Comment on "Simple approach to the creation of a strange nonchaotic attractor in any chaotic system"

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We address the problem of existence of strange nonchaotic attractors (SNAs) in quasiperiodically forced dynamical systems. Recently, Shuai and Wong [Phys. Rev. E **59**, 5338 (1999)] suggested a universal method for constructing a SNA in an arbitrary system possessing chaos. We demonstrate here that, in general, this method fails. For arbitrary systems, it gives a SNA only in a vicinity of transition to chaos. We discuss also a special example, where the method by Shuai and Wong indeed produces a SNA.

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Strange nonchaotic attractors (SNAs) typically accompany the transition from ordered to chaotic dynamics in quasiperiodically forced systems. SNAs share some properties of simple and strange attractors. Similar to regular attractors, they do not demonstrate any sensitivity to initial conditions: all Lyapunov exponents are negative [1]. However, their geometrical structure is fractallike of typical chaotic attractors. SNAs have been first described by Grebogi *et al.* [2], and since then investigated both numerically [3] and experimentally [4].

Criteria for the existence of SNA have been formulated in [5]. Below we use the following one: A SNA exists if there are trajectories with positive finite-time Lyapunov exponents for arbitrary large time. Such positive finite-time Lyapunov exponents indicate that in the phase space there are unstable regions that are visited by a trajectory. In the average, contraction dominates so that the time-averaged Lyapunov exponent is negative, but the unstable pieces of a trajectory spoil regularity of the attractor and make it fractal.

Recently, Shuai and Wong [6] have proposed an approach to produce a strange nonchaotic attractor in any system demonstrating regular and chaotic dynamics. The proposed method is based on a control technique applied to a system parameter C; it can be described by the following steps:

(i) Take two values of the system parameter C, so that for C_1 the dynamics is chaotic, and for C_2 it is regular.

(ii) Add a small quasiperiodic forcing not destroying the type of the dynamics at these parameter values. Then one observes a strange chaotic attractor for C_1 and a smooth quasiperiodic attractor (torus) for C_2.

(iii) Switch periodically between the chosen values of C, by operating the system for time T_1 with parameter value C_1 (chaotic epoch), and then operating it for time T_2 with parameter value C_2 (regular epoch). Note that the times T_1 and T_2 are suggested to be chosen large enough to ensure that the Lyapunov exponents calculated for these time intervals are close to their asymptotic values [cf. Eq. (3) below].

Shuai and Wong claim that if T_1 and T_2 are chosen in such a way that the overall Lyapunov exponent is negative, then the attractor of the system will be strange nonchaotic. In this Comment we show, in contrast to the results of [6], that this construction yields either a strange nonchaotic attractor or a smooth torus. We demonstrate that for existence of a SNA, some additional conditions on the parameters of the method should be fulfilled. Qualitatively, our arguments are the following: the criterion of the SNA formulated above requires that positive finite-time Lyapunov exponents must exist for arbitrary large time. However, the method described provides only positive exponents for finite times of order $t \approx T_1$. If these positive finite-time Lyapunov exponents (that naturally appear during the chaotic stage) are fully compensated by negative Lyapunov exponents during the regular stage, then there are no positive finite-time Lyapunov exponents for times $t=k(T_1+T_2)$ with any $k \geqslant 1$. Thus, the attractor is not strange, but smooth.

For illustration we will use the system presented by the authors of [6]. They considered a quasiperiodically forced logistic map,

$$\begin{aligned} x(t+1) &= f_C[x(t),\theta(t)] \\ &= ax(t)[1-x(t)] + A\sin[2\pi\theta(t)] + C, \qquad (1) \\ \theta(t+1) &= \theta(t)+\omega \quad (\mathrm{mod}\ 1), \end{aligned}$$

where $a=3.6$, $A=0.001$, and $\omega=(\sqrt{5}-1)/2$ are fixed. Figure 1 shows the dependence of the dynamics of system (1) on the value of the parameter C. It is chaotic for $C > -0.012$ and regular otherwise. According to the procedure of [6], we switch the dynamics between two values C_1 and C_2 in these domains. In order to check the existence of SNAs, we have to examine whether there are positive finite-time Lyapunov exponents for arbitrary large times as it has been formulated in [5]. When applying this criterion to system (1), one should take into account that this system is inhomogeneous in time, because it is a combination of alternating intervals T_1 and T_2 having different dynamics. Thus, to apply the criterion above, one should first make the system homogeneous. This can be accomplished by looking at every T_1+T_2 iteration, and therefore by writing out the map after T_1+T_2 time steps. This map F will give the overall dynamics after visiting chaotic and regular domain, and can be treated like other quasiperiodically forced maps:

$$x(t+T_1+T_2) = F[x(t),\theta(t)] = (f_{C_2}^{T_2}\cdot f_{C_1}^{T_1})[x(t),\theta(t)], \qquad (2)$$

