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Transcritical loss of synchronization in coupled chaotic systems

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Abstract

The synchronization transition is described for a system of two asymmetrically coupled chaotic oscillators. Such a system can represent the two-cluster state in a large ensemble of globally coupled oscillators. It is shown that the transition can be typically mediated by a transcritical transversal bifurcation. The latter has a hard brunch that dominates the global dynamics, so that the synchronization transition is normally hard. For a particular example of coupled logistic maps a diversity of transition scenaria includes both local and global riddling. In the case of small non-identity of the interacting systems the riddling is shown to turn into an exterior or interior crisis.

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Synchronization of chaotic systems is a subject of intensive research. More than a decade ago it was shown that in the presence of dissipative coupling the states of two chaotic oscillators may become identical, while remaining chaotic in time [1,2]. This effect has been observed in experiments with lasers and various electronic circuits [3–5]. Moreover, it has been proposed to utilize synchronization as a tool for constructing new types of communication schemes [6–9]. Usually when considering synchronization of two chaotic oscillators, one assumes the full symmetry of the problem. This imposes important restrictions on the possible transition scenaria. The cases of non-identity are then often treated using perturbation approaches. In this paper we focus on a situation where the coupled chaotic systems are identical (or nearly identical), but the coupling term is essentially asymmetrical. We demonstrate that appearing here transcritical transversal bifurcation is dominated by the hard branch, making the synchronization transition hard.

Asymmetrical coupling naturally appears in the dynamics of clusters in large ensembles of globally coupled chaotic oscillators. The basic model, proposed by Kaneko [10,11], has the form

$$x_i(t+1) = (1-\varepsilon) f(x_i(t)) + \frac{\varepsilon}{N} \sum_{j=1}^{N} f(x_j(t)), \qquad (1)$$

where the single system is governed by a one-dimensional mapping $x \to f(x)$. Examples of globally cou-

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pled oscillators include lasers, neuronal ensembles, and Josephson junction arrays [12–15]. As observed in Refs. [10,11,16], the ensemble of N oscillators (1) often demonstrates clustering: Variables of some subsystems are identical. In the simplest case of two clusters we have $x_1 = x_2 = \ldots = x_{N_1} = x$, $x_{N_1+1} = \ldots = x_N = y$, where variables x and y obey the equations

$$x(t+1) = f(x(t)) + \varepsilon p[f(y(t)) - f(x(t))], \tag{2}$$

$$y(t+1) = f(y(t)) + \varepsilon(1-p)[f(x(t)) - f(y(t))]. \tag{3}$$

Here parameter p describes the distribution of the oscillators over the clusters: $p = 1 - N_1/N$. Note that we do not assume the number of oscillators to be large (thermodynamic limit), e.g. for $N = 3$ and $N_1 = 2$ we will have $p = 1/3$, and all the results below are applicable.

The system (2,3) includes also the previously considered cases of symmetrical ($p = 1/2$, see e.g. [17–19]) and unidirectional ($p = 0$) coupling. Our attention is concerned with the general case $p \neq 1/2, 0$. We want to describe, how the desynchronization transition [i.e. the transition from the synchronous state $x(t) = y(t)$ to the asynchronous one $x(t) \neq y(t)$] occurs. A remarkable peculiarity of synchronization transitions, as transitions inside chaos, is that they can be viewed both topologically and statistically. The topological approach is based on the bifurcation theory, while the statistical approach deals with average characteristics of the chaotic system.

From the statistical point of view, the desynchronization transition happens when a typical synchronous trajectory $x(t) = y(t)$ looses stability in the transversal direction (symmetry-breaking or blowout transition). Quantitatively, this corresponds to the point where the transverse Lyapunov exponent $\lambda_\perp$, governing the linear evolution of the transversal (i.e., $\propto (x-y)$) perturbations, changes its sign. For the system (2,3) the transversal Lyapunov exponent is p-independent: $\lambda_\perp = \log|1-\varepsilon| + \lambda$, where λ is the Lyapunov exponent of the synchronous chaotic oscillations. Correspondingly, the blowout transition occurs at $\mathrm{e}_{\mathrm{bl}} = 1 \pm \mathrm{e}^{-\lambda}$.

