Transition to Coherence in Populations of Coupled Chaotic Oscillators: A Linear Response Approach

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We consider the collective dynamics in an ensemble of globally coupled chaotic maps. The transition to the coherent state with a macroscopic mean field is analyzed in the framework of the linear response theory. The linear response function for the chaotic system is obtained using the perturbation approach to the Frobenius-Perron operator. The transition point is defined from this function by virtue of the self-excitation condition for the feedback loop. Analytical results for the coupled Bernoulli maps are confirmed by the numerics.

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Ensembles of globally coupled nonlinear oscillators attracted much attention recently. Such models arise in the study of Josephson junctions [1], multimode lasers [2], and charge density waves [3]. In the living world one uses similar models to describe chirps of grasshoppers [4], neurons [5,6], and yeast cells [7]. A particular interest attracted ensembles of chaotic oscillators [8–12]. Recently, an experimental investigation of 64 globally coupled chaotic electrochemical oscillators have been performed [13]. These studies have revealed that already coupling of identical chaotic oscillators demonstrates nontrivial synchronization patterns.

The first step in the emerging of order and coherence in an ensemble of globally coupled elements is the appearance of a macroscopic mean field. Indeed, the global coupling can be represented as the coupling through the mean field, and the absence of the latter means the absence of interaction and full independence of the elements. In the thermodynamic limit, where the number of elements tends to infinity, one can interpret the appearance of the mean field as a nonequilibrium phase transition. Such a transition is well described for ensembles of noise-driven oscillators [14,15], by virtue of the bifurcation analysis of the nonlinear Fokker-Planck equation. In the case of chaotic deterministic systems one has to consider the nonlinear Frobenius-Perron or Liouville equations (in the cases of discrete and continuous dynamics, correspondingly), which is by far a more difficult task.

In this Letter, we develop an approach to the description of the transition in the ensembles of chaotic elements, based on the response theory for chaos. The main idea is to "break" the feedback loop due to coupling in the ensemble, and to consider the effect of a small periodic force on a chaotic oscillator. With the help of the linear response theory (cf. [16,17]) we can find the linear response of the distribution function of chaos to this force. Then, for each type of coupling the response function can be calculated. After that we can again "close" the feedback loop, reducing the problem of the onset of the mean field to the analysis of stability of a linear discrete dynamical system with a given response characteristic. Note that a similar approach for noise-driven periodic oscillators has been developed in [18].

Our basic model is the system of N globally coupled chaotic maps

$$x_i(t+1) = f(x_i(t)) + \varepsilon g(x_i(t)) a(t), \qquad a(t) = \frac{1}{N}\sum_{i=1}^{N} q(x_i(t)). \tag{1}$$

Here ε is the coupling constant. Note that the coupling performs via the mean field a. We write the coupling in a general form, using arbitrary functions $g(x)$ and $q(x)$. The only natural condition is that in the thermodynamic limit the mean field vanishes for $\varepsilon = 0$, i.e., $\langle q(x)\rangle_0 = 0$, where $\langle\,\rangle_0$ denotes the average over the stationary distribution of the map $x \mapsto f(x)$. This condition ensures that the disordered state with vanishing mean field a exists for all couplings ε (as will be shown below, the instability of this state leads to the transition to ordered state with nonzero mean field a).

In previous investigations [19–22] it has been demonstrated numerically that in system (1) a transition from vanishing to a macroscopic mean field a can occur at some critical coupling strength ε_c. Generally, this transition can be interpreted as a bifurcation in the self-consistent nonlinear Frobenius-Perron equation for the probability density [20,21]. However, except for a special case of noisy homographic maps [22], no analytical approach to the description of the transition has been developed. Below, we develop such an approach, basing it on the linear response theory of chaos.

