Locking-Based Frequency Measurement and Synchronization of Chaotic Oscillators with Complex Dynamics

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We propose a method for the determination of a characteristic oscillation frequency for a broad class of chaotic oscillators generating complex signals. It is based on the locking of standard periodic self-sustained oscillators by an irregular signal. The method is applied to experimental data from chaotic electrochemical oscillators, where other approaches of frequency determination (e.g., based on Hilbert transform) fail. Using the method we characterize the effects of phase synchronization for systems with ill-defined phase by external forcing and due to mutual coupling.

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Phase synchronization has been demonstrated for basic models of chaotic dynamics [1,2] and observed in laboratory experiments with electronic generators, gas discharge, lasers, and electrodissolution of metals [3,4]; for reviews, see [5,6]. The usual approach is based on the introduction of a phase $\phi(t)$ of a continuous-time autonomous (self-sustained) chaotic oscillator; this allows one to detect phase locking and frequency entrainment where the frequency is calculated as the mean velocity of phase rotation. Essential is the separation of the phase dynamics from those of the amplitudes: the phases of two or many systems can be locked due to a weak coupling, whereas the amplitudes remain weakly correlated. However, as yet there is no general way to introduce a phase for an arbitrary irregular oscillator; existing approaches can be exploited only for "good" systems with a rather simple topology of the attractor.

Phase can be straightforwardly introduced if one can find a two-dimensional projection of the attractor in which all trajectories revolve around some origin. For such projections one can, e.g., define phase as an angle in polar coordinates or using the Hilbert transform. Sometimes, a proper projection can be achieved with a coordinate transformation (e.g., using the symmetry properties of the attractor, as in the Lorenz system) [2,6]. Estimation of the average frequency of individual oscillators $\langle\dot{\phi}\rangle$ then allows one to characterize the degree of synchronization. Contrary to these cases of well-defined phase, chaotic oscillators with "wild," nonrevolving trajectories are often termed as those with ill-defined phase. Here only indirect indications for phase synchronization exist (based, e.g., on the ensemble averages [2,7]), but no direct calculation of the phase and the frequency could be performed so far.

In this Letter, we propose a method that allows one to reveal synchronization of systems with ill-defined phases by estimating the average frequency of the observed signals. This method, based on the use of *auxiliary limit cycle oscillators,* can characterize synchronization of two or many coupled systems; we demonstrate it below on numerical examples and experimental data.

To introduce the method, let us consider an ensemble of *uncoupled* limit cycle oscillators with natural frequencies ω_k distributed in an interval $[\omega_{\min}, \omega_{\max}]$. Let each oscillator of this ensemble be driven by a common periodic force of a frequency $\nu \in [\omega_{\min}, \omega_{\max}]$. It is well known that the force synchronizes those elements of the ensemble which have frequencies close to ν. This can be demonstrated by plotting the frequencies of the driven limit cycle oscillators Ω_k, called hereafter the observed frequencies, vs the natural frequencies ω_k: the synchronization manifests itself in the appearance of a horizontal plateau (more precisely, one expects to observe a devil's staircase structure with infinitely many plateaus), where the frequency of entrained elements is equal to ν. Hence, an *unknown* frequency of the drive can be revealed by the analysis of the Ω_k vs ω_k plot. The idea of our approach is to use the ensemble of auxiliary oscillators as a *device for measuring the frequency of complex signals.*

A simple implementation of the method is to drive the array of Poincaré oscillators with a signal $X(t)$

$$\dot{A}_k = (1 + i\omega_k)A_k - |A_k|^2 A_k + \varepsilon X(t). \tag{1}$$

Separating the real amplitude R and the phase ϕ from the complex amplitude $A = Re^{i\phi}$, we obtain for the phase $\dot{\phi}_k = \omega_k - R_k^{-1}\varepsilon X(t)\sin\phi_k$. Noting that for small ε the amplitude R is close to unity and neglecting its fluctuations, we can write equations for our measuring oscillators as pure phase equations:

$$\dot{\phi}_k = \omega_k - \varepsilon X(t)\sin\phi_k,$$
$$\Omega_k = \lim_{t\to\infty}[\phi_k(t) - \phi_k(0)]/t. \quad (2)$$

In calculations below, we normalize the signal $X(t)$ to have zero mean and unit variance so that the coupling constant ε is the only parameter of the method [8].

