Phase Synchronization of Chaotic Rotators

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We demonstrate the existence of phase synchronization of two chaotic rotators. Contrary to phase synchronization of chaotic oscillators, here the Lyapunov exponents corresponding to both phases remain positive even in the synchronous regime. Such frequency locked dynamics with different ratios of frequencies are studied for driven continuous-time rotators and for discrete circle maps. We show that this transition to phase synchronization occurs via a crisis transition to a band-structured attractor.

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Phase synchronization of chaotic oscillators, first described in [1,2], has attracted large interest recently [3–6] (for a review, see [7]). It has been found that due to a periodic external forcing or to coupling with another chaotic oscillator, the phase of a chaotic system can be locked, very much similar to the phase locking of periodic oscillations. This effect has been observed in several experiments ranging from electronic circuits via plasma tube to laser systems [8–10].

The main feature that makes the synchronization of the phase in a chaotic oscillator feasible is that the Lyapunov exponent corresponding to the phase is zero. This means that the phase shifts are neutral (as should be indeed the time shifts in an autonomous system) and are therefore rather sensitive to external pacing. This is in contrast to the other main type of chaotic synchronization—complete synchronization. The latter occurs when the states of two chaotic systems coincide due to coupling; the interaction should be strong enough to suppress the chaotic instability and to make one of the Lyapunov exponents of the system negative [11–13].

Contrary to this, the entrainment of the phase can happen already for relatively small coupling or forcing. Based on the qualitative arguments presented above, it has been argued that phase synchronization of driven oscillators is impossible. Indeed, chaos in a nonautonomous driven system in general possesses no zero Lyapunov exponent. This means that the phase of such an oscillator is not free but is connected to the forcing and thus cannot be entrained.

In this Letter, we describe a class of systems that provide a counterexample to these arguments: here the phases are chaotic with a positive Lyapunov exponent but can nevertheless be entrained. These systems are chaotic rotators; i.e., the dynamical variable here is the angle variable. Typical examples of this class are a periodically driven pendulum, a periodically driven Josephson junction (see, e.g., [14,15] for study of chaos in these systems), and also phase-locked loops can be described as driven rotators [16]. We argue below that the synchronization of chaotic phases is a novel phenomenon that cannot be reduced either to phase synchronization of autonomous chaotic oscillators or to complete or generalized synchronization of general chaotic systems. In particular, it does not correspond to a change in the Lyapunov spectrum, but to a change in topological structure of the chaotic attractor.

Qualitatively, the possibility of phase synchronization of coupled chaotic rotators can be understood as follows. If the angle variable in these systems (hereafter called phase) is chaotic, its dynamics can be generally represented as a biased random walk: the mean velocity corresponds to the mean frequency, and the stochastic variations can be characterized via the diffusion constant. This dynamically generated biased diffusion has been intensively studied for different chaotic rotators [14], as well as for the corresponding discrete system, the circle map [17,18]. For synchronization it is essential that the mean frequency in a continuous way depends on the system parameters (e.g., on the torque for the driven pendulum and on the external current for the Josephson junction) [19]. This means that the mean frequency can be easily adjusted with a small variation of these parameters or due to a small interaction with another chaotic rotator. Thus, phase synchronization in the sense of the adjustment of mean frequencies is possible. Note that this phase synchronization does not necessarily mean the appearance of a stable relation between the phases through the change of the sign of one of the Lyapunov exponents in the system. Both phases can remain chaotic, but their mean rotation velocities are entrained.

To illustrate the possibility of phase synchronization of chaotic phases, we analyze a simple system of two coupled rotators

$$\ddot{\psi}_1 + \gamma_1\dot{\psi}_1 + f_1(\psi_1) = F_1(t) + \varepsilon(\dot{\psi}_2 - \dot{\psi}_1), \quad (1)$$

$$\ddot{\psi}_2 + \gamma_2\dot{\psi}_2 + f_2(\psi_1) = F_2(t) + \varepsilon(\dot{\psi}_1 - \dot{\psi}_2). \quad (2)$$

Here $f_{1,2}$ are 2π-periodic functions and ε is the coupling constant. Both rotators are driven periodically $F_{1,2}(t) = \alpha_{1,2} + \beta_{1,2}\sin(\omega_{1,2}t + \varphi_{1,2})$. In general, all the parameters of the two systems, including the frequencies of the periodic forces $\omega_{1,2}$, can be different. In the context of the Josephson junctions [20,21], system (1),(2) describes resistively coupled junctions, which are driven by the external currents having dc components $\alpha_{1,2}$ and ac components with amplitudes $\beta_{1,2}$. In the calculations below we use functions $f_{1,2}(\psi) = \exp[10(\cos\psi - 1)]\sin(8\psi)$ which model localized in the phase space nonlinearities.