To apply the linear response theory we consider a supplementary problem of the effect of small periodic force (with frequency ω) on a single map:

$$x(t+1) = f(x(t)) + \alpha g(x(t)) e^{i\omega t}. \tag{2}$$

The forcing affects the dynamics, and our goal is to find the variations of the probability distribution density in the first order in $\alpha \ll 1$. Denoting the right-hand side of (2) as $F_t(x)$, we write the Frobenius-Perron operator for the density $\rho_t(x)$

$$\rho_{t+1}(x) = \int \delta[x - F_t(y)]\rho_t(y)\,dy\,. \qquad (3)$$

Considering the map as defined on the interval $[0, 2\pi)$ (this can always be achieved with a normalization), we can introduce the Fourier transform of the density $\rho_t(x) = \sum \psi_t(k)e^{ikx}$ and to obtain from (3) the corresponding Frobenius-Perron operator in the Fourier space:

$$\begin{aligned}\psi_{t+1}(k) &= \textstyle\sum_l R_t(k,l)\psi_t(l)\,,\\ R_t(k,l) &= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \exp[ilx - ikF_t(x)]\,dx\,.\end{aligned} \qquad (4)$$

Taking into account that $F_t(x) = f(x) + \alpha g(x)e^{i\omega t}$ we can write up to the first order in α

$$R(k,l) = R^0(k,l) + \alpha R^1(k,l)e^{i\omega t}, \qquad (5)$$

where

$$R^0(k,l) = \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \exp[ilx - ikf(x)]\,dx\,,$$

$$R^1(k,l) = \tfrac{-ik}{2\pi}\textstyle\int_0^{2\pi} g(x)\exp[ilx - ikf(x)]\,dx\,.$$

Substituting this in (4) and writing $\psi_t(k) = \psi^0(k) + \alpha e^{i\omega t}\psi^1(k)$ we obtain the equation for the complex amplitude of the perturbation ψ^1

$$e^{i\omega}\psi^1 = \sum_l [R^0(k,l)\psi^1(l) + R^1(k,l)\psi^0(l)]\,. \qquad (6)$$

One can formally write a solution of this linear system, but from this formal expression it is difficult to judge whether the solution is nonsingular. Indeed, as is well known in the theory of chaos (see, e.g., [16,23]), the response to small perturbations can be singular. This happens, e.g., in structurally unstable chaotic systems. In such systems, small changes of a parameter lead to a topologically nonequivalent dynamics, which can, e.g., be seen in the symbolic description or in the representation via unstable periodic orbits. Note that to this class belong even many systems where chaos persists in the whole parameter range (e.g., the Lorenz attractor and the tent map), let alone such nonhyperbolic examples where small perturbation can lead to a periodic window (like in the logistic map). Response of the structurally unstable system is expected to be singular [23], which, in particular, can be seen from the fractal dependence of some statistical characteristics on a parameter [24]. Therefore, in order to be in the realm of validity of the linear response theory, we have to consider a structurally stable map $x \mapsto f(x)$.

Below, we study the simplest such map—the Bernoulli map $f(x) = 2x \bmod 2\pi$. In this case $R^0(k,l) = \delta(2k - l)$ and $\psi^0(k) = \delta(k)(2\pi)^{-1}$; thus Eq. (6) reduces to

$$\psi^1(k)e^{i\omega} = \psi^1(2k) - \frac{ik}{2\pi}G(2k)\,, \qquad (7)$$

where $G(l)$ is the Fourier harmonics of $g(x)$,

$$G(l) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ilx}g(x)\,dx\,. \qquad (8)$$

The solution of (7) gives the final expression for the response of the probability density:

$$\psi^1(k) = \frac{-ik}{4\pi}\sum_{m=1}^{\infty} 2^m e^{-im\omega}G(2^m k)\,. \qquad (9)$$

The next step is the closing of the feedback loop in model (1). To this end we have to find which mean field is generated in the first order in α in system (2). In the thermodynamic limit $N \to \infty$ we can calculate the mean field as the average over the probability density:

$$\lim_{N\to\infty}\sum_{i=1}^{\infty}\frac{1}{N}q(x_i(t)) = \langle q(x(t))\rangle = \int_0^{2\pi}\rho_t(x)q(x)\,dx\,.$$

