section the coupling function $f(\varphi) = \sin\varphi$. This simplest choice also brings additional symmetries, to be discussed below.

The main quantities of interest are the observed frequencies of the oscillators defined as the mean rotation velocities $\Omega_k = \langle\dot{\varphi}_k\rangle$. For the oscillators forming a cluster, these frequencies coincide. Thus, an appearance of a cluster can be easily seen from the bifurcation diagram Ω_k vs. ε. We present these diagrams for several values of the chain length N in Fig. 1.

The synchronization diagrams reveal several features:

(i) With increasing ε the rotators successively group into clusters of equal frequencies. The last transition to a single cluster occurs at ε_c, which can be calculated according to Eqs. (4) and (5). In the case $\Delta_k = 1$ the solution (4) can be written ex-

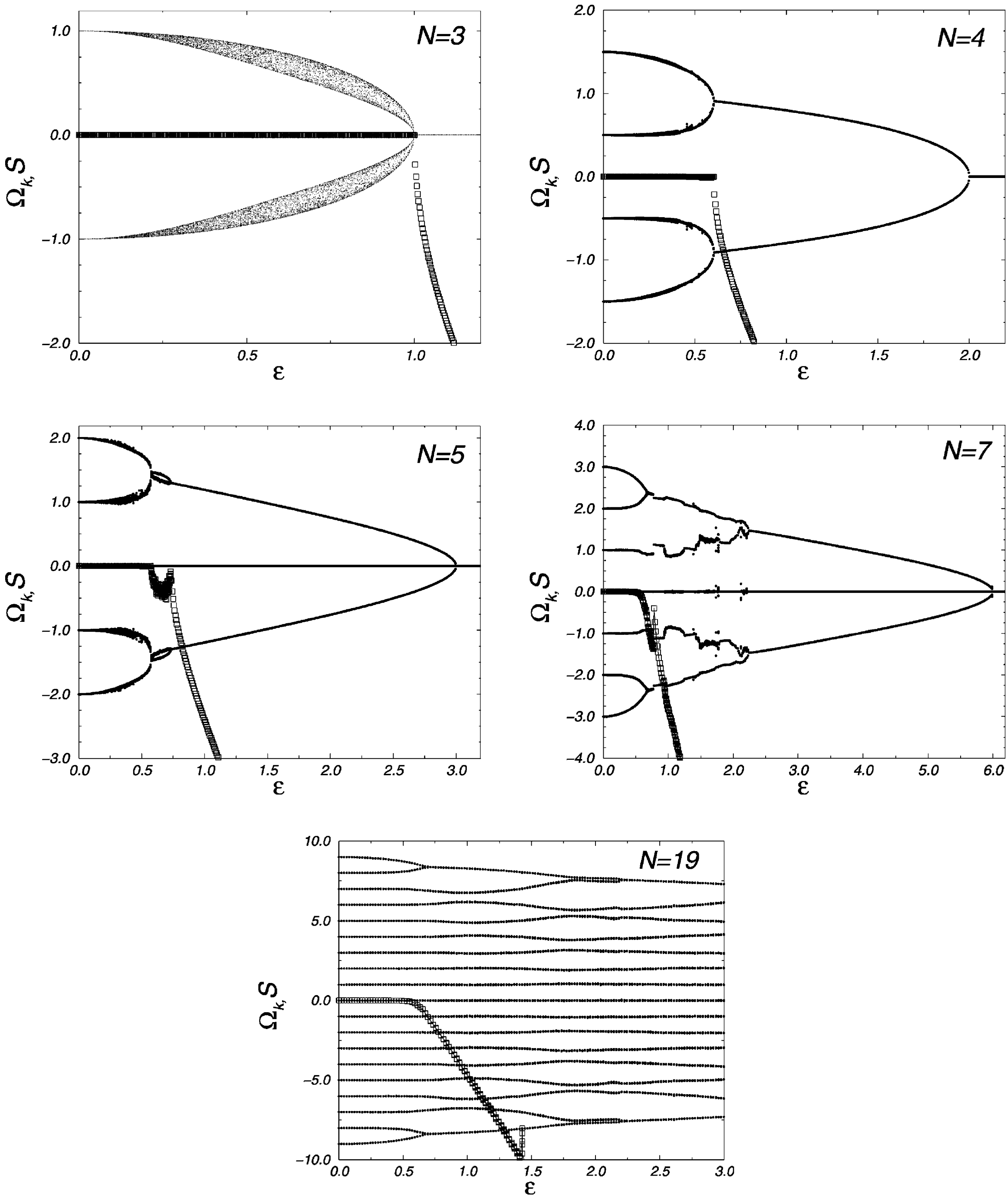

Fig. 1. Observed frequencies Ω_k vs. coupling strength ε for oscillator chains of different length. The bifurcation diagrams were produced by choosing randomly 10 initial points for each ε and plotting the resulting frequencies with dots on one graph. The smeared regions that are seen for small ε indicate the dependence of the frequencies on the initial conditions. On these graphs also the average contraction rate S (see Eq. (6)) of the phase volume is shown with squares.