In Fig. 1 we show the dependence of the difference between the variables $\psi_1 - \psi_2$ on different couplings. One can see that sufficiently large coupling results in a nearly constant phase difference. Notably, in all the regimes shown in Fig. 1 the two largest Lyapunov exponents are positive. Thus, this example demonstrates the existence of synchronization of chaotic phases.

To get a deeper insight in this effect, it is convenient to study a discrete in time model. The simplest model yielding chaotic rotations is the circle map

$$\phi(t+1) = \phi(t) + b + F(\phi(t)), \qquad t = 0, 1, 2, \ldots. \tag{3}$$

Here $F(\cdot)$ is a 2π-periodic function, and b is a parameter governing the frequency of rotations. The mean rotation frequency in map (3) is nothing else but the rotation number

$$\rho = \lim_{T\to\infty} \frac{\phi(T) - \phi(0)}{T}. \tag{4}$$

In our consideration below we take, following [22], the piecewise linear nonlinearity: $F(\phi) = c\phi, -\pi < \phi < \pi$. The advantage of this choice [compared to the other popular map with $F(\phi) = c\sin\phi$] is that it ensures chaos for all $c > 1$ without periodic windows. Our numerical simulations show that the choice of the nonlinearity is not crucial: similar regimes can be observed with other functions $F(\phi)$ as well.

We describe symmetrically coupled circle maps (3), i.e., the two-dimensional system

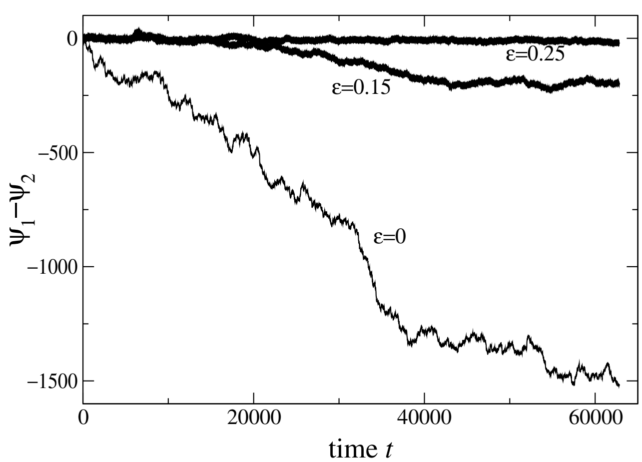

FIG. 1. Phase dynamics in coupled chaotic rotators (1),(2) for $\gamma_{1,2} = 0.1$, $\omega_{1,2} = 0.5$, $a_{1,2} = 0.01$, $\beta_1 = 1.4$, and $\beta_2 = 1$. The phase difference grows slower with increasing of the coupling and is bounded for $\varepsilon = 0.25$. The largest Lyapunov exponent in the system does not essentially depend on the coupling and for all presented regimes is $\lambda_1 \approx 0.12$. The second Lyapunov exponent decreases with the coupling and is $\lambda_2 = 0.04$ for $\varepsilon = 0.15$ and $\lambda_2 = 0.02$ for the synchronous regime at $\varepsilon = 0.25$.

$$\begin{aligned}\phi_1(t+1) = \phi_1(t) + b_1 + F_1(\phi_1(t)) \\ + \varepsilon G_1(\phi_1(t), \phi_2(t)),\end{aligned} \tag{5}$$

$$\begin{aligned}\phi_2(t+1) = \phi_2(t) + b_2 + F_2(\phi_2(t)) \\ + \varepsilon G_2(\phi_2(t), \phi_1(t)).\end{aligned} \tag{6}$$

Here ε is the coupling constant, and the parameters of two maps $b_{1,2}$ and $c_{1,2}$ are, in general, different. To characterize synchronization, we calculate the rotation numbers $\rho_{1,2}$ for both phases according to (4). If both rotation numbers coincide and both Lyapunov exponents of the system (5),(6) are positive, then we have synchronous chaotic rotations.

We study first the case of dissipative coupling, when $G_1 = F_2(\phi_2) - F_1(\phi_1), G_2 = F_1(\phi_1) - F_2(\phi_2)$. For the piecewise linear maps the Lyapunov exponents in the coupled system can be found analytically:

$$\lambda_{1,2} = \log\left|1 + (1-\varepsilon)c_+ \pm \sqrt{\varepsilon^2 c_+^2 + (1-2\varepsilon)c_-^2}\,\right|,$$

where $c_\pm = c_1 \pm c_2$. Determining the rotation numbers $\rho_{1,2}$ numerically, we identify an interval of coupling ε where these numbers coincide (Fig. 2). In this synchronization region both phases are chaotic and obey an irregular motion consisting of regular bias and diffusion. Because of the coupling, the mean velocities are adjusted yielding synchronization, while the diffusion is not suppressed (Fig. 3).