Substituting here the expression for the density

$$\rho_t(x) = \sum_k [\psi^0(k) + \alpha e^{i\omega t}\psi^1(k)]e^{ikx},$$

we obtain

$$\langle q\rangle = \alpha K(\omega)e^{i\omega t}, \qquad K(\omega) = \sum_k Q(-k)\psi^1(k)\,, \qquad (10)$$

where $Q(k)$ are the Fourier harmonics of the function $q(x)$ defined similar to (8). Here we have taken into account that the unperturbed invariant density does not contribute to the mean field. In the particular case of the Bernoulli map we obtain from (9) and (10)

$$K(\omega) = \sum_{k=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{-ik}{4\pi}2^m e^{-im\omega}G(2^m k)Q(-k)\,. \qquad (11)$$

The function $K(\omega)$ is the linear response function of system (2) for the observable $q(x)$. In terms of Eq. (1) it describes the linear dynamics of the mean field a in the spectral representation. The condition of the generation of a in the original ensemble (1), i.e., the condition of instability of the state $a = 0$, can be formulated as the condition for nondecay of perturbations within the feedback loop (the self-consistency condition). The total amplification is the product of the factors ε and $K(\omega)$. Thus the instability threshold is defined by the relation

$$\varepsilon K(\omega) = 1\,. \qquad (12)$$

This relation determines the frequency of oscillations at the instability threshold [from the imaginary part of (12)] and the critical value of coupling ε_c [from the real part of (12)].

As a particular example we consider the following coupling in the Bernoulli map:

$$g(x) = \sin 2x + \sin 4x, \qquad q(x) = \cos x\,. \qquad (13)$$

Substituting this in (11) yields

$$K(\omega) = -\frac{1}{4\pi}(2e^{-i2\omega} + e^{-i\omega})\,, \qquad (14)$$

and from (12) we obtain the following critical values for the coupling:

$$\varepsilon_{cl} = 2\pi, \qquad \omega_1 = \arccos(-\tfrac{1}{4}); \qquad \varepsilon_{c2} = -\tfrac{4\pi}{3}, \qquad \omega_2 = 0\,. \tag{15}$$

These results of the theory are confirmed in numerical simulations, presented in Figs. 1 and 2.

Essential features of the transitions can be deduced already from the linear analysis above. The transition to coherence at ε_{c1} has nonzero frequency, so it should be a Neimark (discrete time Hopf) bifurcation. Numerics (Figs. 1a and 2) shows that it is supercritical. The transition at ε_{c2} occurs with the unit multiplier; thus one can expect here a transcritical transition to a nonzero fixed point (Fig. 1b) (a pitchfork bifurcation is excluded due to the absence of symmetry $a \mapsto -a$). Note that the scaling laws expected for these transitions ($\langle a^2 \rangle \sim |\varepsilon - \varepsilon_c|$ for the Neimark one and $\langle a^2 \rangle \sim |\varepsilon - \varepsilon_c|^2$ for the transcritical one) are distorted by finite-size effects.

The linear response approach above can be obviously generalized to the case when the mean field has its own dynamics. Such a situation appears, e.g., in a series array of Josephson junctions coupled by means of an external load [25]. The junctions are coupled via the common current, which obeys an additional equation (for the oscillatory circuit load considered in [25] this is the equation of a driven damped linear oscillator). In the linear response theory this additional dynamics of the mean field can be easily incorporated just by multiplying the response function of the chaotic map $K(\omega)$ with the response function of the mean field dynamics $L(\omega)$, thus yielding the stability condition

$$\varepsilon L(\omega) K(\omega) = 1$$

instead of (12). As a simple example let us consider the ensemble (1) with the inertial dynamics the mean field

$$a(t) = \gamma a(t-1) + \frac{1}{N} \sum_{1}^{N} q(x_i(t))\,.$$

In this case $L(\omega) = (1 - \gamma e^{i\omega})^{-1}$ with an obvious modification of the transition values (15).