plicitly: $\alpha_m = m(N-m)/2$. Taking the maximum of this expression, we obtain

$$\varepsilon_c = \max_m \alpha_m = \begin{cases} \dfrac{N^2}{8} & \text{even } N, \\ \dfrac{(N-1)(N+1)}{8} & \text{odd } N. \end{cases}$$

(ii) There are regions where the diagram is "smeared". Most visible is this region for the case of three oscillators. In the smeared region the averaged frequency depends on initial conditions, what means that the system does not have a single attractor, but, presumably, many invariant states. Note that these regions mostly appear in small lattices at small couplings, prior to the first clustering. The investigation of this state is the main purpose of this paper.

2.4. Quasi-Hamiltonian dynamics for small couplings

To reveal the dynamics of the lattice, we have calculated the Lyapunov exponents. The continuous-time system (2) has one zero Lyapunov exponent for non-constant solutions (and, correspondingly, in system (1) two Lyapunov exponents are exactly zero). The calculations of the exponents give the results shown in Fig. 2. For small couplings we always obtain a sign-symmetric picture of the exponents: they appear in pairs having the same absolute value and

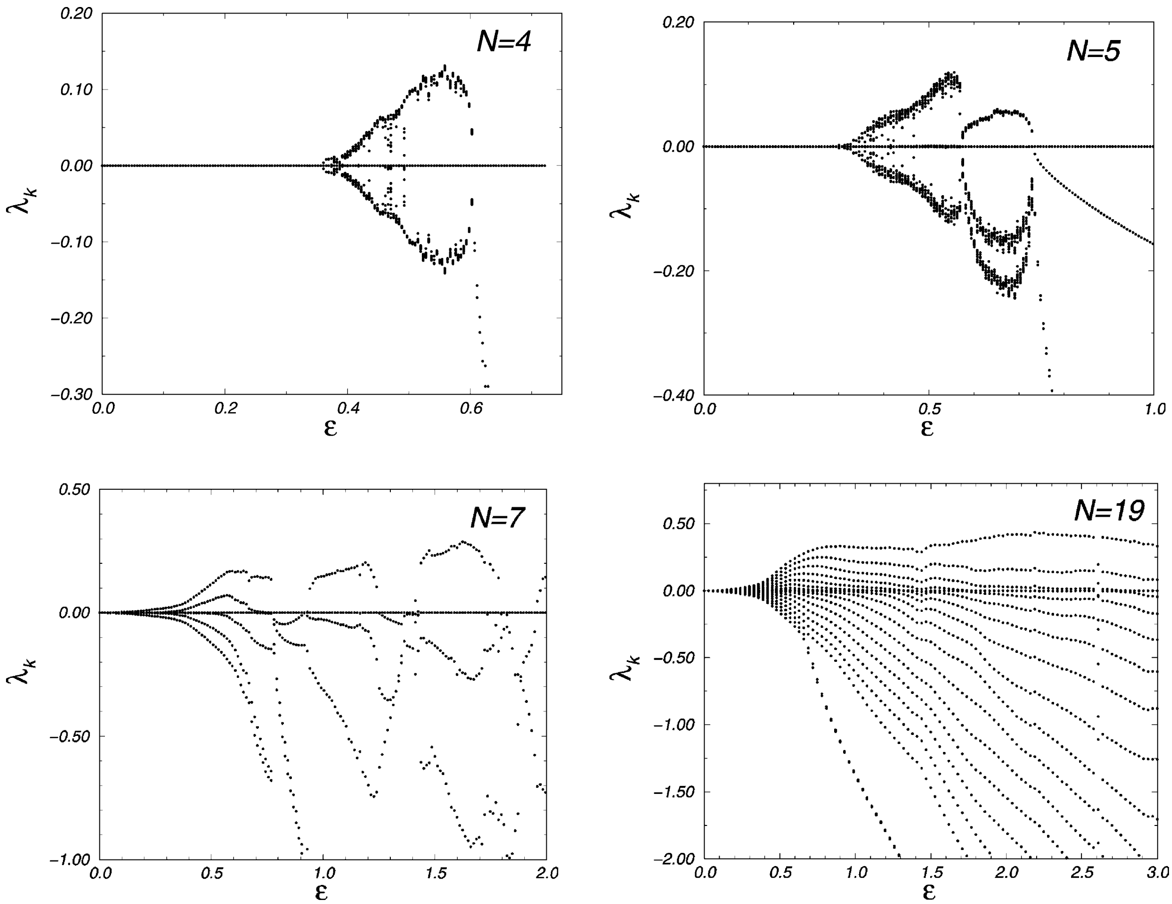

Fig. 2. Lyapunov exponents vs. coupling strength ε for the same lattices as in Fig. 1 (the case $N = 3$ is not shown, here all Lyapunov exponents vanish for $\varepsilon < 1$ and two are negative for $\varepsilon > 1$).

