

## Noise-Induced Dynamics in Bistable Systems with Delay

L. S. Tsimring

*Institute for Nonlinear Science, University of California–San Diego, La Jolla, California 92093-0402*

A. Pikovsky

*Department of Physics, University of Potsdam, Postfach 601553, D-14415 Potsdam, Germany*

(Received 5 July 2001; published 28 November 2001)

Noise-induced dynamics of a prototypical bistable system with delayed feedback is studied theoretically and numerically. For small noise and magnitude of the feedback, the problem is reduced to the analysis of the two-state model with transition rates depending on the earlier state of the system. Analytical solutions for the autocorrelation function and the power spectrum have been found. The power spectrum has a peak at the frequency corresponding to the inverse delay time, whose amplitude has a maximum at a certain noise level, thus demonstrating coherence resonance. The linear response to the external periodic force also has maxima at the frequencies corresponding to the inverse delay time and its harmonics.

DOI: 10.1103/PhysRevLett.87.250602

PACS numbers: 05.40.–a, 02.50.Ey, 05.40.Ca

The effects of random noise on bistable systems and the related phenomenon of stochastic resonance have received great attention in the last decade. As a result, a comprehensive theory and a whole range of experimental observations have emerged (for a recent review see [1]). In many physical as well as biological systems, the time-delayed feedback plays a significant role in the dynamics. These systems in the absence of noise have been thoroughly investigated using the theory of delay-differential equations [2]. The theory of stochastic delay-differential equations, in which effects of noise and time delay are combined, remains much less studied. Meanwhile, it appears that the combination of these features is ubiquitous in nature. Examples include biophysiological dynamics [3] and laser dynamics in optical cavities [4]. It is also believed that the combined effects of noise, bistability, and delay play an important role in gene regulatory networks [5].

The delayed stochastic bistable systems have been a subject of several recent papers [6–8]. In Ref. [6], a systematic statistical description of a certain class of stochastic delay-differential equations was developed in the limit of small time delay. More interesting, however, is the case of a large time delay which is comparable with the mean Kramers transition time determined by the noise intensity and the potential barrier height. In this case, resonant phenomena may occur which would lead to spontaneous oscillations of the system with a certain preferred frequency. In Refs. [7], Ohira and co-workers studied the related phenomenon of delayed random walks. In that model, the hopping probability depends on the position of the particle a given number of hops in the past. In certain cases, the particle diffusion is limited, and it exhibits quasiregular oscillations near the origin. In Ref. [8], Ohira and Sato studied a discrete-time two-state system in which the occupancy probabilities of the two states depended on the state of the system some  $N$  time steps before. While that model also showed some interesting resonant features,

it appears to us somewhat unrealistic, since the states of the system at two consecutive iterations are completely uncoupled, and its dynamics is, in fact, identical to that of a superposition of  $N$  independent one-dimensional maps affected by random noise. In most practically relevant cases, however, the state of the system should be affected in the first place by its immediate past, with additional correction arising from the time-delayed feedback.

In this Letter we study the effects of the thermal activation on bistable systems with additional time-delayed feedback. Our prototypical model is the overdamped particle motion in the double-well quartic potential  $U(x(t), x(t - T))$ , described by the Langevin equation

$$\begin{aligned} \frac{dx(t)}{dt} &= -\frac{\partial U(x(t), x(t - T))}{\partial x(t)} + \sqrt{2D} \xi(t) \\ &\equiv x(t) - x^3(t) + \epsilon x(t - T) + \sqrt{2D} \xi(t). \end{aligned} \quad (1)$$

Here  $T$  is the delay and  $\epsilon$  is the strength of the feedback, and  $\xi(t)$  is a Gaussian white noise with  $\langle \xi \rangle = 0$  and  $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$ . At small  $D$  and  $\epsilon$ , the particle spends most of the time near potential minima  $x = \pm 1$ , only occasionally jumping from one to another.

In our analytic description we neglect the small intrawell fluctuations and approximate (1) with a two-state (dichotomic) system, in which the dynamical variable  $s(t)$  takes two values  $s = \pm 1$  corresponding to  $x > 0$  and  $x < 0$ , respectively. This reduction has been successfully used in studies of the stochastic resonance [9]. The dynamics of  $s$  is determined by the switching rates, i.e., by the probabilities to switch  $s \rightarrow -s$ . Because of the delay, we have two switching rates depending on the state  $s(t - T)$ :  $p_1$  if the state at time  $t - T$  is the same as at time  $t$ , and  $p_2$  otherwise. Thus, the switching rate can be written as

$$p(t) = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2} s(t)s(t - T). \quad (2)$$

This rate dependence on the state at time  $t - T$  makes this process essentially non-Markovian.