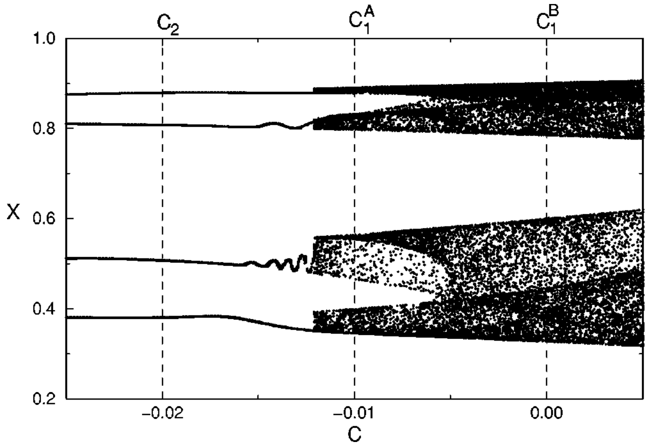

FIG. 1. The bifurcation diagram for the quasiperiodically forced logistic map in dependence on the additive system parameter C governing different kind of dynamics (quasiperiodic motion on a torus or chaotic motion). Two cases of switching the parameter C are discussed in the text: (A) $C_1^A=-0.01$, $C_2=-0.02$, and (B) $C_1^B=0$, $C_2=-0.02$.

$$\theta(t+T_1+T_2)=\theta(t)+(T_2+T_1)\omega \quad (\mathrm{mod}\ 1).$$

Below we discuss two representative cases, which differ in the choice of parameter value C_1.

Case A. $C_1^A=-0.01$, $C_2=-0.02$.

Case B. $C_1^B=0$, $C_2=-0.02$.

In both cases we have switchings between regular and chaotic dynamics, the corresponding Lyapunov exponents are: $\lambda_1^A=0.03545$, $\lambda_1^B=0.172$, and $\lambda_2=-0.114$. The difference between cases A and B is in the topological structure of chaos: for A it has four bands, while for B it has two bands. We will see that this results in different features of the combined dynamics.

We are going to show first that, in general, the proposed method of [6] is not able to construct a SNA in any chaotic system. In particular, Fig. 1 of [6] is not valid for an arbitrary choice of parameter values C_1 and C_2. However, for some special choice of parameter values the occurrence of a SNA was found and also reported in [6].

If one plots the attractors of system (2) in the phase plane (x,θ), they look for both cases A and B like smooth tori, because the last T_2 iterations correspond to the contracting dynamics. To determine the nature of the dynamical regime in dependence on the iteration lengths T_1 and T_2, further investigations concerning finite-time Lyapunov exponents are necessary.

1. Case A

According to the step 3 of the proposed method, the times T_1 and T_2 of the expanding and contracting dynamics have to be adjusted so that the overall dynamics is contracting. This means that the Lyapunov exponent must be negative. For large T_1,T_2 one can neglect the transients at the switchings between the two dynamical regimes, and approximate the largest Lyapunov exponent of Eq. (2) as

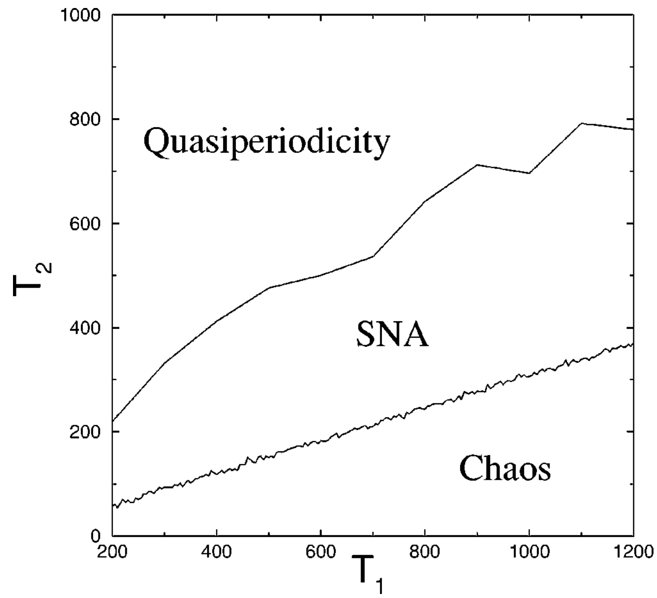

FIG. 2. Phase diagram for the system (2) in the T_1-T_2 plane with $C_1^A=-0.01$ and $C_2=-0.02$. The slope of the transition line between the chaos and SNA is fitted to be 0.311, the transition line between the dynamical regimes of the SNA and torus is estimated roughly on the basis of the distribution of local Lyapunov exponents.