opposite signs (additionally, some Lyapunov exponents can be zero). This means that the phase volume is conserved on time average: its mean divergence

$$S = \sum_k \left\langle \frac{\partial \dot{\psi}_k}{\partial \psi_k} \right\rangle = -2\varepsilon \sum_k \langle f'(\psi_k) \rangle \qquad (6)$$

is the sum of the Lyapunov exponents, and it vanishes. We have checked this by calculating the average (6) directly, and found it to be nearly zero (apart from statistical fluctuations). These results are presented in Fig. 1 (see also detailed calculations in Fig. 5).

The symmetrical Lyapunov exponents and the conservation of the phase volume are the hallmarks of the Hamiltonian dynamics. Thus we call the dynamics of the lattice at small couplings quasi-Hamiltonian.

3. Reversibility of regular and chaotic regimes

3.1. Reversibility

Our explanation of the quasi-Hamiltonian behavior is based on the reversibility of the dynamics (see [20,21] for mathematical definitions). Reversibility means that there exists an involution $\boldsymbol{R} : \Psi \mapsto \Psi$ (involution means that $\boldsymbol{R}^2$ is identical transformation; Ψ here denotes the set of variables $\Psi = (\psi_1, \dots, \psi_n)$) which together with the time reversal transformation $\boldsymbol{T} : t \mapsto -t$ leaves the system invariant. Reversibility yields that the trajectories of a dynamical system come in symmetric pairs. Indeed, for every point of the phase space $\Psi(0)$ there is the symmetric point $\boldsymbol{R}\Psi(0)$, and the trajectory $\Psi(t)$ starting from $\Psi(0)$ is symmetric to the trajectory $\boldsymbol{R}\Psi(-t)$ running backward in time and starting from $\boldsymbol{R}\Psi(0)$. In the terms of trajectory stability, these symmetric trajectories have inverse Lyapunov spectra, because the Lyapunov exponents change sign with the time inversion.

The fact that a system is reversible (i.e. it possesses an involution as described above) still does not say anything on the dissipativity/conservativity of the observed dynamics; it means only that if there is an attractor, there should be the corresponding symmetric repeller. Particularly important is the case when some symmetric trajectories coincide, i.e. if the involution $\boldsymbol{R}$ transforms a trajectory to itself. It will be the case if (but not *only if*) this trajectory crosses the set Fix $\boldsymbol{R}$ of the invariant points of the involution ($\Psi \in$ Fix $\boldsymbol{R}$ means that $\boldsymbol{R}\Psi = \Psi$). Such a trajectory we call reversible. Properties of periodic reversible trajectories are like those of in Hamiltonian systems: the Lyapunov exponents are sign-symmetric and the phase space volume in their vicinity is conserved on average (in particular, the local Poincaré map is area-preserving).

In general, reversible trajectories may be non-periodic, and coexist with non-reversible ones. Here we can distinguish two cases. A reversible non-periodic trajectory can connect an attractor and a repeller, being heteroclinic. Otherwise, it can be non-wandering, in particular, if it can repeatedly return to a vicinity of the set Fix $\boldsymbol{R}$. In the latter case the average properties are qualitatively similar to those of periodic reversible trajectories, and in particular the Lyapunov exponents are sign-symmetric. This property is very important if we consider complex (quasi-periodic or chaotic) invariant sets. If such a set is ergodic, and at least one typical trajectory belonging to it is reversible, then the invariant measure is $\boldsymbol{R}$-symmetric and the dynamics of the system is quasi-Hamiltonian on this set. Note that this property does not require any symplectic structure and hence does not depend on evenness/oddness of the underlying phase space.

We now argue that in order for periodic and non-wandering reversible trajectories to exist, the set Fix $\boldsymbol{R}$ should be large enough. Let us consider the evolution of Fix $\boldsymbol{R}$ in time. A reversible periodic exists if the sets $\mathcal{F}^t$(Fix $\boldsymbol{R}$) and Fix $\boldsymbol{R}$ intersect, where $\mathcal{F}^t$ is the evolution operator of the dynamical system. This intersection generally can occur in a n-dimensional phase space if the topological dimension of Fix $\boldsymbol{R}$ is large enough, at least $[n/2]$, i.e. $n/2$ for even and $(n-1)/2$ for odd n, the dimension of $\mathcal{F}^t$(Fix $\boldsymbol{R}$) being then $[n/2] + 1$. Based on the continuity arguments, we obtain the same estimate for a general existence of non-wandering trajectories, because in the latter case the distance between $\mathcal{F}^t$(Fix $\boldsymbol{R}$) and Fix $\boldsymbol{R}$ has to nearly vanish at some times. Contrary to this, if the dimension of the set Fix $\boldsymbol{R}$ is small, generally there are no non-wandering reversible trajectories. From these arguments, it follows that not all possible involutions $\boldsymbol{R}$ can explain the quasi-Hamiltonian behavior, but only those having a high-dimensional invariant set Fix $\boldsymbol{R}$.