Another straightforward generalization of the theory above is including an additive noise in the dynamics. In this case Eq. (1) is rewritten as

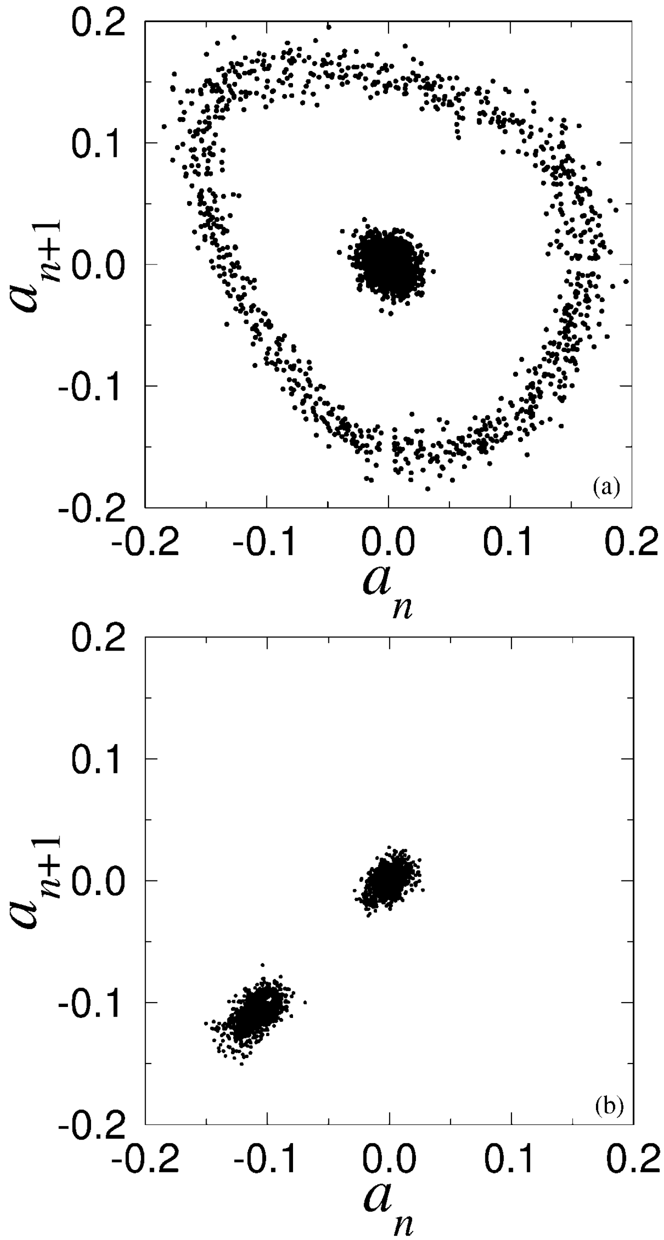

FIG. 1. Dynamics of the mean field in the ensemble of $N = 10^4$ coupled maps represented as a recurrence plot a_{n+1} vs a_n. (a) The transition at ε_{c1}. In the disordered state at $\varepsilon = 5$ the mean field vanishes up to finite-size fluctuations (a cloud around $a_n = a_{n+1} = 0$); in the coherent state ($\varepsilon = 7$) nearly periodic oscillations are observed. (b) The transition at ε_{c2} is one from a zero equilibrium point ($\varepsilon = -3$) to a nearly constant mean field at $\varepsilon = -5$.