$$\Lambda=\frac{T_1\lambda_1+T_2\lambda_2}{T_1+T_2}. \tag{3}$$

Consequently, there exists a transition line in the parameter plane (T_1,T_2) separating the phases of chaos and SNA, what was also found by the authors of [6]. This transition line is obtained numerically (see Fig. 2) and its slope 0.311 is in a good agreement with the theoretically expected one resulting from Eq. (3). According to [6], above this transition line only strange nonchaotic behavior exists. We show now, that a SNA exists only in a small band above the line of transition to chaos.

We use the criterion above, according to which a SNA is present if the Lyapunov exponent is negative, but positive finite-time Lyapunov exponents λ_T do exist. Thus, we are going to find an approximation for the maximum finite-time Lyapunov exponent $\lambda_{T_1+T_2}^{max}$ after T_1+T_2 iterations. Similar to Eq. (3) we can write

$$\lambda_{T_1+T_2}^{max}=\frac{T_1\lambda_{T_1}^{max}+T_2\lambda_{T_2}^{max}}{T_1+T_2}, \tag{4}$$

where $\lambda_{T_1}^{max}>0$ and $\lambda_{T_2}^{max}<0$ are maximal possible finite-time Lyapunov exponents of map (1) for times T_1,T_2 in regimes with C_1,C_2, respectively. Numerically, we have estimated these maximal finite-time exponents for 2000 randomly chosen initial conditions (x,θ). Now the condition $\lambda_{T_1+T_2}^{max}>0$ can be used as a condition for the existence of a SNA. This line, depicted in Fig. 2, is only a crude approximation of the transition line separating the regimes of the SNA and quasiperiodicity, because in Eq. (4) we neglect all

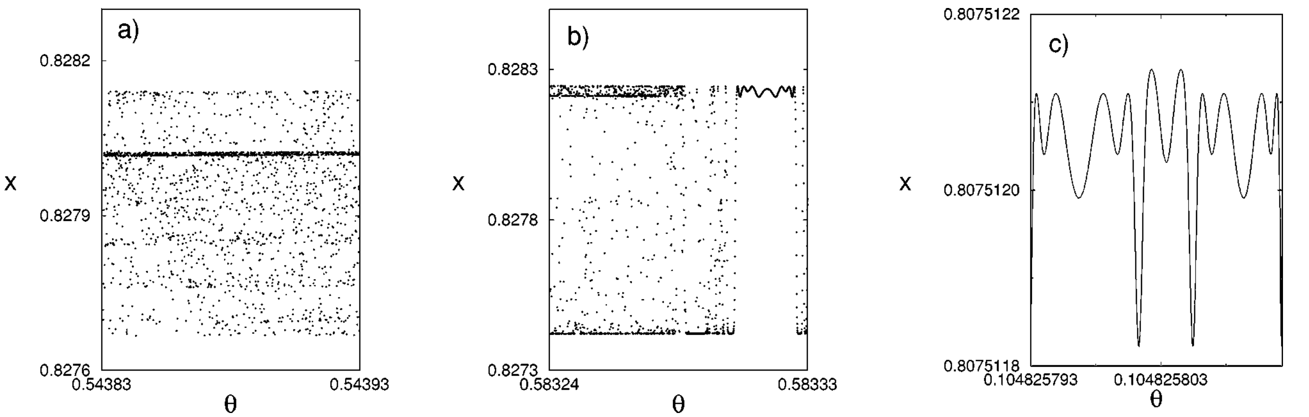

FIG. 3. Phase portraits for the system (2) with $C_1^A = -0.01$, $C_2 = -0.02$, and $T_1 = 200$. (a): chaos for $T_2 = 48$. (b): SNA for $T_2 = 68$. (c): smooth torus for $T_2 = 102$.

correlations and as a result the real finite-time exponents are smaller than those predicted by Eq. (4). Nevertheless, we demonstrated that the proposed method is not able to construct a strange nonchaotic attractor, except for a finite region near the border to chaos.