3.2. Reversibility of the oscillator lattice

We now proceed to apply this concept to the lattice of n oscillators (2). The involution yielding reversibility here is

$$\boldsymbol{R} : \psi_k \mapsto \pi - \psi_{n-k}. \tag{7}$$

One can see that this transformation can be represented as the product $\boldsymbol{R} = \boldsymbol{P} \circ \boldsymbol{Q}$ of two involutions:

$$\boldsymbol{P} : \psi_k \mapsto \psi_{n-k}, \tag{8}$$

and

$$\boldsymbol{Q} : \psi_k \mapsto \pi - \psi_k. \tag{9}$$

These transformations reflect the symmetry of the distribution of the natural frequencies ($\boldsymbol{P}$ requires $\Delta_k = \Delta_{n-k}$) and the symmetry of the coupling function $f(\cdot)$ ($\boldsymbol{Q}$ requires that the odd function f has only odd harmonics in its expansion in sine Fourier series). Involution (7) has an invariant set $\psi_k + \psi_{n-k} = \pi$ of dimension $[n/2]$, thus we can expect periodic and non-wandering reversible trajectories to exist. This is not the case for the involution $\boldsymbol{Q}$: its invariant set $\psi_k = \pi/2$ is one point.

Below we consider implications of the reversibility described for some particular lattices.

3.3. Three coupled oscillators

The chain of three rotators has the simplest non-trivial dynamics. System (2) reduces to only two equations:

$$\dot\psi_1 = 1 - 2\varepsilon \sin\psi_1 + \varepsilon \sin\psi_2,$$
$$\dot\psi_2 = 1 - 2\varepsilon \sin\psi_2 + \varepsilon \sin\psi_1. \tag{10}$$

For $\varepsilon > \varepsilon_c = 1$ there exists a stable point solution, corresponding to synchronization. We now demonstrate that the whole dynamics for $\varepsilon < \varepsilon_c$ is reversible.

Involution (7) for the system (10) has invariant line Fix $\boldsymbol{R}$: $\psi_1 + \psi_2 = \pi$. It is clear that on the two-dimensional phase plane (ψ_1, ψ_2) every rotating trajectory crosses this line many times, thus all trajectories are periodic and reversible, and the desynchronized state is quasi-Hamiltonian. We show the phase portrait in Fig. 3. It represents a typical for an integrable Hamiltonian system family of periodic orbits. These orbits have different periods, and this explains the diversity of frequencies in Fig. 1. A difference in Hamiltonian phase portraits can also be easily seen: because the phase volume is conserved in average, but not locally, different regions on the phase plane are filled with different densities. The transition to clusters occurs via an inverse saddle-node bifurcation, at which a stable and unstable points appear at $\psi_1 = \psi_2 = \pi/2$ from the condensation of trajectories.

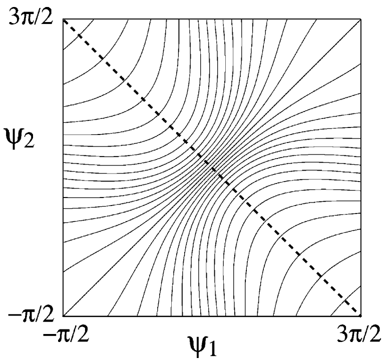

Fig. 3. The phase portrait of system (10). The line Fix $\boldsymbol{R}$ is shown as bold dashed one; it is crossed by all trajectories.

3.4. Four coupled rotators

The system of four coupled rotators reads

$$\dot\psi_1 = 1 - 2\varepsilon \sin\psi_1 + \varepsilon \sin\psi_2,$$
$$\dot\psi_2 = 1 - 2\varepsilon \sin\psi_2 + \varepsilon \sin\psi_1 + \varepsilon \sin\psi_3,$$
$$\dot\psi_3 = 1 - 2\varepsilon \sin\psi_3 + \varepsilon \sin\psi_2. \tag{11}$$

Applying involution (7), we obtain that the set Fix $\boldsymbol{R}$ is the line $\psi_1 + \psi_3 = \pi$, $\psi_2 = \pi/2$. The phase trajectories in a three-dimensional phase space generally do not intersect a given line, so we cannot expect reversibility for all trajectories. In this case we observe a non-trivial transition from the quasi-Hamiltonian to the dissipative dynamics, to be described below.