To show how the method works, we consider a model quasiharmonic process with mean frequency ω_0 and slowly varying amplitude and phase: $X(t) = 2[1 + a(t)]\cos[\omega_0 t - \theta(t)]$. Substituting this in (2) and averaging over the period of fast oscillations $2\pi/\omega_0$, we obtain for the slowly varying phase difference $\psi = \phi - \omega_0 t + \theta$ the equation $\dot{\psi} = \omega - \omega_0 + \dot{\theta}(t) - \varepsilon[1 + a(t)]\sin\psi$; for a harmonic signal ($\dot{\theta} = a = 0$) it has for $\varepsilon \geq |\omega - \omega_0|$ the synchronized solution $\psi_0 = \arcsin[(\omega - \omega_0)/\varepsilon]$. For weak modulation we can linearize around this state and obtain for the deviations $\delta\psi$:

$$\frac{d(\delta\psi)}{dt} = \dot{\theta} - a(t)(\omega - \omega_0) - \sqrt{\varepsilon^2 - (\omega - \omega_0)^2}\,\delta\psi.$$

Assuming that θ and a are independent random processes, we can express the power spectrum of the phase fluctuations through the spectra of these processes:

$$S_{\delta\psi}(\sigma) = \frac{\sigma^2 S_\theta(\sigma) + (\omega - \omega_0)^2 S_a(\sigma)}{\varepsilon^2 - (\omega - \omega_0)^2 + \sigma^2}.$$

One can see that the fluctuations are small only in the middle of the synchronization region (for $\omega \approx \omega_0$); here only the phase fluctuations S_θ contribute. Modeling S_θ by the Lorentzlike spectrum $S_\theta = (2\Delta V_\theta)/[(\sigma^2 + \Delta^2)\pi]$, where V_θ and Δ are the variance and the characteristic maximal frequency of fluctuations of θ, we obtain $V_{\delta\psi} = \int_0^\infty S_{\delta\psi}(\sigma)d\sigma = V_\theta\Delta(\varepsilon + \Delta)^{-1}$. This final formula shows that good synchronization (i.e., small variance of $\delta\psi$) can be achieved if ε is sufficiently larger than Δ, i.e., if the coupling constant is larger than the characteristic frequency of phase fluctuations.

To illustrate the approach, we consider the Rössler system with a funnel attractor [Fig. 1(a)].

$$\dot{x} = -y - z, \qquad \dot{y} = x + 0.4y,$$
$$\dot{z} = 0.25 + z(x - 8.5). \quad (3)$$

Clearly, we cannot find an origin around which all trajectories revolve. The power spectrum for the variable x [Fig. 1(b)] is broad; it does not contain a dominating maximum. Because of these properties, there is no direct way to introduce the phase for this system and to characterize its synchronization [2]. The frequencies of the oscillators in the measuring device driven by $X(t) = x(t)$ are shown with the solid curve ($E = 0$) in Fig. 1(c). The resulting plateau in the Ω_k vs ω_k plot gives $\Omega^p \approx 0.94$ [9]. One can see that this characteristic frequency cannot be directly associated with a peak in the power spectrum [Fig. 1(b)]. We also see that Fig. 1(c) does not show the

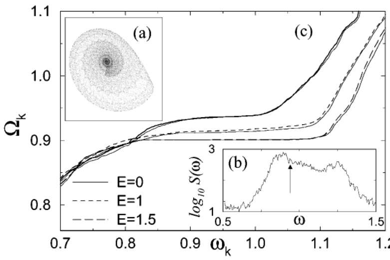

FIG. 1. (a) Funnel attractor in the Rössler system (3). (b) Power spectrum of $x(t)$. The arrow shows the characteristic frequency as determined from (c). (c) Output of the frequency measuring device (2) with $\varepsilon = 0.5$ as a function of the natural frequencies ω_k at three forcing amplitudes. Thin lines show the corresponding results obtained without subtraction of the mean value from $x(t)$; the difference is less then 0.1%.

devil's staircase structure, but only one, smeared plateau. This is due to the chaotic nature of the process $x(t)$, so that, similar to the case of narrow-band noisy signals, the high-order phase-locked regions are not observed [6,10].