We have found synchronization of chaotic rotators for different types of coupling. As an illustration we also present results for the nonlinear coupling functions $G_1(\phi_1, \phi_2) = -G_2(\phi_2, \phi_1) = \sin(\phi_2 - \phi_1)$. In this case the Lyapunov exponents cannot be found analytically and should be determined numerically. An example of a synchronous regime where both Lyapunov exponents are

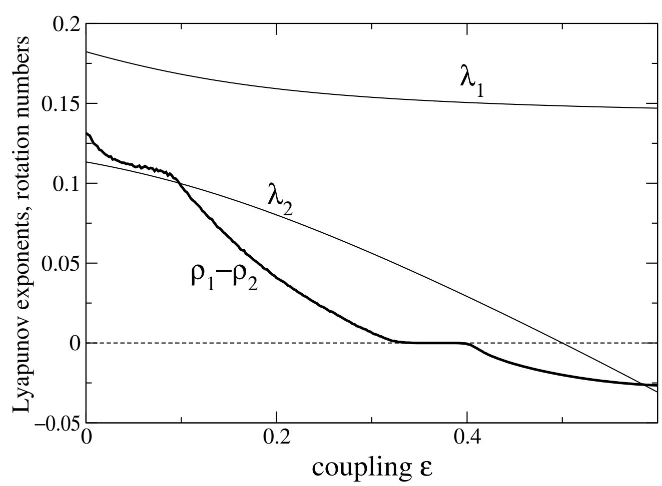

FIG. 2. Lyapunov exponents and the difference between the rotation numbers for the dissipative coupled circle maps (5),(6) with parameters $b_1 = 0.57$, $b_2 = 0.55$, $c_1 = 0.2$, and $c_2 = 0.12$. In the interval of couplings $0.33 < \varepsilon < 0.4$ the rotation numbers coincide and the phases are synchronous.

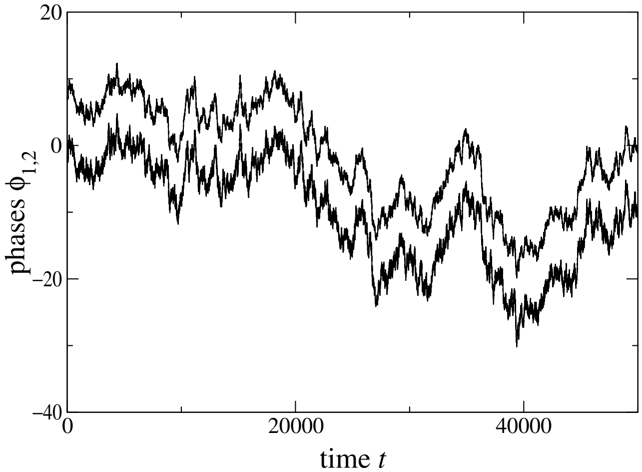

FIG. 3. Diffusion of the phases in the synchronous regime (the same parameters as in Fig. 2 and $\varepsilon = 0.38$) demonstrates coherence on the large scale.

positive is shown in Fig. 4. We encounter here not the simplest coincidence of the frequencies, but a high-order 3:1 synchronization; i.e., in the synchronous state the ratio between the rotation numbers is $\rho_1/\rho_2 = 3$.

It is instructive to study which structural changes in the phase space correspond to this synchronization transition. As this transition is not accompanied by a change of the sign of Lyapunov exponents, it is a transition inside chaos. In Fig. 5 we show the phase portraits inside and somewhat outside (to see some remnants of the bands) of the 3:1 synchronization region of Fig. 4. We find that in the synchronous regime the attractor consists of a few bands,

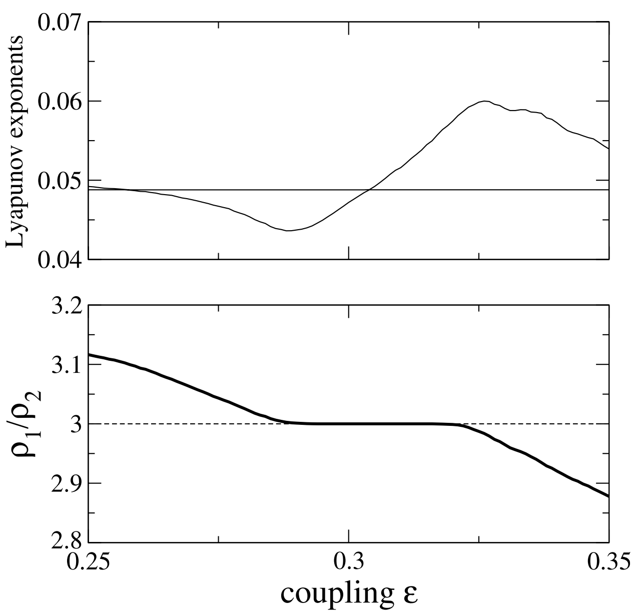

FIG. 4. The 3:1 synchronization region in chaotic circle maps coupled by *sinus* functions $G_{1,2}$. Both Lyapunov exponents in this region are positive. The parameters are $b_1 = 2$, $b_2 = 0.6$, and $c_{1,2} = 0.05$.