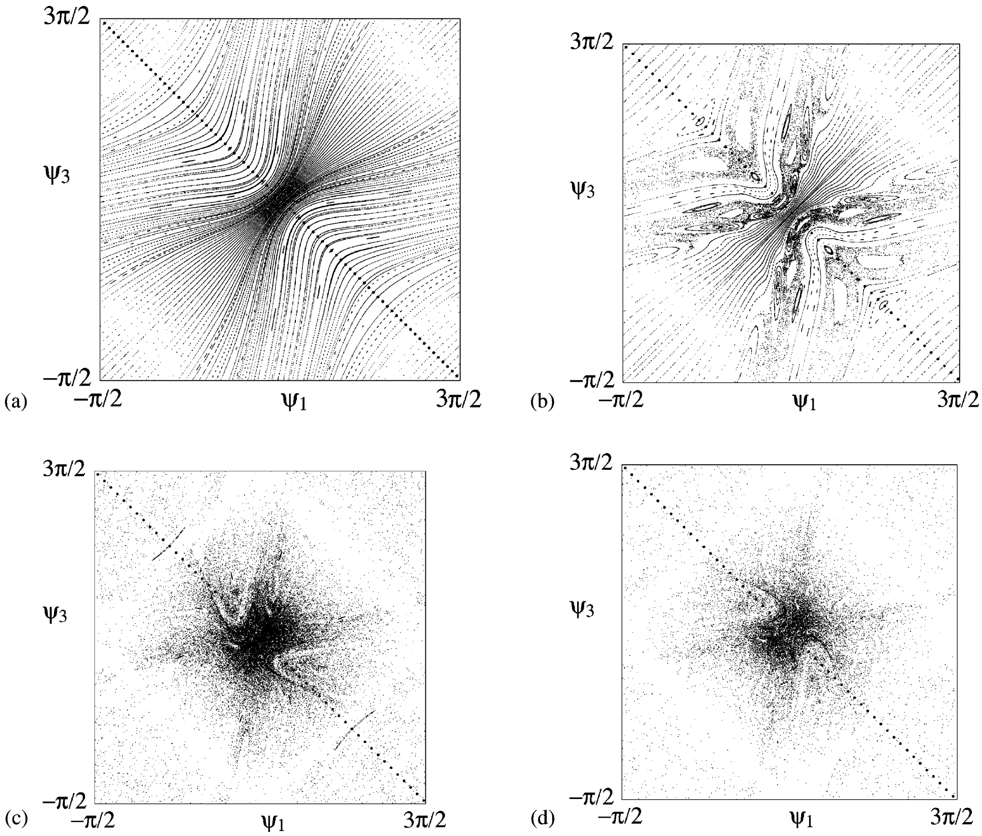

Fig. 4. The Poincaré map for system (11) for different values of coupling. The maps are constructed by choosing the initial conditions on the line $\psi_1 + \psi_3 = \pi$ (filled circles) and plotting 2000 their iterations. (a) $\varepsilon = 0.3$: the quasi-periodic states dominate. (b) $\varepsilon = 0.39$: chaotic and quasi-periodic regimes coexist. The attractor and the repeller for $\varepsilon = 0.49$ are shown in (c) and (d), respectively.

To visualize the dynamics we constructed the Poincaré maps. The Poincaré section was chosen by the condition $\psi_2 = \pi/2$ so that the invariant set of the involution is the line $\psi_1 + \psi_3 = \pi$ on this plane. The Poincaré maps for different values of parameter ε are presented in Fig. 4. They are constructed by iterations of initial points lying on the line $\psi_1 + \psi_3 = \pi$, i.e. belonging to Fix $\boldsymbol{R}$.

To verify whether the dynamics is quasi-Hamiltonian or not, we calculated the average over a very large time (up to $T = 1.5 \times 10^7$) divergence of the phase volume S. Only the values of S that are nearly the same for the averaging times $T/2$ and T have been considered to be distinguishable from zero. The data are presented in Fig. 5 together with the calculations for larger lattices.

3.4.1. *Quasi-periodic dynamics for small ε*

In the case shown in Fig. 4a the dynamics appears to be quasi-periodic, and the phase space appears to be foliated by tori. All these tori cross the line Fix $\boldsymbol{R}$, thus on each torus there is a reversible non-wandering trajectory. This ensures reversibility of the tori, and the whole dynamics is quasi-Hamiltonian. The average divergence S in this case is indistinguishable from zero.