In the topological picture, one interprets the transition as transversal bifurcations of synchronous orbits; usually the periodic orbits embedded in the chaotic attractor are considered as a skeleton of the chaotic set. These bifurcations generally occur at different values of the coupling parameter ε, and the transition thus occupies a whole interval of the coupling parameter $\varepsilon_{\mathrm{rid}} \le \varepsilon \le \varepsilon_{\mathrm{bl}}$ to be specified below. In particular, the state when some synchronous periodic orbits are already transversally unstable, while the transversal Lyapunov exponent is negative is characterized by riddling [21]: In the vicinity of the synchronous state $x = y$ there are transverse perturbations that grow, although typical perturbations decay. The riddled state is especially sensitive to perturbations and noise [22,23].

The analysis of local bifurcations of the periodic points should be complemented by a study of the global dynamics of the system. An important role in this connection is played by the absorbing area [19,20], – an invariant or semi-invariant region in state space, which can enclose the chaotic attractor and thus restrict bursts of the trajectories out of synchrony.

The concrete type of riddling and blowout transition depends on the type of transversal bifurcation of the synchronous periodic orbits of the symmetric state. For two identical coupled oscillators, and in the case of symmetric coupling $p = 1/2$, the symmetry in the transversal direction is preserved and the bifurcation will also be symmetrical: A pitchfork bifurcation (if the transversal multiplier is $+1$ at the bifurcation point) or a period doubling (if the multiplier is -1). Both these bifurcations of a symmetrical fixed point P_0 can be super- or subcritical. Let us now consider, how asymmetry of the system affects these bifurcations. The period-doubling bifurcation is preserved even for $p \neq 1/2$ (Fig. 1a). Due to symmetry with respect to time shifts the quadratic terms in the normal form cancel. This means that the standard period doubling occurs when the transversal multiplier passes through -1 independently on the symmetry of the system [24,25]. Contrary to this, the supercritical pitchfork bifurcation transforms to a transcritical bifurcation, as it is schematically shown in Fig. 1b.

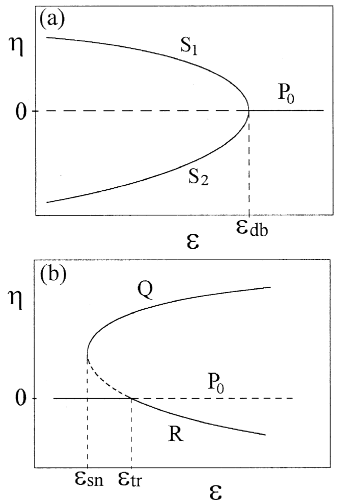

Fig. 1. A schematic view of the period-doubling (a) and the transcritical (b) bifurcations in the asymmetrical case. In the latter case the hard branch P_1 with finite $\eta = y - x$ appears far from the symmetric state $\eta = 0$.

At the transcritical bifurcation there are two transversally stable supercritical branches: one appears softly from the stable state $\eta \overset{\text{def}}{=} y - x = 0$, but the other appears far away from this state via a saddle-node bifurcation at some subcritical value ε_{sn} of the coupling parameter. We refer to these branches as the soft and the hard branch, respectively, and argue that in general the hard branch controls the transition. Let us look on the normal form of the transcritical bifurcation of type Fig. 1b in the case of continuous time, it includes linear, quadratic, and cubic terms [24,25]

$$\dot{\eta} = a\eta(\varepsilon + b\eta - c\eta^2).$$

This equation can be written in a potential form $\dot{\eta} = -\partial_\eta V = -\partial_\eta[-a\varepsilon\eta^2/2 - ab\eta^3/3 + ac\eta^4/4]$. At the transition point the potential V has one extremal point $\eta = 0$ and the minimum $\eta_{min} = b/c$. It is easy to check that $V(\eta_{min}) < V(0)$, i.e. the absolute minimum of the potential is at the point η_{min} away from $\eta = 0$. So in the presence of small noise the hard branch η_{min} will dominate.