Next we study a synchronization of the system (3) by a periodic forcing. The first equation of (3) now reads $\dot{x} = -y - z + E\sin(\nu t)$. Performing measurements with "device" (2) for different values of the forcing amplitude E, we see that the measured frequency Ω^p approaches the external frequency $\nu = 0.9$, giving a clear picture of frequency entrainment [Fig. 1(c)]. This adjustment of the oscillator frequency with increase of the forcing is exactly what is observed for oscillators with well-defined phase [2]; here we have been able to demonstrate it for the case of an ill-defined phase as well. It is important to mention that the shift of the plateau is due to the entrainment of the chaotic oscillations and is not an effect of the presence of a periodic component in the signal $X(t)$. This was checked by using a mixture of an unforced oscillation $x(t)$ and a periodic force $E\sin(\nu t)$ for $X(t)$ in (2); in this case no shift of the plateau has been observed.

We have applied the method to the experimental data obtained from the ensemble of 64 globally coupled chaotic electrochemical oscillators [4,11]; see [11] for the details of the experiment. The oscillators have been subjected to a mutual coupling stronger than that required for phase synchronization but weaker than that necessary for complete synchronization. The array was forced periodically and the oscillations have been recorded for several values of the forcing amplitude. Because of the coupling, several of the oscillators in this parameter range demonstrate complex patterns of oscillations so that with the Hilbert transform method it was impossible to define the phase straightforwardly (see the inset in Fig. 2). Nevertheless, with applying our method we were able, without any special adjustment, to determine the frequencies of all oscillators in the array and to show that with increasing of the forcing amplitude they become phase synchronized with the external force, Fig. 2.

Our next application is the analysis of two coupled oscillators with ill-defined phases. The scalar signals

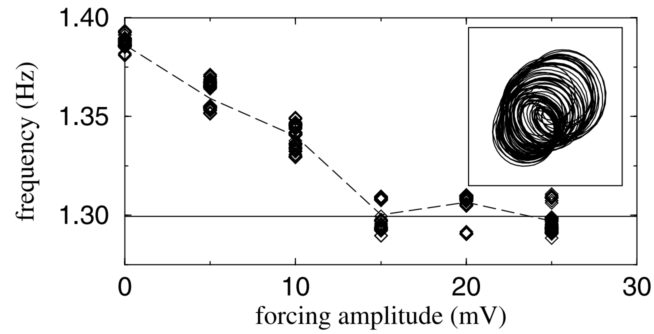

FIG. 2. Frequencies of all 64 chemical oscillators for different amplitudes of the forcing (the frequency of the external force is shown with a horizontal line). The dashed line shows the frequency averaged over the ensemble. The inset shows the representation of one of the oscillators in the coordinates "signal—its Hilbert transform," for the forcing amplitude 20 mV.

$x_{1,2}$ from these systems are used as inputs for two measuring devices, i.e., $x_{1,2}$ drive two identical ensembles (1) or (2). The outputs of the devices are two frequencies $\Omega^p_{1,2}$. The onset of the equality $\Omega^p_1 = \Omega^p_2$ with the increase of coupling will reflect the synchronization of the complex systems under consideration. As a particular example, we consider two weakly coupled Rössler systems with funnel attractors:

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2} y_{1,2} - z_{1,2} + \eta(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + 0.22 y_{1,2} + \eta(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= 0.1 + z_{1,2}(x_{1,2} - 8.5), \end{aligned} \quad (4)$$

where $\omega_1 = 0.98$, $\omega_2 = 1.03$. Application of the method (Figs. 3 and 4) reveals synchronization for coupling parameter $\eta \gtrsim 0.05$.

The particular parameter values in (4) allow us to compare our approach with direct phase measurements. Indeed, for most η (except for an interval $0.04 \gtrsim \eta \gtrsim 0.03$), the trajectory in the coordinates $(\dot{x}, \dot{y})$ rotates around the origin and the phase $\phi = \arctan(\dot{y}/\dot{x})$ is well defined [12]; the resulting frequencies are also shown in Fig. 4. Some discrepancy with our techniques is seen for the coupling below synchronization threshold due to two reasons: the failure of the direct phase measurement and not well-expressed plateaus in Ω_k vs ω_k plots. Nevertheless, synchronous regimes are perfectly revealed by our method.

