Resolving Clusters in Chaotic Ensembles of Globally Coupled Identical Oscillators

A. Pikovsky,[1] O. Popovych,[1,2] and Yu. Maistrenko[1,2]

[1]*Department of Physics, University of Potsdam, Postfach 601553, D-14415 Potsdam, Germany*
[2]*Institute of Mathematics, National Academy of Sciences of Ukraine, 01601 Kiev, Ukraine*

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Clustering in ensembles of globally coupled identical chaotic oscillators is reconsidered using a twofold approach. Stability of clusters towards "emanation" of the elements is described with the evaporation Lyapunov exponents. It appears that direct numerical simulations of ensembles often lead to spurious clusters that have positive evaporation exponents, due to a numerical trap. We propose a numerical method that surmounts the spurious clustering. We also demonstrate that clustering can be very sensitive to the number of elements in the ensemble.

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Many systems in physics and natural sciences can be represented as ensembles of coupled nonlinear elements. Such models arise in the study of Josephson junctions, multimode lasers, and charge density waves [1]. A particular interest attracted ensembles of chaotic oscillators [2]. Recently, an experimental investigation of 64 globally coupled chaotic electrochemical oscillators have been performed [3]. These studies have revealed that already coupling of identical chaotic oscillators demonstrates nontrivial synchronization patterns.

One of the intriguing effects in ensembles of globally coupled identical chaotic oscillators is clustering [4–8], observed also in experiments [3]. It appears that initially randomly chosen initial states in the course of evolution come close to each other and eventually become identical. The final configuration consists of several (or many) clusters of equal states; these clusters may vary in time regularly or in a chaotic manner. As have been numerically demonstrated in [4,5,8], typically for large couplings there exist one or few clusters.

Typically, clustered states have been studied by means of direct numerical simulation of large ensembles [4–7,9]. In these studies two important aspects are usually not taken into account. From one side, spurious clusters can appear due to a numerical procedure of simulations. Indeed, because the interacting elements are identical, their individual dynamics are identical as well, provided that their states coincide within the computer representation of real numbers. This means that if two states differ by a distance less than the numerical precision, these states form a cluster. To check if the appeared cluster is an eventual state, one has to study the stability of it, and this is the second aspect of cluster dynamics that is barely discussed in the literature (cf. [7,10,11]).

In this paper we investigate the clustering in ensembles of coupled maps surmounting the drawbacks of direct numerical simulations. First, we show that the stability of clusters can be characterized with the *evaporation Lyapunov exponent* that measures the stability of clusters towards "emanation" of individual elements. Only clusters with negative evaporation exponents can be considered as stable objects, and we demonstrate that in many cases direct numerical simulations can lead to unstable clusters. Second, we suggest a numerical method that overcomes the problem of numerical precision and yields only stable clusters. Furthermore, applying these methods we describe an interesting effect of ensemble size sensitivity: for some parameter values one observes clusters for certain specific numbers of elements in the ensemble, and does not observe any clustering for other ensemble sizes.

Our basic model is the ensemble of N globally coupled maps [4,5,10,11]

$$x_i(t+1) = (1-\varepsilon) f(x_i(t)) + \frac{\varepsilon}{N} \sum_{j=1}^{N} f(x_j(t)) . \quad (1)$$

Here ε is the coupling constant; throughout the paper we consider the coupled parabolic (logistic) maps $f(x) = a - x^2$. If the states in the ensemble form M clusters, then we can write

$$x_1 = x_2 = \cdots = x_{m_1} = X_1 ,$$
$$x_{m_1+1} = x_{m_1+2} = \cdots = x_{m_1+m_2} = X_2 ,$$
$$\vdots$$
$$x_{m_1+\cdots+m_{M-1}+1} = \cdots = x_N = X_M ,$$

where $m_1, m_2, \ldots, m_M$ are the sizes of the clusters. Surely, $\sum_1^M m_l = N$. Following the dynamics of the clusters, we reduce N-dimensional system (1) to the M-dimensional system for the cluster variables X_l:

$$X_l(t+1) = (1-\varepsilon) f(X_l(t)) + \varepsilon \sum_{k=1}^{M} p_k f(X_k(t)) , \quad (2)$$

where $p_l = m_l/N$ is the portion of the elements belonging to the lth cluster; $\sum_1^M p_l = 1$.