Below we discuss the properties of the transcritical synchronization transition in a particular case of coupled logistic maps, and show that here the hard branch indeed determines the dynamics. In the system (2,3) with the logistic map $f(x) = ax(1-x)$ the fixed point $P_0 = (x^*, x^*)$, $x^* = 1 - a^{-1}$, is first to loose its stability in the transversal direction. Note that the longitudinal (along the direction $x = y$) multiplier at this point $f'(x^*)$ is negative, and the transversal multiplier is

$$\mu_\perp = (1-\varepsilon) f'(x^*). \tag{4}$$

Thus, the transcritical bifurcation occurs for $\varepsilon > 1$, while for $\varepsilon < 1$ the transversal bifurcation is the symmetrical period doubling even if the coupling is asymmetrical. Below we fix the parameter of the logistic map $a = 3.8$, and consider possible synchronization transitions in dependence on the coupling ε and asymmetry p (see Fig. 2 where, due to the obvious symmetry, only the region $p < 1/2$ is shown). The point of transcritical bifurcation of the fixed point P_0 is, according to (4), p-independent, the same holds for the transversal Lyapunov exponent. Thus the riddling and the blowout transitions are horizontal lines $\varepsilon_{rid} \approx 1.555...$ and $\varepsilon_{bl} \approx 1.65...$

Depending on the value of p, different types of hard branch are observed corresponding to different types of saddle-node bifurcation at $\varepsilon = \varepsilon_{sn}$. In the region of large p ($0.239... < p < 0.5$) the fixed points arising at ε_{sn} are unstable in the longitudinal direction, and have different stability in the transversal direction, so one point is a saddle and the other is a

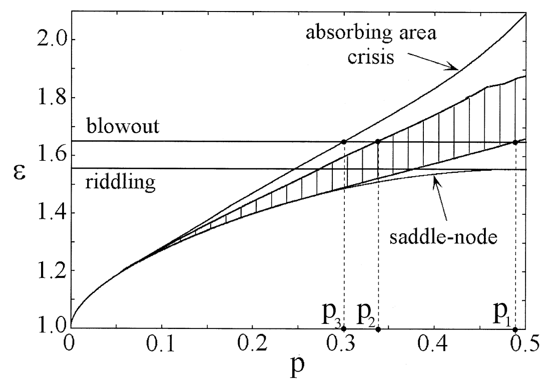

Fig. 2. Bifurcation diagram for two coupled logistic maps. In the crosshatched region an asymmetrical attractor exists outside the synchronous state. The lines of the riddling and blowout bifurcations, the saddle-node bifurcation, and absorbing area crisis are also shown.

repellor. Here we denote the most unstable object to appear (in the present case the repellor) as R, and the less unstable one (in the present case the saddle) as Q. With increasing ε, the saddle Q undergoes an inverse period-doubling, and becomes attracting (this bifurcation line is depicted in Fig. 2 as a lower boundary of crosshatched region). With further increase of e the attractor becomes a focus, then quasiperiodic and chaotic. Finally it turns into a chaotic saddle through an exterior crisis (upper boundary of the crosshatched region in Fig. 2).

For $0.0329... < p < 0.239...$ two fixed points appear at $\varepsilon_{\mathrm{sn}}$, which are stable node and saddle (again we denote them by Q and R). For larger ε the saddle R turns into a repellor through a supercritical period-doubling, and the situation becomes the same as discussed for larger values of p above. Finally, for very small asymmetry parameters $0 < p < 0.0329...$ a repellor (unstable node) and a saddle fixed point appear at $\varepsilon_{\mathrm{sn}}$. While the saddle point turns into repellor R through a supercritical period-doubling, the repellor is not stabilized, so that the points from its vicinity escape to infinity after the boundary crisis of the absorbing area (contact bifurcation of its boundary with the basin of infinity). Summarizing these findings, we can conclude that prior to the riddling transition, nontrivial objects outside of the diagonal $x = y$ appear. One is the repelling fixed point R., Depending on the parameter p, the other object Q can be a saddle, a stable fixed point, a more complex attractor, a chaotic saddle, or the basin of attraction to infinity. This richness of transitions makes the system under consideration exemplary.