A quantitative relation between rate process (2) and original model (1) can be easily established for small  $D$  and  $\epsilon$  by virtue of the Kramers formula for the escape rate [10]  $r_K = (2\pi)^{-1} \sqrt{U''(x_m)U''(x_0)} \exp[-\Delta U/D]$ , where  $x_m$  and  $x_0$  are the positions of the minimum and the maximum of the potential, respectively, and  $\Delta U$  is the potential barrier to cross over. For small  $D$ , the switching rates are small compared to the intrawell equilibration rate, and the probability density distribution is close to a narrow Gaussian distribution centered around  $x_m$ , and so the adiabatic approximation applies. For small  $\epsilon$ ,  $|x_m| = 1 \pm \epsilon/2$  depending on the sign of  $x(t)x(t - T)$ ,  $x_0 = 0$ , and in the first order in  $\epsilon$  we obtain

$$p_{1,2} = \frac{\sqrt{2 \pm 3\epsilon}}{2\pi} \exp\left[-\frac{1 \pm 4\epsilon}{4D}\right]. \quad (3)$$

Without loss of generality let us assume that the system is at state  $s = 1$  at time 0. We define  $n_{\pm}(t)$  to be the probability of attaining value  $\pm 1$  at time  $t$ . The master equations for  $n_{\pm}(t)$  is written in a usual way,

$$\begin{aligned} \dot{n}_+(t) &= -W_1(t)n_+(t) + W_1(t)n_-(t), \\ \dot{n}_-(t) &= -W_1(t)n_-(t) + W_1(t)n_+(t), \end{aligned} \quad (4)$$

where  $W_1(t) dt$  is the probability of transition from  $-1$  to  $+1$  within time interval  $(t, t + dt)$  and vice versa. In our stochastic model with time-delayed feedback,

$$\begin{aligned} W_1(t) &= p_1 n_+(t - T) + p_2 n_-(t - T), \\ W_1(t) &= p_2 n_+(t - T) + p_1 n_-(t - T). \end{aligned} \quad (5)$$

Substituting (5) in (4) and making use of the normalization condition  $n_-(t) + n_+(t) = 1$ , we obtain

$$\dot{n}_+(t) = p_2 n_-(t) - p_1 n_+(t) - (p_2 - p_1)n_-(t - T), \quad (6)$$

$$\dot{n}_-(t) = -p_1 n_-(t) + p_2 n_+(t) - (p_2 - p_1)n_+(t - T). \quad (7)$$

The correlation function  $C(\tau)$  is determined as

$$C(\tau) = \langle s(\tau)s(0) \rangle = \langle s(\tau) \rangle = n_+(\tau) - n_-(\tau) \quad (8)$$

[we recall that the initial state is  $s(0) = 1$ ]. Thus, replacing  $t$  with  $\tau$  and subtracting (7) from (6), we obtain

$$\frac{dC(\tau)}{d\tau} = -(p_1 + p_2)C(\tau) + (p_2 - p_1)C(\tau - T). \quad (9)$$

This equation should be complemented with the symmetry  $C(-\tau) = C(\tau)$  and the normalization  $C(0) = 1$  conditions.

The solution of this linear equation on the interval  $(0, T)$  can be found using ansatz  $C(\tau) =$

$A \exp(-\lambda\tau) + B \exp\lambda(\tau - T)$ . Plugging this ansatz in Eq. (9) yields  $\lambda = 2\sqrt{p_1 p_2}$ ,  $B = A(\sqrt{p_2} - \sqrt{p_1})/(\sqrt{p_2} + \sqrt{p_1}) \exp(-2\sqrt{p_1 p_2} T)$ . The constant  $A$  is found from the condition  $C(0) = 1$ , and we obtain

$$C(\tau) = \frac{(\sqrt{p_1} + \sqrt{p_2})e^{-\lambda\tau} + (\sqrt{p_2} - \sqrt{p_1})e^{\lambda(\tau - T)}}{\sqrt{p_1} + \sqrt{p_2} + (\sqrt{p_2} - \sqrt{p_1})e^{-\lambda T}}. \quad (10)$$

Using (9) and (10), one can easily calculate  $C(\tau)$  at all  $\tau > T$ ,

$$\begin{aligned} C(nT + \tau') &= e^{-(p_1 + p_2)\tau'} C(nT) + (p_2 - p_1) \\ &\times \int_0^{\tau'} C((n-1)T + t) e^{(p_1 + p_2)(t - \tau')} dt, \end{aligned} \quad (11)$$

where  $n = 1, 2, \dots$ , and  $0 < \tau' < T$ .