while with the loss of synchronization these bands disappear. Hence, the transition between these states is an interior crisis [23].

If analyzed in the extended phase space $-\infty < \phi_{1,2} < \infty$, the banded structure corresponds to a diffusion along a strip $\phi_1 \propto 3\phi_2$. The system performs a biased random walk along the strip, accompanied with bounded chaotic fluctuations in the transverse direction. It is clear that the mean frequencies of the two rotations coincide. Outside the synchronization region, the diffusion is unbounded in both directions; thus there is no limitation on the mean velocities and they are different.

This effect of synchronization of chaotic rotators can be understood in the framework of the general synchronization theory [7]. The main idea is that chaotic rotations have two scales in the phase space (this separation has been successfully used in the theory of chaotic diffusion; see [22,24]). The small scale is related to the motion within one periodicity cell $0 \leq \phi < 2\pi$; here the dynamics is essentially chaotic. On the large scale ($\gg 2\pi$) the motion consists of random transitions between the cells, and here only the transitions rates are important. Thus, on the large scale the phase dynamics is equivalent to the dynamics of the phase of a noise-driven periodic oscillator, where one also encounters a biased random walk. Consequently, on the large scale the synchronization of chaotic rotators appears very much similar to the phase locking of periodic rotators or oscillators in the presence of noise. In particular, the banded structure of Fig. 5 can be interpreted as a smeared stable periodic orbit on the torus that represents the 3:1 phase locking. We also note that a similar separation between chaos on a small scale and noisy dynamics on a large scale has been discussed for certain examples of self-organized criticality [25].

In conclusion, we have demonstrated the existence of phase synchronization of chaotic rotators, at which both phases are chaotic [26]. It can be observed both in continuous-time systems and discrete models. In the synchronous state both phases exhibit a random walk with,

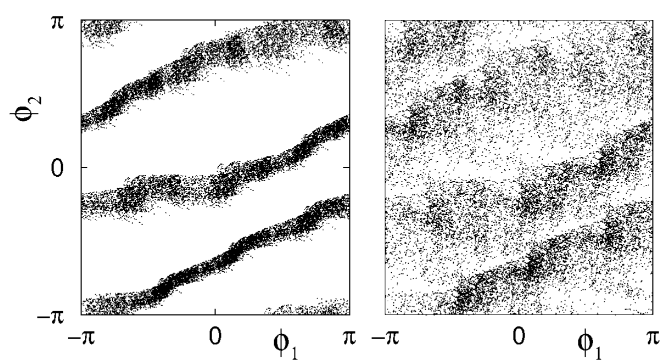

FIG. 5. The phase portraits of the system of two coupled circle maps for the same parameters as in Fig. 4. Left panel: synchronous state at $\varepsilon = 0.3$. Right panel: outside of the synchronization region $\varepsilon = 0.275$.

in general, a rational relation between mean velocities. We have presented particular examples of 1:1 and 3:1 synchronizations. In the phase space the synchronous regime corresponds to a banded structure of the attractor. This synchronization transition occurs via an interior crisis, at which the bands disappear. We have shown that this novel type of synchronization relies heavily on the existence of two time scales in the phase dynamics: a short time scale is responsible for chaos, while the longer one is responsible for a biased phase diffusion. The synchronization occurs at this longer time scale and can thus be considered as a "coarse grained" phenomenon. It gives therefore an interesting example of nontrivial interaction between microscales and macroscales, what is now of great interest in the context of irreversible statistical mechanics [24], self-organized criticality, etc. The observed phenomenon could potentially be investigated experimentally in many practically important systems of coupled chaotic rotators, e.g., Josephson junctions, and continuous and digital phase-locked loops. Moreover, we expect that synchronization of chaotic phases can happen in forced interacting oscillators, in particular, in neural relaxation oscillators. Investigation of the presented effect in networks of such elements should be the focus of a future study.

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[26] Of course, a more trivial case when only one Lyapunov exponent is positive is also possible; in this case the phase synchronization appears as a manifestation of the generalized one.