To get an insight in the properties of phase synchronization in systems with an ill-defined phase, we computed the Lyapunov exponents (LEs) as a function of the coupling (Fig. 4). For systems with a well-defined phase and weak phase diffusion, the frequency locking transition is known to occur approximately at the value of coupling at which one of zero LEs becomes negative. As follows from Fig. 4, for coupled systems (4) synchronization sets in for essentially larger coupling, but nevertheless prior to the coupling at which one of the positive LEs becomes negative. Thus, we indeed have an interval of coupling values where the phase synchronization takes place (in the sense

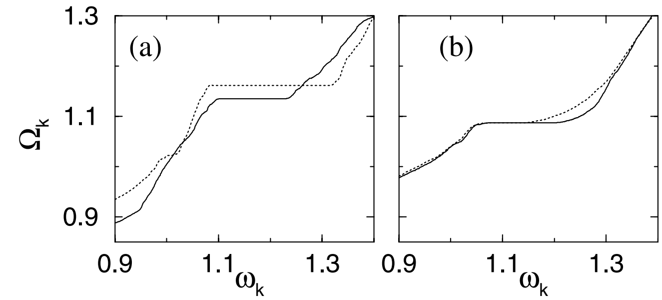

FIG. 3. Output of the frequency measuring device for two Rössler systems (4). (a) No coupling, (b) coupling, $\eta = 0.05$. Note that the common frequency in (b) lies below both frequencies in (a); such a frequency shift is usual for diffusive coupling; it appears due to dependence of the frequency on the amplitudes, the latter being reduced due to coupling.

that the phases are adjusted while there exist two positive LEs corresponding to the amplitudes). To explain why the locking sets in for relatively large coupling, we note that the phase dynamics in chaotic systems is qualitatively similar to those in noisy systems, and in the systems with ill-defined phase, fluctuations of the latter are extremely strong. Thus, one needs strong stability of the phase difference to suppress the divergence of the phases—as a result the synchronization onsets for finite negative LE. In other words, for systems with ill-defined phase the synchronization should be interpreted as a statistical effect that does not directly correspond to changes in the LEs.

Summarizing this example, we can say that the calculation of the LEs and the calculation of the frequencies deliver complementary characteristics of the dynamics. LEs characterize intrinsic microscopic organization of a strange attractor; they are difficult to obtain from experimental data. The frequency is a macroscopic statistical measure of the process; it can be easily obtained from experimental observations and characterizes only the oscillatory aspect of the dynamics. It is therefore not surprising that transitions in these two measures in general do not coincide. Only when some macroscopic variables can be directly associated with certain LEs (e.g., in

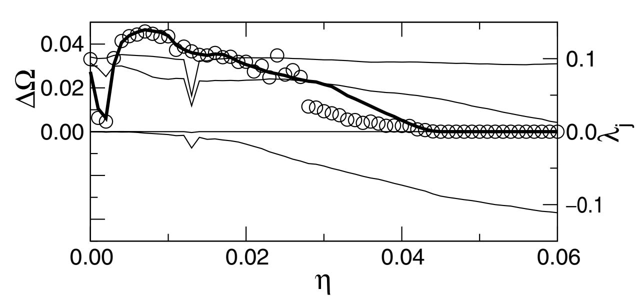

FIG. 4. Synchronization transition in system (4) traced by means of the frequency difference $\Delta\Omega = \Omega^p_2 - \Omega^p_1$ (circles) and by counting the number of rotations in the projection of the attractor onto the $(\dot{x}, \dot{y})$ plane (bold line). Four largest Lyapunov exponents λ_j are shown with solid lines.

systems with well-defined phases, the latter correspond to the zero LEs), the correspondence between the two characteristics is good.

We expect that for chaotic systems with very complicated topology of the phase space the plateau may be not seen at all and hence the frequency may be not found. This limitation of the presented method is due to the fact that systems with strong effective noise do not have a characteristic frequency and are not capable of synchronization. Like noisy systems, chaotic oscillators with ill-defined phase do not allow an unambiguous definition of synchronization; therefore, the distinction between the systems that can and cannot synchronize is smeared. Note also that with our method we define frequency of a signal, not of an oscillator. So, frequencies for different observables from one system can differ.