We now demonstrate how the objects Q and R determine the synchronization transition. At the riddling transition the fixed point P_0 in the diagonal looses its transversal stability through a transcritical bifurcation which can be viewed as a collision of P_0 with the repellor R. The repellor R brings with it part of the stable manifold of Q. Thus, riddling means that there is a dense set of points near $x = y$ that come close to Q. We can distinguish different types of riddling. If Q is a finite attracting set (for $0.273... < p < 0.377...$) or infinity (for $0 < p < 0.245...$), then transition to global riddling occurs (at $\varepsilon = \varepsilon_{\mathrm{rid}}$) with a possibility for some nearly synchronized states to be attracted to Q or to escape to infinity. The former situation is illustrated in Fig. 3. Most trajectories from the vicinity of the synchronous state are eventually attracted to it, but some set of the points escapes to another attractor. With small noise, almost all the points will eventually escape [21–23]. If Q is a saddle embedded into the basin of synchronous state (chaotic for $0.245... < p < 0.273...$ or point cycle for $0.377... < p < 0.5$), the riddling occurring at $\varepsilon = \varepsilon_{\mathrm{rid}}$ is local. In this case some exceptional set of points near the diagonal $x = y$ can go far away, but eventually almost all of them return back to the synchrony. With small noise one observes bursts of finite size at a finite rate.

From the above decsription it is also clear that the soft branch (that appears during exchange of stability of the point P_0 and the repellor R) is less important for the dynamics. This is evident in the case when the hard branch Q is an attractor, but even in the case when Q is a saddle it dominates the intermittent dynamics because the bursts of $x - y$ towards Q are much larger than the bursts in the direction of the soft branch. Moreover, our analysis of global dynamics inside absorbing areas which envelopes the synchronous attractor shows that points in the vicinity of the soft branch can visit the hard branch as well. This means that even perturbations that push the synchronous state towards the soft branch eventually result in a hard transition at the transcritical point.

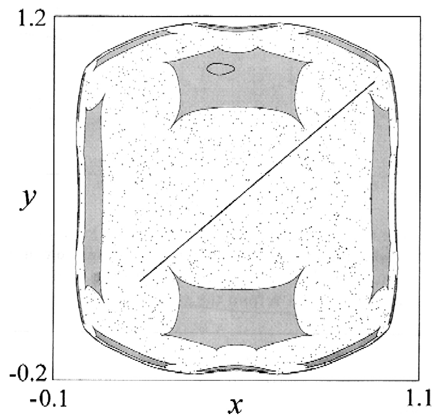

Fig. 3. The synchronized attractor (a piece of the line $x = y$) coexists with the asymmetrical quasiperiodic attractor Q (invariant curve at $y \approx 1, x \approx 0.5$). The basin of the synchronized attractor, shown in white, is riddled: everywhere dense there are pieces of the basin of Q (shown in grey). Parameters $a = 3.8$, $\varepsilon = 1.6$, $p = 0.33$.

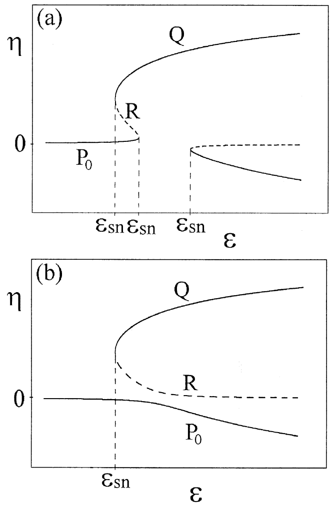

Fig. 4. Two possible ways of destruction of the transcritical riddling bifurcation by a small parameter mismatch. The latter is due to deviations of the parameter of the logistic map: (a) $a_2 = 0.998\,a_1$, (b) $a_2 = 1.002\,a_1$.

Hence, global two-dimensional dynamics of the system always has to be taken into account to explain the transcritical riddling transition, unless $p \neq 0$.

Numerical experiments show that the structures Q in the phase space that appear and develop outside the diagonal $x = y$ determine the blowout transition as well. For a nearly symmetrical case $0.488... < p < 0.5$ we have a soft slightly asymmetrical transition. For $0.336... < p < 0.488...$ the blowout transition is hard and the loss of synchronization leads to a stable strongly asymmetrical regime developed from Q. For $0.3... < p < 0.336...$ the blowout transition is topologically hard (large asymmetrical deviations from the synchrony are possible) but metrically soft, as a trajectory repeatedly returns to synchrony $x = y$. Finally, for very large asynchrony $p < 0.3...$, after the blowout bifurcation, almost all trajectories escape to infinity.