We plot the correlation function (10) and (11) as a function of normalized time  $\tau/T$  and dimensionless parameters  $p_{1,2}T$  in Fig. 1. Its structure differs depending on whether the feedback is positive [ $p_2 > p_1$ , which corresponds to a positive  $\epsilon$  in (1)] or negative ( $p_2 < p_1$ ,  $\epsilon < 0$ ). For positive feedback the correlation function is positive, and has maxima at  $\tau \approx nT$ . For negative feedback the peaks at  $\tau \approx nT$  have alternating signs. It is interesting to note that the peaks of the correlation function are always delayed with respect to  $nT$ . For  $\lambda T \gg 1$ , the time interval corresponding to the first peak is

$$\tau_1 = T + (\sqrt{p_2} - \sqrt{p_1})^{-2} \ln \frac{(\sqrt{p_1} + \sqrt{p_2})^2}{2(p_1 + p_2)}. \quad (12)$$

From the correlation function we can also determine the power spectrum  $S(\omega) = \int_{-\infty}^{\infty} C(\tau) \cos(\omega\tau) d\tau$ . It is convenient to derive the expression for  $S$  directly from Eq. (9). Denoting  $L(\omega) = \int_0^{\infty} C(\tau) \exp[i\omega\tau] d\tau$  and substituting here (9) we obtain

$$\begin{aligned} -C(0) - i\omega L &= -(p_1 + p_2)L \\ &+ (p_2 - p_1)e^{i\omega T}[L - I(\omega)], \end{aligned} \quad (13)$$

where

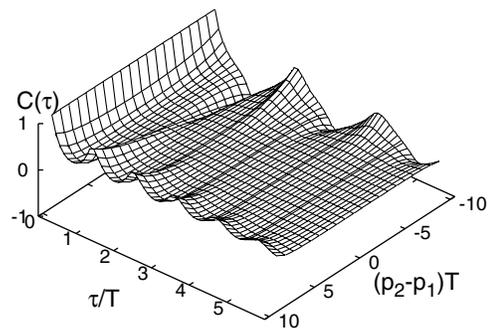


FIG. 1. The autocorrelation function in the two-state model as a function of the time lag  $\tau$  and the feedback strength  $p_2 - p_1$ , for  $(p_1 + p_2)T = 10$ .

$$I(\omega) = \int_0^T C(\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{(\sqrt{p_1} + \sqrt{p_2})[1 - e^{(-i\omega - \lambda)T}](i\omega + \lambda)^{-1} + (\sqrt{p_2} - \sqrt{p_1})[e^{-i\omega T} - e^{-\lambda T}](i\omega - \lambda)^{-1}}{\sqrt{p_1} + \sqrt{p_2} + (\sqrt{p_2} - \sqrt{p_1})e^{-\lambda T}}.$$

Using  $C(0) = 1$  and  $S(\omega) = 2 \operatorname{Re} L(\omega)$ , we obtain

$$S(\omega) = 2 \operatorname{Re} \frac{1 + (p_2 - p_1)e^{i\omega T} I(\omega)}{p_1 + p_2 - (p_2 - p_1)e^{i\omega T} - i\omega}. \quad (14)$$

We compare this analytic result with numerical simulations of the bistable oscillator (1) in Fig. 2 and find a very good agreement between theory and simulations. It can be further improved if one takes into account the neglected intrawell fluctuations which give rise to an additional Lorentzian contribution to the power spectrum with the width corresponding to the frequency of oscillations in one potential well, and the amplitude proportional to noise intensity  $D$  (see, e.g., [1]).