To confirm that the phase diagram Fig. 2 is valid, we investigated representative attractors from the three domains, for a fixed value $T_1 = 200$ and appropriate values of T_2. As already mentioned, the attractors of system (2) in any case look like smooth tori, even for parameter values T_1 and T_2 inside the chaotic and the strange nonchaotic regime. To resolve the strange structure of those attractors one has to go into very small scales. The reason for this is the following: because at the last T_2 iterations of map (2) the dynamics is contracting, the size of the attractor in the x direction can be estimated as $\Delta x \approx D \exp(\lambda_2 T_2)$. Here $D \approx 0.1$ is the size of the band of the chaotic attractor with C_1 (see Fig. 1), and $\lambda_2 < 0$ is the Lyapunov exponent of the regular dynamics. This estimate yields $\Delta x \approx 10^{-4}$. Thus, to see if the attractor of the map (2) is strange or not, we have to zoom the picture, resolving it at the level $\Delta \theta \approx 10^{-4}$, or finer. This is done in Fig. 3. In this resolution one can readily distinguish between the strange [cases (a),(b)] and smooth [case (c)] attractors. Additional calculation of the Lyapunov exponent allows us to characterize Fig. 3(a) as chaos, Fig. 3(b) as SNA, and Fig. 3(c) as smooth torus.

The approximation of the line seperating the phases of strange nonchaotic behavior and quasiperiodic motion was obtained by estimating the maximum value of finite-time Lyapunov exponent after one period $T = T_1 + T_2$ based on the distribution of finite-time Lyapunov exponents λ_{T_1} and λ_{T_2} for an ensemble of 2000 randomly chosen initial conditions, while neglecting correlations between chaotic and regular dynamics. However, the criterion for the SNA is the existence of positive finite-time Lyapunov exponents for arbitrary large times. Now we are going to show that this criterion is fulfilled for any points of the parameter plane (T_1, T_2) inside the region limited by the both numerically estimated transition lines. Let us first introduce an alternative time k according to $t = k(T_1 + T_2)$, then the criterion for the SNA is the existence of positive finite-time Lyapunov exponents for arbitrary large k. According to the thermodynamic formalism [5], one expects that for large k the probability $P_k(\Lambda)$ that the time-k Lyapunov exponent has value Λ scales as $P_k(\Lambda) \sim \exp[k\phi(\Lambda)]$. Here $\phi(\Lambda)$ is a scaling function having a maximum $\phi = 0$ at the negative time-∞ Lyapunov exponent. According to this, the probability to observe a positive time-k Lyapunov exponent is

$$P_+(k) = \int_0^\infty P_k(\Lambda) d\Lambda \sim \exp(-ak), \quad a = -\phi(0). \quad (5)$$

To check this relation, we calculated $P_+(k)$ using the ensemble of 4.24×10^7 pieces of trajectories for the point $(T_1 = 400, T_2 = 200)$ inside the region of the SNA on the parameter plane (T_1, T_2) (see Fig. 2); the results are presented in Fig. 4. One can see that the numerics fits very well relation (5) even for small k, with $a \approx 0.84$. This allows to estimate, with which probability one can observe a positive finite-time Lyapunov exponent for every k. The check of the functional

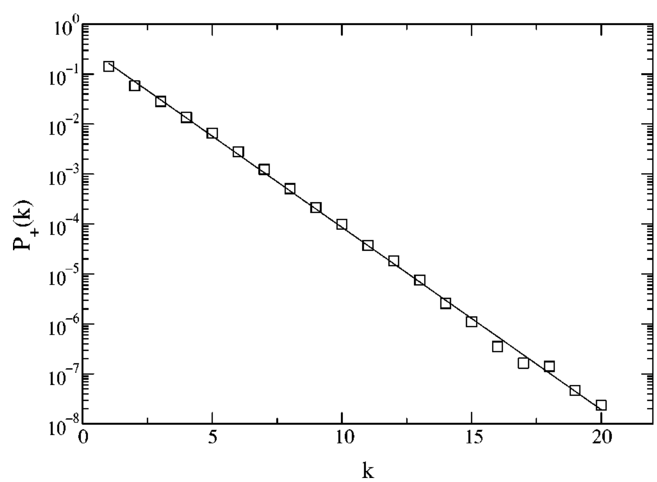

FIG. 4. The numerically estimated probability $P_+(k)$ to observe a positive time-k Lyapunov exponent. For calculations an ensemble of 4.24×10^7 pieces of trajectories was used at parameter values $T_1 = 400$, $T_2 = 200$ with $C_1^A = -0.01$ and $C_2 = -0.02$. The line $P_+ = 0.38 \exp(-0.84k)$ is the best fit.