We now discuss the effect of a small mismatches between both systems (i.e. slightly different mappings for the x and y variables in (2,3)) on the synchronization transition. With such a mismatch, the state $x = y$ is no longer invariant, but a nearly synchronous attractor (apparently chaotic) where $x \approx y$ still exists. We demonstrate that its fate is mediated by the “broken transcritical bifurcation”. Indeed, the transcritical bifurcation is structurally unstable [24,25] and is destroyed even by a small mismatch.

If we restrict our attention to the fixed point P_0, there are two possible scenaria depending on the sign of the mismatch, as shown in Fig. 4. In both, the synchronous one-dimensional attractor transforms into a thin two-dimensional invariant attracting region, as is schematically shown in Fig. 5. Its boundary is created by the unstable manifolds of P_0. In the case of Fig. 4a the saddle point P_0 that is on the boundary of the nearly synchronous attractor $x \approx y$ collides with the repellor P_2 in a saddle-node bifurcation, and disappears (Fig. 5a). For the nearly synchronous attractor this is the point of exterior or interior crisis depending on whether P_1 is attracting or not.

The situation of Fig. 4b appears at the first glance to be different, as here the point P_0 does not bifurcate at all. However, in this case a crisis occurs as well: At some critical value of coupling e_c the

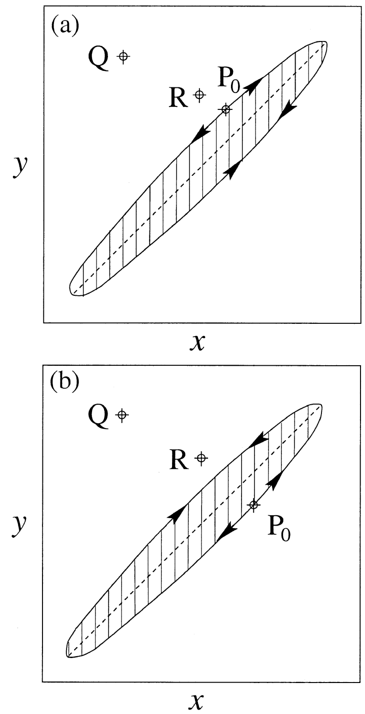

Fig. 5. A sketch of the phase portraits of the system with a small parameter mismatch before collision of the nearly synchronous attractor (crosshatched) with the repelling fixed point P_2. The two cases correspond to Fig. 4.

unstable point P_2 collides with the unstable manifold of the point P_0 (this manifold constitutes a part of the border of nearly synchronous attractor, see Fig. 5b). Beyond e_c the nearly synchronous attractor is destroyed and the attracting set is P_1 (if it is stable), the absorbing area, or infinity (if the system is beyond the absorbing area crisis). We see that any mismatch transforms the riddling loss of synchronization into a crisis transition to an asynchronous state.

In conclusion, we have discussed peculiarities of the desynchronization transition for a system with symmetrical synchronous state and asymmetrical coupling. The transitions above can be directly followed in ensembles of coupled oscillators of type (1) as clustering transitions. Although we have considered only the transitions between fully synchronized and two-cluster states, the main ideas can be applied to more complex configurations as well. Indeed, suppose that in system (1) there is a K-cluster state. Any of existing clusters can be splitted in new ones, yielding a state with $K+1$ clusters. This transition can be described using a $(K+1)$-dimensional mapping similar to (2,3), and in this high-dimensional system it may be a transcritical transition, because a symmetric (with two variables coinciding) solution looses its symmetry in a non-symmetric way. If this transition is mediated by a periodic orbit that first becomes transversally unstable, then the description above is fully applied.

We have demonstrated that the asymmetry of the coupling is important if, at the stability threshold, the transversal multiplier is $+1$ (which corresponds to a pitchfork bifurcation in the fully symmetrical case), while the period doubling transition, for multiplier -1, is rather insensitive to asymmetry. The pitchfork bifurcation is transformed into a transcritical one due to asymmetry, and the riddling transition is shown to be typically determined by the hard, i.e. far located, branch. In practice, this means that due to noise, asynchronous bursts of finite level may be observed. In the presence of a small mismatch between the coupled oscillators the transcritical bifurcation is destroyed and the riddling transition changes its progressing. Indeed, the nearly synchronized state disappears via exterior or interior crises.

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