3.4.2. *Mixed quasi-Hamiltonian dynamics*

In the case shown in Fig. 4b the dynamics appears to be quasi-Hamiltonian with chaotic and quasi-periodic components. In some components the images of Fix $\boldsymbol{R}$ appear to be dense. This allows us to speak on "reversibility in average". Note that due to ergodicity the

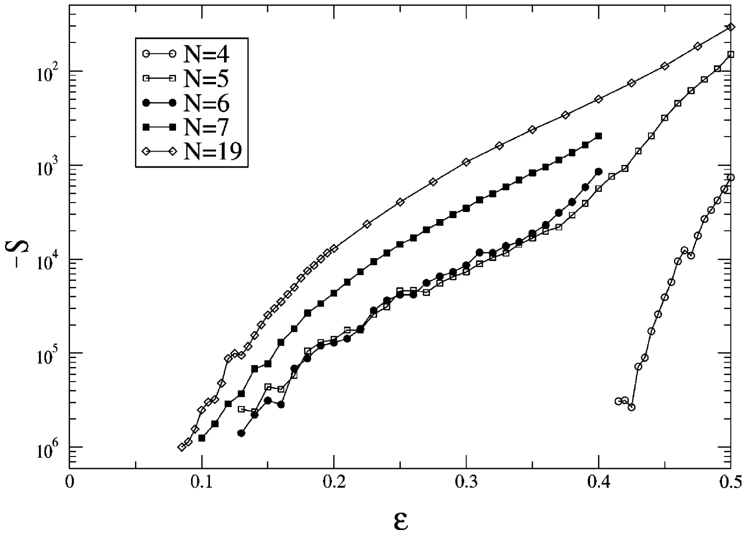

Fig. 5. The average divergence of the phase volume for lattices of different sizes. The lowest values at $|S| \approx 10^{-6}$ correspond to the remaining statistical uncertainty achieved after averaging over times as large as 10^7. Up to this uncertainty, the threshold for the transition from the quasi-Hamiltonian to reversible behavior appears to lie at $\varepsilon \approx 0.1$ for lattices with $N > 4$.

mean frequency is the same for all typical trajectories in the chaotic sea, but has different values for different tori. Thus, the mean frequency depends on initial conditions. From the other hand, there are components having no overlap with images of Fix $\boldsymbol{R}$, they are nevertheless symmetric with regard to the involution.

3.4.3. Chaotic dissipative dynamics

As one can see from the detailed calculations of the mean divergence of phase volume (Fig. 5), for $\varepsilon > 0.43$ the divergence is non-zero, although very small. Accordingly, we have to characterize the observed chaotic state as an attractor. Surely, there exists also the symmetric to the attractor repeller—it can be easily obtain via backward integration of Eq. (11). We present the phase portraits of the attractor and the repeller in Fig. 4c and d. From visual inspection of these pictures one may conclude that the attractor and the repeller "overlap". However, according to the Birkhoff ergodic theorem, the invariant measures of these invariant sets should be mutually singular.[1] The contradiction resolves if one takes into account that although the attractor and the repeller look like possessing positive Lebesgue measure, in reality they are fractals having Lebesgue measure zero. Because the mean divergence of the phase volume is very small, the dimensions of these fractals are extremely close to 2, therefore, it is difficult to distinguish them from quasi-Hamiltonian dynamics. On visual inspection of Fig. 4(c) and (d), it appears that iterations of Fix $\boldsymbol{R}$ (both forward and reverse in time) return arbitrary close to Fix $\boldsymbol{R}$. Why this does not ensure reversibility, remains an open question.

We emphasize that for some values of coupling we observed non-chaotic, periodic attractors in the system. The situation appears to be similar to other cases of non-hyperbolic chaotic dynamics (e.g. in the Henon map), where stable orbits with relatively short periods appear and disappear as a parameter is varied. Numerically, it is difficult to distinguish whether in these situations the chaotic attractor transforms to a chaotic saddle and the only attractors is the regular one, or there is a bistability "chaos–periodic orbit".

3.4.4. Clustering transition

With increasing the coupling, at $\varepsilon \approx 0.604$ a pair of stable and unstable fixed points appears in the Poincaré map. On the stable periodic solution of system (11)

[1] We are thankful to D. Turaev for this remark.

the mean rotation frequencies of the variables ψ_3 and ψ_1 coincide, what corresponds to the appearance of the cluster (cf. Fig. 1). The set Fix $\boldsymbol{R}$ is now attracted to the stable orbit which is a global attractor of system (11), and the dynamics on this attractor is no more reversible.

3.5. Large number of rotators

In the case of a large number of rotators we can characterize the dynamics with averaged quantities like Lyapunov exponents, but it is rather difficult to reveal the topological structure in the phase space. Calculations of the Lyapunov exponents show that for small couplings ε they are coming in sign-symmetric pairs and the phase volume is conserved in average, i.e. the system is quasi-Hamiltonian. The dimension of the invariant set of the involution Fix $\boldsymbol{R}$ is exactly $[n/2]$ and thus is large enough to make reversible orbits possible.