The stability of the ensemble dynamics is twofold. From one side, one can investigate dynamical properties of the M-dimensional mapping (2). This investigation naturally leads to M Lyapunov exponents that characterize the dynamics. These exponents, however, do not give the full description of the stability in the original ensemble (1), as they do not describe perturbations that destroy the clustering partition. These perturbations lead to other Lyapunov

exponents, which we call "evaporation exponents" (EE) (sometimes one speaks on split or transversal ones [6,11]). They show whether the cluster is stable towards emanation of some of its elements [12]. From the mathematical viewpoint these exponents characterize perturbations that are transversal to the invariant subspace corresponding to the clustering partition.

To define the EEs, let us consider two variables x_i and x_j belonging to one cluster $x_i = x_j = X_l$. If we make a perturbation that does not change the sum $x_i + x_j$, namely if we take $x_i + \Delta, x_j - \Delta$, then this perturbation does not induce any perturbations in other clusters. Thus, it is an eigenmode corresponding to the evaporation. For Δ we obtain $\Delta(t+1) = (1-\varepsilon)f'(X_l(t))\Delta(t)$ and the growth rate of Δ is given by the EE

$$\Lambda_l = \langle \log|(1-\varepsilon)f'(X_l)| \rangle . \quad (3)$$

Clearly, due to the degeneracy of the system (1), all $m_l - 1$ EEs of the cluster l are equal to (3).

The evaporation Lyapunov exponents provide a tool to characterize the internal stability of the cluster states. Only clusters with negative EEs correspond to the attractors of the ensemble (although these attractors can be Milnor attractors if, despite negative EEs, some trajectories on them are unstable with respect to evaporation). Now, we can find possible M-cluster states by numerically simulating mapping (2) and picking out the attractors that have negative EEs. The results of such an analysis for $M = 2$ are presented in Fig. 1. On the plane (ε, p) (for 2-cluster state we denote p_1 as p, because $p_2 = 1 - p$ is uniquely defined) we see regions of existence of stable (towards evaporation) 2-clusters.

From the structure of the stable regions in Fig. 1 follows an interesting effect of *system size sensitivity,* it occurs for some ranges of the coupling parameter ε. Let us fix $a = 2$ and the parameter of the coupling $\varepsilon = 0.45$. As it results from Fig. 1a, for this coupling there are several tiny regions (windows) of stable 2-clusters at small values of p; i.e., the stable clusters are highly asymmetric. The largest of the windows is $0.041\,06 < p < 0.042\,89$. Consider an ensemble of some fixed size N, then the parameter p can take not all possible values, but only the ratios $1/N, 2/N, \ldots, (N-1)/N$; i.e., this parameter is quantized. If for given N none of these possible values falls into the stability regions, no stable 2-clusters can be observed. If we restrict ourselves to ensemble sizes less than 100, then the only possible values N allowing stable 2-clusters for $\varepsilon = 0.45$ are $N = 24, 47, 48, 70{-}73, 81, 94{-}97$; there are no stable 2-clusters for $N = 100$.

The results of the analysis of 2-cluster states above are not confirmed by direct numerical simulations of the ensemble (1), due to the effect of *finite precision of calculations*. In the simulations performed for ensembles of $N = 100$ maps for $\varepsilon = 0.45$, $a = 2$, we always observed a convergence (although slow, cf. [9]) to a 2-cluster state; see Fig. 2. The reason unstable states are observed in