In Fig. 3 we show the dependence of the power spectrum on the noise intensity  $D$ , while the feedback parameter  $\epsilon$  is kept constant [the switching rates  $p_{1,2}$  are calculated according to (3)]. The peak at the main frequency  $\omega \approx 2\pi/T$  has a maximum at a certain noise level (for negative feedback the picture is similar, with the maximum near  $\omega \approx \pi/T$ ). This is a characteristic feature of the coherence resonance [11]: the coherence in the noise-driven system attains maximum at a “resonant” noise temperature. In the present case the underlying physical mechanism is the resonance between the Kramers rate and the delay. If the Kramers rate is small (for small noise intensity), a characteristic interval between the switches is larger than the delay time, and the latter is not displayed in the spectrum because the process is a purely Poissonian one (with a renormalized one due to the feedback switching rate). For an intermediate Kramers rate the switchings are highly influenced by the feedback, with a preferable periodicity with the delay time  $T$  being manifested as a peak in the spectrum. For large noise intensity the effect of the feedback decreases again, because the relative magnitude

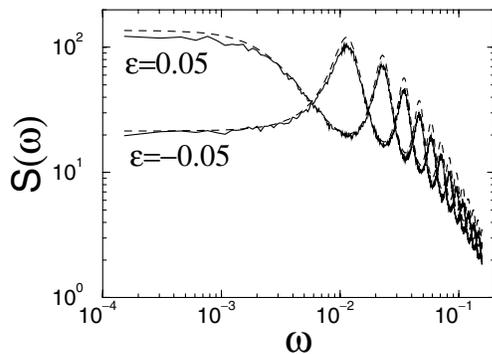


FIG. 2. Comparison of the power spectrum in Eq. (1) for  $D = 0.1$ ,  $T = 250$ ,  $\epsilon = \pm 0.05$  (solid line) with theory (14) (dashed line). Note that for positive and negative  $\epsilon$  the main peaks are near  $2\pi/T$  and  $\pi/T$ , respectively, in accordance with Fig. 1.

of the delayed feedback  $(p_2 - p_1)/(p_1 + p_2)$  is proportional to  $D^{-1}$ . In particular, for the parameters of Fig. 3, the maximum of the main peak is achieved at  $\omega_{\max} \approx 0.024$ ,  $D \approx 0.0537$ . The Kramers time without feedback, according to (3), is  $T_K \approx 467$ . For negative feedback  $\epsilon = -0.1$  we obtain  $\omega_{\max} \approx 0.0116$ ,  $D \approx 0.0444$ , and  $T_K \approx 1233$ .

Let us discuss now the response of the time-delay stochastic system to a periodic external force. Similar to Ref. [9], we assume that the transition rates (5) are modulated with a frequency  $\Omega$  according to the Arrhenius rate law,

$$W_1(t) = [p_1 n_+(t - T) + p_2 n_-(t - T)] e^{\gamma(t)},$$

$$W_2(t) = [p_2 n_+(t - T) + p_1 n_-(t - T)] e^{-\gamma(t)},$$

where  $\gamma(t) = \mu D^{-1} \cos(\Omega t + \phi)$ . The equation for the quantity  $\sigma(t) = n_+(t) - n_-(t)$  (which now is not the autocorrelation function) now reads

$$\frac{d\sigma}{dt} = -(p_1 + p_2)(n_+ e^{\gamma(t)} - n_- e^{-\gamma(t)}) + (p_2 - p_1)(n_+ e^{\gamma(t)} + n_- e^{-\gamma(t)})\sigma(t - T).$$

In the linear approximation  $\mu \ll 1$  this reduces to

$$\frac{d\sigma}{dt} = -(p_1 + p_2)[\sigma + \gamma(t)] + (p_2 - p_1)\sigma(t - T)[1 + C(t)\gamma(t)].$$

Now writing  $\sigma = \sigma_0 + \mu D^{-1} \sigma_1$  where  $\sigma_0$  is the solution (10) and (11), we obtain for the first-order correction  $\sigma_1$

$$\frac{d\sigma_1}{dt} = -(p_1 + p_2)\sigma_1 + (p_2 - p_1)\sigma_1(t - T) + [(p_2 - p_1)\sigma_0(t)\sigma_0(t - T) - (p_1 + p_2)] \times \cos(\Omega t + \phi). \quad (15)$$

We are interested in the response at the frequency  $\Omega$  for  $t \rightarrow \infty$ , because only this part contributes to the delta peak in the spectrum at this frequency. For  $t \rightarrow \infty$ ,  $\sigma_0(t) \rightarrow 0$ , so we can neglect the corresponding term