Numerically, it appears that the transition from quasi-Hamiltonian to dissipative dynamics for a large number of oscillators is not as abrupt as for $N = 3$, and does not coincide with the point of the first clustering, but is similar to the continuous transition described above for $N = 4$. This can be seen from the calculations of the average divergence of the phase space S (6) presented in Fig. 5. Because of large statistical fluctuations we were not able to determine S with accuracy better than 10^{-6}, and with this accuracy the threshold for the transition lies at $\varepsilon \approx 0.1$. This number is expected to be the same for all chain lengths because first clustering always appears at the ends of the chain, nearly at $\varepsilon = 0.6$ for all chains with $N > 3$. Due to high dimensionality of the system, we could not follow any topological transition in the structure of chaos at this point.

4. Violations of reversibility

Obviously, the involution (7) which is responsible for the reversibility is based on the high symmetry in the system lattice. This symmetry is due both to the particular distribution of the frequencies Ω_k and to the symmetry of the coupling function. We demonstrate here that violations of these symmetries lead to non-reversible dynamics.

4.1. Non-uniform frequency distribution

The involution (7) requires that the frequency differences in (2) are symmetric

$$\Delta_k = \Delta_{n-k}, \quad k = 1, \dots, \tfrac{1}{2}n, \tag{12}$$

but not necessarily equal. We illustrate this in Fig. 6a. The phase volume here is conserved in average, and the dynamics remains reversible and quasi-Hamiltonian. Contrary to this, when we take a distribution of frequency differences that violates the symmetry, we obtain a strange attractor instead of quasi-Hamiltonicity (Fig. 6b). We emphasize that also in this latter case the system is reversible under involution (9). The dimension of the invariant set of this involution is, however, too low to ensure reversibility of the dynamics.

We note that if the symmetry (12) is only slightly violated, the dynamics remains nearly quasi-Hamiltonian: the convergence of the phase space volume is small. In the chaotic case this means that the dimension of the attractor is close to the dimension of the phase space. In the periodic case like Fig. 3a weak dissipation means that the Poincaré map is a circle map close to the identity. It is known that in such maps a vast majority of states is quasi-periodic, i.e. they have zero Lyapunov exponents and are therefore not distinguishable from the quasi-Hamiltonian ones.

If the symmetry (12) is strongly violated, then only strongly dissipative regimes can be expected in the oscillator lattice. Extensive numerical studies with random distribution of frequencies [10–13] have demonstrated that a typical picture is a successive merging of oscillators in larger and larger clusters. Particular features of the clustering hierarchy depend on the (random) set of frequencies. As has been shown in [22], the one-cluster state in an array with a random distribution of the frequencies is observed in a broader range of parameters than in an array with a linear distribution. Clustering of phase dynamics is also observed for chaotic oscillators, e.g., see [13–15]. In conclusion of this discussion we would like to

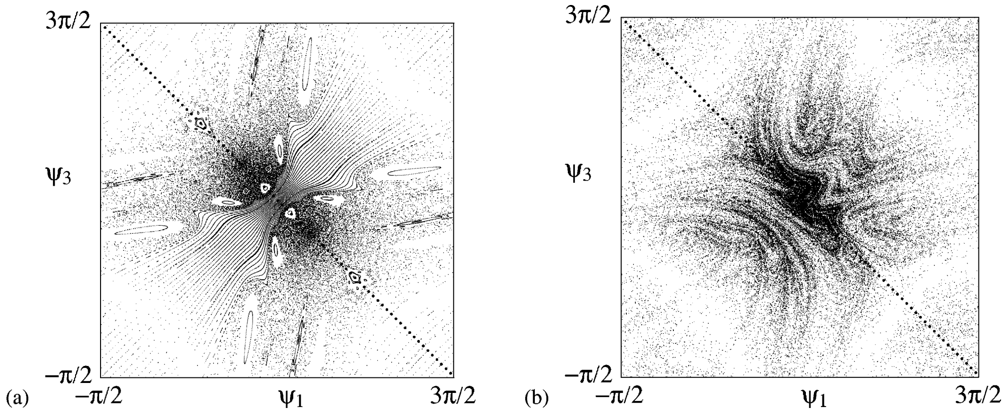

Fig. 6. The dynamics of system (2) with $N = 4$, $\varepsilon = 0.4$: (a) $\Delta_1 = \Delta_3 = 1$, $\Delta_2 = 1.3$. Here a violation of equality of frequencies that does not destroy involution (7) preserves quasi-Hamiltonian dynamics. (b) $\Delta_1 = \Delta_2 = 1$, $\Delta_3 = 1.2$. The involution (7) is broken; the dynamics is dissipative with a strange attractor.