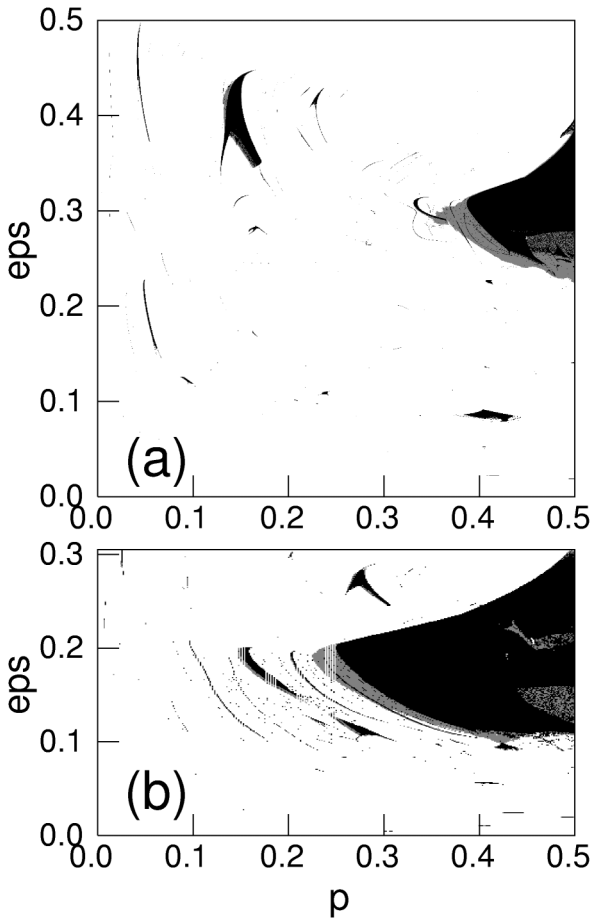

FIG. 1. Regimes with two clusters that are stable towards evaporation (i.e., having negative evaporation exponents) on the plane (ε, p), (a) $a = 2$, (b) $a = 1.6$. Some of these states are regular (black), and some are chaotic or quasiperiodic in time (grey).

direct simulations is the numerical trap (cf. [11,13]). Suppose that during the evolution two elements in the ensemble come so close to each other that their states are represented in the computer by the same numbers. Then they form a cluster and never diverge. This is how fake clusters are formed.

We now propose a numerical method allowing us to overcome the numerical trap and to confirm the theory based on the evaporation exponent. The main idea is in the use of logarithmic variables when two (or more) elements are nearly equal. More precisely, if $|x_j - x_i| < 10^{\delta_1}$, we introduce the difference $y_{ji} = x_j - x_i$ and write in the first approximation $y_{ji}(t+1) = (1-\varepsilon)f'(x_i(t))y_{ji}(t)$. For the logarithm of this difference $z_{ji} = \log_{10}|y_{ji}|$ we obtain an equation that does not contain any numerical peril:

$$z_{ji}(t+1) = \log_{10}|(1-\varepsilon)f'(x_i(t))| + z_{ji}(t) \quad (4)$$

(of course, we have also to calculate in parallel the sign of y_{ji} to be able to restore y from z when $|x_j - x_i|$ becomes not small again). After this transformation we, instead of iterating x_i and x_j, are iterating the variable x_i and the

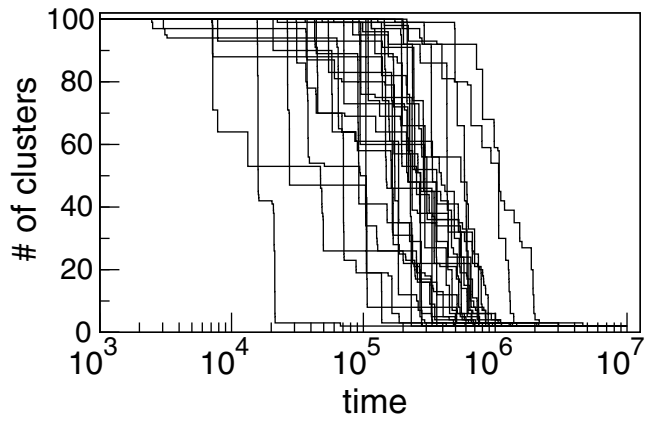

FIG. 2. Evolution of an ensemble of 100 coupled logistic maps for $\varepsilon = 0.45$, obtained via direct simulations with quadruple precision. Thirty independent runs for randomly chosen initial conditions are shown. Two elements are united in a cluster if their states coincide within the numerical precision of calculations.