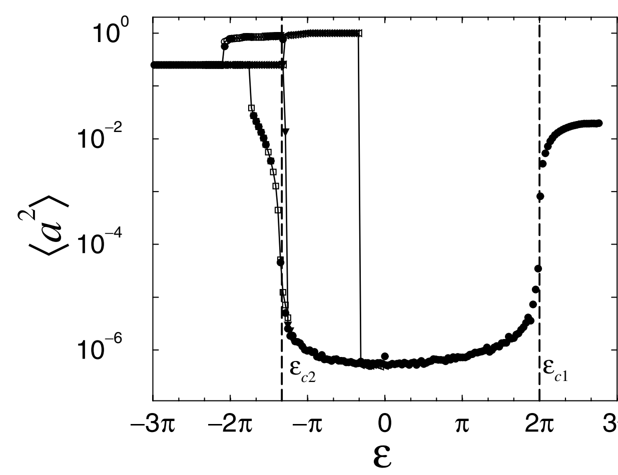

FIG. 2. The dependence of $\langle a^2 \rangle$ in the ensemble of $N = 10^6$ maps on the coupling ε reveals transitions at the critical values predicted by the linear theory (15) (dashed lines). Two large-amplitude branches at $\varepsilon < \varepsilon_{c2}$ correspond to one-cluster solution (largest branch, this solution is a fixed point $x_i = 0$ for small $|\varepsilon|$ and exhibits period doublings for larger $|\varepsilon|$) and to a two-cluster period-two solution (middle branch). Jumps to these branches make the transition at ε_{c2} hysteretic.

$$x_i(t+1) = f(x_i(t)) + \xi_i(t) + \varepsilon g(x_i(t))a(t), \quad a(t) = \frac{1}{N}\sum_1^N q(x_i(t)), \tag{16}$$

where $\xi_i(t)$ are independent equally distributed random variables. The Frobenius-Perron equation (3) is generalized to this case by including the convolution of ρ with the probability density of ξ (see, e.g., [26]). In the Fourier space one simply multiplies the operator R (4),(5) with the characteristic function of noise:

$$\tilde{R}(k,l) = w(k)R(k,l),$$

where $\tilde{R}, R$ are the operators with and without noise, respectively, and $w(k)$ is the Fourier transform of the probability density W_ξ of the random noise:

$$w(k) = \int_{-\infty}^{\infty} W_\xi(x)e^{-ikx}\,dx.$$

With this modification, all the linear response theory holds. Moreover, the presence of the fast decaying factor $w(k)$ regularizes the Frobenius-Perron equation, so that a nonsingular solution for the response function can be expected even when the deterministic dynamics is structurally unstable, or even when in the deterministic dynamics periodic windows are present. In the particular case of Bernoulli maps, the final expression for the response function K (11) is modified to

$$K(\omega) = \sum_{k=-\infty}^{\infty} \frac{-ikQ(-k)}{4\pi} \sum_{m=1}^{\infty} 2^m e^{-im\omega} G(2^m k) \times \prod_{l=0}^{m-1} w(2^l k).$$

Formally, the generalization of the theory above to the case of continuous-time oscillators is also simple. One just writes the Liouville equation instead of the Frobenius-Perron one, or the Fokker-Planck equation if the noise is present. However, analytic solution of the perturbation problem appears to be hardly feasible. Nevertheless, one can still proceed numerically, determining the linear response function from the simulations [17]. Because the fluctuations-dissipation theorem does not hold for generic chaotic systems, one has to rely on direct simulations. Namely, one has to numerically integrate the periodically driven system [in the discrete case to iterate the map (2)] and to calculate the spectral component of the output at the driving frequency. The amplitude and the phase of this component yield the linear response function $K(\omega)$. Then the condition similar to (12) gives the threshold of instability and the frequency of the appearing oscillations of the mean field. Examples of this analysis will be presented elsewhere.

In conclusion, we have developed a theory of the transition to coherent collective behavior in ensembles of globally coupled chaotic maps. The main idea is to define the linear response function according to the mode of coupling. Noteworthy, this function can be obtained from the analysis of a single system. The critical coupling follows then from the stability condition for the feedback loop of the mean field (12). We have also outlined several straightforward generalizations of the method, e.g., to the noisy systems. Less obvious is a generalization to the case of nonidentical interacting systems; it is the subject of current work.

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