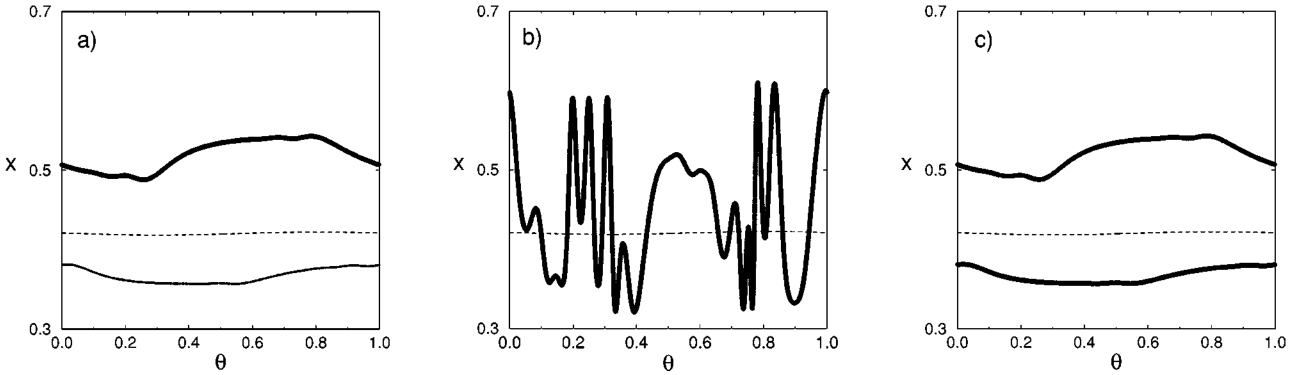

FIG. 5. Illustration of the topological construction leading to strange nonchaotic behavior for case B with $C_1^B=0.0$, $C_2=-0.02$, $T_1=450$, $T_2=750$. Only two branches out of four are shown for clarity. See text for description of the panels.

relation (5) can be used as a numerical method for a precise estimation of the border ''SNA torus.'' At this transition a tends to infinity.

2. Case B

In contrast to the case A, here the estimation of the maximum finite-time Lyapunov exponent by Eq. (4) gives erroneous results. This is due to a special topological property of the dynamics, which we describe in the following: in the derivation of Eq. (4), we took finite-time Lyapunov exponents from the attractors of Eq. (1) at the parameters C_1 and C_2, and neglected transients to these attractors. In the case A it was justified, because durations of these transients were limited. Now the situation is different, because a repeller comes to play in the stable regime with parameter C_1^B. Indeed, between the branches of stable tori in Fig. 1 there are unstable tori, separating the corresponding basins of attraction. The crucial point is that the chaotic attractor of Eq. (1) with C_2 covers both the stable tori and the unstable torus (repeller). This means, that some initial points in the regular dynamics with C_1^B have very long transients (if they lie near the repeller), or even remain on the repeller within the time T_1. As a result, these trajectories have positive finite-time Lyapunov exponents, because both the finite-time Lyapunov exponent of chaos and that of the repeller are positive. According to the criterion above, we have to conclude that a SNA exists here for arbitrary large T_1.

We illustrate the topological mechanism for existence of an SNA in Fig. 5. Here we restrict the picture to the two branches of the attractor (out of four). Although the attractor in Fig. 5 appears as a smooth one, we demonstrate that in fact a mixing of the points between the branches occurs, meaning that it is a strange nonchaotic attractor. At the end of the regular part of the iterations one obtains an attractor consisting of two seemingly smooth branches (a). In panel (a) the points on the upper branch are selected and shown with bold dots. The unstable torus is denoted by the dashed line and the lower branch of the stable attractor by small dots. The bold points from the upper branch are iterated according to the chaotic dynamics, with parameter value C_1^B. After 20 iterations the formerly smooth curve is wrinkled [panel (b)] and intersects the unstable torus at various points. Therefore, different bold dots belong to basins of different branches shown in panel (a). Thus, after the chaotic epoch is over and the regular dynamics with C_2 is switched on, the bold dots will be spreaded between the two branches, as shown in panel (c). In principle, there should be also points in between the two branches—these are the points that fall nearly exactly on the repeller. However, a probability to meet such a point is extremely small.

To summarize our results, in general it is not possible to construct a SNA by applying the method proposed by authors of [6]. Only in the special case B it is possible to get an SNA because of the topological arrangement (the unstable torus placed inside the chaotic band of the attractor) that is not valid in general. We conclude that the proposed method of Ref. [6] does not construct a SNA in any chaotic system. The method works either near the transition to chaos (and it follows from the general theory [5] that near the transition to chaos one typically observes an SNA), or requires a special topological arrangement. It remains an open question how to construct a SNA in any chaotic system.

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