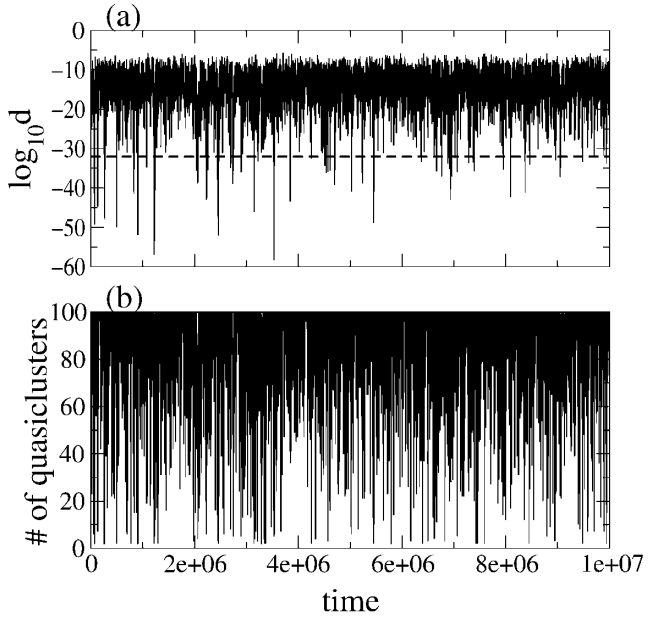

FIG. 3. The same calculations as in Fig. 2, but performed using the method described in the text. (a) The minimal over the ensemble distance between the states $d = \min_{i \neq j}|x_i - x_j|$; (b) the number of quasiclusters (see text). Evolution for one set of randomly chosen initial conditions is shown. After 10^7 iterations no clusters are observed, in agreement with the stability theory developed. The dashed line shows the accuracy of the quadruple precision calculations.

logarithmic variable z. This goes on until z either reaches a large value δ_2 and then we return to the usual variables x_i and x_j and system (1) or z becomes so small ($z < \delta_3$) that we identify x_j and x_i in a real cluster and do not follow the perturbation y_{ji} any more. Practically, we use quadruple precision and $\delta_1 = -15$, $\delta_2 = -13$, $\delta_3 = -3000$. To summarize, in the proposed method we switch from time to time between the initial system (1) and the system (1),(4) where a part of variables is replaced with the logarithms. Comparing to the direct simulations, we lose in accuracy due to linearization when transforming to (4), but we win in the ability to resolve the states as close as 10^{δ_3}.

Performing the calculations with the new method for the same parameters as in Fig. 2 ($N = 100$, $\varepsilon = 0.45$, we note that at these parameter values there is no stable 2-cluster in the ensemble), we obtain Fig. 3. Now the spurious clusters do not appear and the state of the ensemble remains fully desynchronized. However, we can see that during the evolution the distance d between the elements can be very small, up to 10^{-55}. Clearly, such a distance cannot be resolved with the direct simulations even with the quadruple precision.

To demonstrate how the insufficient numerics may affect the observed clustering, we simulated the system of $N = 100$ in two cases: (i) with $a = 1.6$ and $\varepsilon = 0.3$ and (ii) with $a = 1.9$ and $\varepsilon = 0.3$. The case (i) is most dangerous; here only the 2-cluster 50:50 is stable. In calculations with double precision this true cluster appears with probability ≈ 0.06, while unstable 1-cluster appears with probability ≈ 0.79 (the rest are unstable 2-clusters). The situation improves with the quadruple precision: here no 1-clusters are observed, but the stable 2-cluster appears with probability 0.32 only. In calculations with our method we always observed the stable 2-cluster 50:50, although sometimes it sets on after very long transients ($\approx 10^7$ iterations). The case (ii) is less numerically dangerous: here stable clusters are abundant (cf. Fig. 1a). Nevertheless, in the calculations with the double precision the probability to be trapped by unstable clusters is nonzero, although rather small (0.1%); no such clusters appeared in calculations with the quadruple precision.

In the course of application of the numerical method it is natural to define quasiclusters as the groups of elements which are closer than 10^{δ_2} to each other and which therefore are represented by one usual variable and by the logarithms of small deviations. The number of quasiclusters vs time is shown in Fig. 3b. One can see that sometimes there are only two quasiclusters, which, however, after some time are destroyed due to "evaporation." This agrees with the stability properties discussed above.

If we perform the same computation with the "resonant" size of the ensemble (e.g., $N = 95$ for $\varepsilon = 0.45$), then after some long transient a stable cluster with $m_1 = 4$, $m_2 = 91$ appears. During the transient time many other quasiclusters with "wrong" partitions m_1, m_2 appear and then disappear, but only for one with the "correct" partition will the logarithmic variable eventually become less than $\delta_3 = -3000$ (Fig. 4).

Here we compare our numerical technique with the other, rather simple method to avoid fake clusters — just by adding some small noise to each mapping in (1). Noise destroys perfect clustering, but one may hope to observe "nearly clustered states" if the system without noise possesses stable clusters. This, however, is true only if the