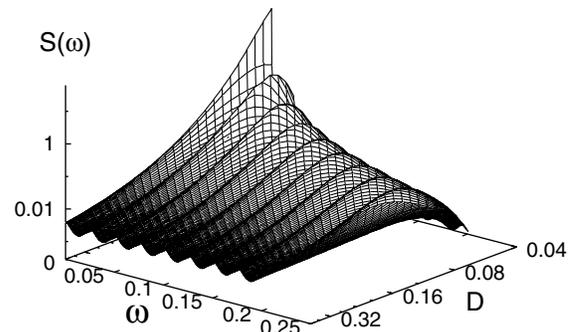


FIG. 3. Power spectrum in the delay system (1) calculated in the two-state approximation for  $T = 250$  and  $\epsilon = 0.1$ .

$(p_2 - p_1)\sigma_0(t)\sigma_0(t - T)$  in (15) and write the solution as

$$\sigma_1(t) = \text{Re} \frac{(p_1 + p_2)e^{i\Omega t + \phi}}{(p_2 - p_1)e^{-i\Omega T} - i\Omega - (p_1 + p_2)}.$$

This is exactly the periodic component at frequency  $\Omega$  in the process  $s(t)$ , and the linear response  $\eta$  is

$$\eta = \frac{1}{2D^2} \frac{(p_1 + p_2)^2}{|(p_2 - p_1)e^{-i\Omega T} - i\Omega - (p_1 + p_2)|^2}. \quad (16)$$

In the absence of delayed feedback, when  $p_1 = p_2 = r_K$ , this expression coincides with that of [9] for the stochastic resonance in the two-state model. With the feedback, the response demonstrates a resonancelike structure in dependence on the driving frequency (contrary to the classical stochastic resonance); see Fig. 4.

In conclusion, we have developed a theory of a prototypical noise-driven bistable system with delayed feedback. In general, such problems are very difficult because of the non-Markovian nature of the dynamics. However, for small noise and small magnitude of the feedback, the problem can be simplified by reduction to the two-state model with certain transition rates which depend on the earlier state of the system. Using this approximation, we were able to derive the analytical formulas for the autocorrelation function and the power spectrum in a very good agreement with direct numerical simulations of the original Langevin equation. The power spectrum has a pronounced peak at the frequency corresponding to the delay time, whose amplitude has a maximum at a certain noise level, thus demonstrating coherence resonance. This level corresponds to the mean switching time comparable to the delay time. We should emphasize the difference between this phenomenology and the stochastic resonance which occurs in periodically driven noisy bistable systems. In the present case, the system is autonomous, and the characteristic time scale corresponding to the main spectral peak is imposed by (but not equal to) the time delay. There is no strictly periodic component in the output signal, so the peaks of the correlation function decay at large times. We also studied the linear response of our system to the external periodic force. It also has maxima at the frequencies corresponding to the inverse delay time and its harmonics.

In a more general context of multistable dynamical systems with memory, the behavior of the system depends on its past through some memory kernel. Such a kernel is equivalent to multiple time delays. A similar analysis of the correlation properties for such systems would be of great interest. Furthermore, in applications, multiple feed-

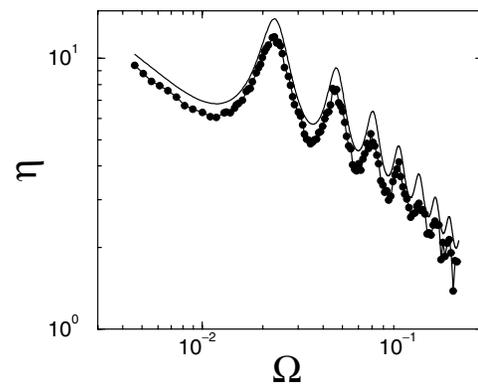


FIG. 4. Linear response of model (1) for  $D = 0.1$ ,  $T = 250$ ,  $\epsilon = 0.05$ , normalized by the variance of the process (circles), compared with theory (16) (line).

back loops with different delay times occur in networks of interacting elements, such as biological neurons, stock traders, or Internet nodes. It is very interesting to study the influence of noise on the dynamics of such networks. We anticipate the emergence of spontaneous oscillations and the resonant features similar to the effects considered in this Letter.

We thank A. Neiman and M. Rosenblum for helpful discussions. L.T. acknowledges support from the U.S. Department of Energy, Engineering Research Program of the Office of Basic Energy Sciences, from the U.S. Army Research Office, and from UC MEXUS/CONACYT.

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