In a general context, we can interpret the ensemble of uncoupled oscillators with a common input [Eqs. (1) or (2)] as a nonlinear filter that picks up a certain frequency from a broad-band input. Indeed, the average velocity of the phase point rotation around the limit cycle in a single oscillator (1) is determined by some average properties of the aperiodic driving force. In particular, the system filters out the action when the signal $X(t)$ is nearly zero, because the point of the oscillator continues to rotate with the natural frequency. In this respect, our device is similar to the phase-locked loop [13]. The latter provides a phase of an input even during epochs when the amplitude of the input is small, of the same order as the underlying noise. This suggests that the method described can be used for estimation of the phase during the dynamic evolution. In particular, taking the natural frequency in (2) in the middle of the plateau, one can use the corresponding phase $\phi_k(t)$ as an "estimate" to the signal's phase. We checked this for the signal from the Lorenz attractor where a comparison with the "true" phase obtained by other methods is possible [2]. The results show that the phase $\phi_k(t)$, although it does not coincide with the true one, follows the same pattern of deviations from the uniform rotation.

An important issue is how the above defined frequency is related to the power spectrum of the signal. As we have shown in Fig. 1(b), it does not coincide with the maximal peak frequency of the spectrum (although we expect closeness for narrow-band signals). The reason is that the calculation of the power spectrum can be represented as signal filtering by a set of linear selective resonators; thus, the spectrum measures the ability of the signal to excite resonantly linear oscillators. Our device performs nonlinear filtering of the signal by an array of self-sustained oscillators, and it measures the ability of the signal to synchronize such oscillators. It is noteworthy that one can use not only self-sustained oscillators, but also other nonlinear elements capable of locking, e.g., Josephson junctions (cf. [14]).

In summary, we have proposed a locking-based method for frequency determination from complex signals. With this technique, we succeed in detecting synchronization between systems that do not allow direct estimation of their phases and frequencies. The method can be used in case of many interacting systems; it can be easily implemented for experimental data.

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[1] A. S. Pikovsky, Sov. J. Commun. Technol. Electron. **30**, 85 (1985); E. F. Stone, Phys. Lett. A **163**, 367 (1992); M. Rosenblum, A. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[2] A. Pikovsky *et al.*, Physica (Amsterdam) **104D**, 219 (1997).

[3] U. Parlitz *et al.*, Phys. Rev. E **54**, 2115 (1996); C. M. Ticos *et al.*, Phys. Rev. Lett. **85**, 2929 (2000); E. Allaria *et al.*, Phys. Rev. Lett. **86**, 791 (2001).

[4] I. Z. Kiss and J. L. Hudson, Phys. Rev. E **64**, 046215 (2001).

[5] A. S. Pikovsky *et al.*, in *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCH, Weinheim, Germany, 1999), pp. 305–328; *A focus issue on phase synchronization in chaotic systems*, edited by J. Kurths [Int. J. Bifurcation Chaos Appl. Sci. Eng. **7** (2000)].

[6] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University, Cambridge, 2001).

[7] A. Pikovsky, M. Rosenblum, and J. Kurths, Europhys. Lett. **34**, 165 (1996).

[8] The mean value can slightly influence the result; see Fig. 1(c).

[9] Practically, the middle point of the plateau Ω_k^p was determined from the minimum of the running variance $\sum_{j=k-L}^{k+L}(\Omega_j - \bar{\Omega}_k)^2$, where $\bar{\Omega}_k = (2L+1)^{-1}\sum_{j=k-L}^{k+L}\Omega_j$. Variation of L from 3 to 10 gave no essential difference. This method provides also a smoothening of the Ω_k vs k curve.

[10] P. S. Landa, *Self-Oscillations in Systems with Finite Number of Degrees of Freedom* (Nauka, Moscow, 1980).

[11] W. Wang, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **86**, 4954 (2001); Chaos **10**, 248 (2000).

[12] It is easy to see that the transformation to the coordinates $(\dot{x}, \dot{y})$ proposed in Chen *et al.* [Phys. Rev. E **64**, 016212 (2001)] is not of general use. So, it also fails for the driven system (3) and another parameter choice in (4), whereas our technique reveals phase synchronization in these cases as well.

[13] R. E. Best, *Phase-Locked Loops* (McGraw-Hill, New York, 1984).

[14] S. P. Benz and C. J. Burroughs, Appl. Phys. Lett. **58**, 2162 (1991).