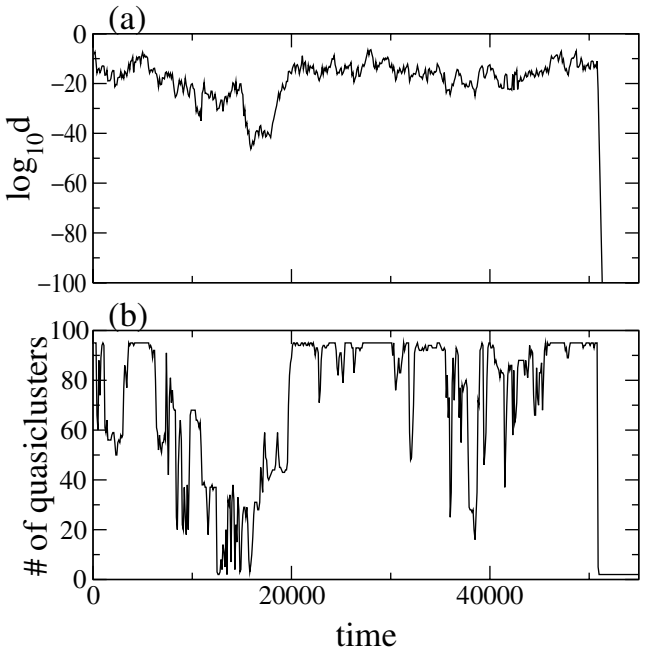

FIG. 4. The same as Fig. 3, but for an ensemble of $N = 95$ elements. Here the 2-cluster state with $p_1 = 4$, $p_2 = 91$ is stable towards evaporation; it establishes after 51 000 iterations. Note that after 12 500 iterations another quasicluster state with a "wrong" partition p_1, p_2 appears as a temporary event.

stability of clusters is uniform. If, on the other hand, the attractors are of Milnor type, i.e., their basin of attraction is riddled (what is often encountered in coupled chaotic oscillators, see [14]), then even a very small noise will produce declustering. Thus, adding noise does not allow one to distinguish unstable and Milnor-stable (riddled) clusters, while our numerical method does. Another drawback of the addition of noise is that it can change the statistical properties of the system, because it influences not only the transversal dynamics, but also the longitudinal one.

In conclusion, we have analyzed the clustering in the ensembles of chaotic oscillators using a twofold approach, based on the stability analysis of clusters with the evaporation exponents, and on the special numerical technique of the ensemble simulations. This study reveals that many cluster states that appear in the course of direct simulations are spurious, because they are unstable towards evaporation. This has nothing to do with the possible Milnor stability of the corresponding M-cluster attractors of mapping (2), but just corresponds to the fact that in the highly symmetrical ensemble of globally coupled maps under consideration the differences between the elements can attain extremely small values. Thus, a correct numerical investigation of complex cluster states, e.g., of glassy states [4,5,9], necessarily requires the application of the special numerical method. An interesting feature of the ensemble of coupled logistic maps is the system size sensitivity, where the clustering appears for some resonant numbers of elements.

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[1] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988); K. Wiesenfeld *et al.,* Phys. Rev. Lett. **65**, 1749 (1990); S. H. Strogatz *et al.,* Physica (Amsterdam) **36D**, 23 (1989).
[2] H. Bohr *et al.,* Parallel Comput. **12**, 113 (1989); A. Pikovsky, M. Rosenblum, and J. Kurths, Europhys. Lett. **34**, 165 (1996); D. H. Zanette and A. S. Mikhailov, Phys. Rev. E **57**, 276 (1998); **58**, 872 (1998).
[3] W. Wang, I. Z. Kiss, and J. L. Hudson, Chaos **10**, 248 (2000).
[4] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989); Physica (Amsterdam) **41D**, 137 (1990).
[5] A. Crisanti, M. Falcioni, and A. Vulpiani, Phys. Rev. Lett. **76**, 612 (1996).
[6] K. Kaneko, Phys. Rev. Lett. **78**, 2736 (1997); Physica (Amsterdam) **124D**, 322 (1998).
[7] N. J. Balmforth, A. Jacobson, and A. Provenzale, Chaos **9**, 738 (1999).
[8] S. C. Manrubia and A. S. Mikhailov, Phys. Rev. E **60**, 1579 (1999).
[9] S. C. Manrubia and A. S. Mikhailov, Europhys. Lett. **50**, 580 (2000); **54**, 451 (2001).
[10] O. Popovych *et al.,* Phys. Rev. E **63**, 036201 (2001).
[11] O. Popovych, Y. Maistrenko, and E. Mosekilde, Phys. Rev. E (to be published).
[12] In a particular case of two clusters these exponents have also been considered in [7].
[13] A. S. Pikovsky, Phys. Rev. Lett. **73**, 2931 (1994).
[14] Y. L. Maistrenko, V. L. Maistrenko, O. Popovych, and E. Mosekilde, Phys. Rev. E **60**, 2817 (1999).