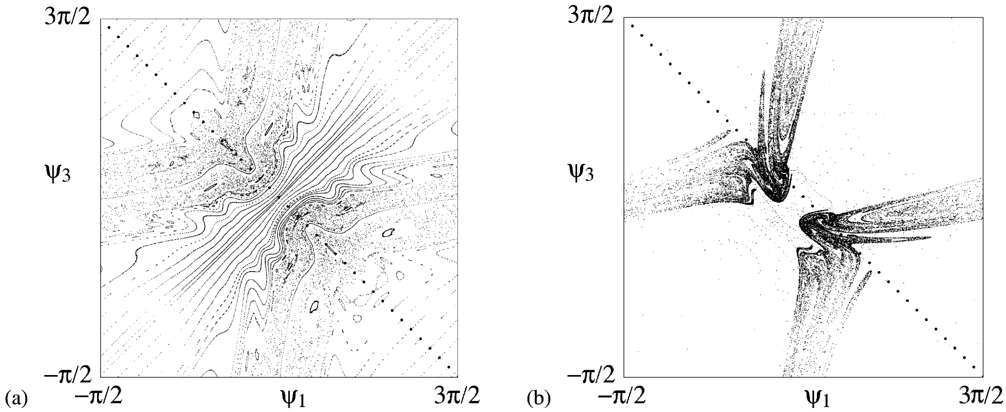

Fig. 7. (a) The dynamics of system (2) with $N = 4$, $\Delta_k = 1$, $\varepsilon = 0.35$ and $f(\psi) = \sin\psi + 0.2\sin 3\psi$ is reversible. (b) The same system as (a), but with a non-symmetric coupling function $f(\psi) = \sin\psi + 0.2\sin 2\psi$ and $\varepsilon = 0.4$ has a strange attractor.

mention that for coupled oscillators of type (1) a distribution of frequencies always plays a destructive role: if the frequencies are equal, the synchronous state is stable for any value of the coupling parameter. This should be contrasted to arrays of chaotic oscillators, where, e.g., identical coupled oscillators may demonstrate a space–time chaos, while a disorder may lead to appearance of regular (non-chaotic) regimes [23].

4.2. Non-symmetric coupling function

Here we demonstrate that violations of the function symmetry (9) lead to break of reversibility. The odd coupling functions invariant under involution (9) are represented by a sine Fourier series with odd harmonics only. Such functions yield reversible dynamics; one example is presented in Fig. 7a. If even harmonics in the Fourier series are present, the dynamics is dissipative as can be seen in Fig. 7b.

5. Conclusion

The extremely simple system of coupled phase oscillators demonstrates extremely rich dynamics. This can be already seen from Figs. 1 and 2. Many regimes in large lattices are chaotic, so the clustering should be described as a transition inside chaos. In this paper, we focused on a particular peculiarity of the dynamics for very small couplings and demonstrated that this dynamics is reversible. This property is responsible for a rather unusual for dissipative systems quasi-Hamiltonian dynamics. Although the reversibility holds for any coupling, only when the clusters are absent the reversible trajectories appear to be dense in the ergodic components; for large couplings they connect a repeller and an attractor, so that the observed dynamics on the attractor is dissipative. One can say that synchronization excludes quasi-Hamiltonicity and vice versa. Although we have started with a rather degenerate case of a linear distribution of the frequencies in the array, a detailed analysis of the underlying symmetry has shown that this condition can be considerably weakened, see Eq. (12). Moreover, we have argued that if the system is "nearly reversible", i.e. its parameters only slightly deviate from the symmetric ones, then the dissipativity can be extremely small, resulting in long transients and high noise sensitivity.

We would like to emphasize that the reversibility differs significantly from the usual symmetry properties of dynamical systems that include only phase space transformations. The main feature of reversibility is that together with a transformation of the phase space one changes the sign of time. This allows one to encounter a time-reversible, quasi-Hamiltonian behavior, what is very unusual for dissipative systems. In particular, in the quasi-Hamiltonian case there are no attractor and repellers, the Poincaré recurrence theorem works, Lyapunov exponents come in symmetric pairs. Typically, such systems have many ergodic components, demonstrating usual for non-hyperbolic Hamiltonian models coexistence of chaotic and regular behaviors.

In investigating the simplest non-trivial case of four coupled phase oscillators we have found a non-trivial transition from the quasi-Hamiltonian to dissipative dynamics. It can be characterized as a spontaneous breaking of the time-reversal symmetry, at which the mean contraction rate smoothly deviates from zero. Such a transition, to the best of our knowledge, was not observed previously.

In discussing reversibility we have argued that the symmetry of the involution that gives rise to reversibility should be large enough. In the particular case considered in this paper, this requires not only the symmetry of the coupling function, but the symmetry of the natural frequencies as well. If the dimension of the invariant set of the involution is low, no reversible dynamics is observed. It would be interesting to apply these ideas to the systems of globally coupled Josephson junctions, where the nonlinear functions are known to have a high symmetry [18,19].

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