Parametric Excitation of Breathers in a Nonlinear Lattice

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We investigate localized periodic solutions (breathers) in a lattice of parametrically driven, nonlinear dissipative oscillators. These breathers are demonstrated to be exponentially localized, with two characteristic localization lengths. The crossover between the two lengths is shown to be related to the transition in the phase of the lattice oscillations.

1. Introduction

Recently discovered localized periodic excitations (breathers) in nonlinear lattices are attracting large interest now [1, 2, 3]. It can be proven that such solutions exist for general nonlinear chains of oscillators, at least in the limit of small coupling between oscillators [4, 5]. Here the idea of the anti-continuous limit, where one starts with the dynamics of the uncoupled oscillators, is important. In the uncoupled chain, a localized solution can be constructed in a straightforward way by setting one oscillator in the periodic regime, while all the others are exactly at rest (at a fixed point). Such a solution can be continued for nonvanishing coupling, provided some non-resonance conditions are fulfilled [4, 5].

The localized solution is exact in the Hamiltonian situation, with dissipation it can only be metastable. The goal of this paper is to show that with parametric excitation breathers can exist as exact solutions even in the presence of dissipation. Qualitatively, this can be understood from considering the anti-continuous limit. With dissipation and without external forcing the only steady state of a nonlinear oscillator is a stable fixed point. To construct a nontrivial localized solution we must have, in addition to the fixed point, another attractor in the phase space of a single oscillator. It is well-known that such a bistability appears if a nonlinear oscillator is excited parametrically. A bistable behavior can be also observed if a nonlinear oscillator is subject to a periodic external force. In this latter case, however, the zero fixed point is not a solution. Thus, in a chain of uncoupled parametrically excited oscillators we can prepare such a state (choosing appropriate initial conditions) that all oscillators but one are at rest. One can expect that for at least small coupling such a state results in a localized breather. Below, we demonstrate this numerically for a particular model.

2. The Model

We consider a chain of nonlinear oscillators subject to a spatially homogeneous time-periodic parametric driving. One can think of a chain of spin oscillators in a periodic magnetic field [6] or of a chain of pendula hanging at a common bar which is moving vertically. The governing equations of motion are

$$\begin{aligned} \dot{q}_i &= p_i, \\ \dot{p}_i &= -2dp_i - (1 - h\cos(\omega_0 t))q_i - \gamma q_i^3 \qquad (1) \\ &\quad + c(q_{i+1} - 2q_i + q_{i-1}) \end{aligned}$$

The boundary conditions can be chosen to be periodic or open. We suppose the local potential to have the form $V(q_i) = \frac{1}{2}q_i^2 + \frac{1}{4}\gamma q_i^4$, another realistic possibility would be the potential $V(q) = \sin q$. The parameters are the strength of driving h, the damping constant d, the driving frequency ω_0, and the coupling constant c. We restrict our consideration to small valuesof h and d, where the behavior is relatively simple [7].

The dynamics of the single oscillator system (of which we have identical copies at each site in setting $c = 0$) for small values of h and d is well-known, it can easily be obtained from the asymptotic method [7]. The state $q = 0$, $p = 0$ exists for all values of the parameters; it is unstable within the region of parametric resonance $\omega_{0,1} < \omega_0 < \omega_{0,2}$. A non-

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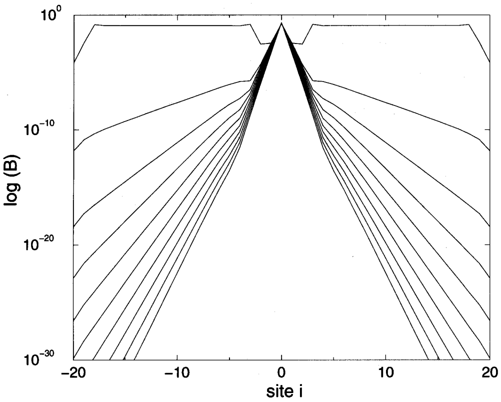

Fig. 1. Spatial decay profile for different couplings c in the lattice described by (1). The parameters are $d = 0.05$, $h = 0.3$, $\omega_0 = 2.25$. From bottom to top, the couplings are $c = 0.016$, $c = 0.018$, $c = 0.020$, $c = 0.022$, $c = 0.025$, $c = 0.028$, $c = 0.032$, $c = 0.035$, $c = 0.040$, $c = 0.046$. For couplings above $c_{cr} \approx 0.04$ the localization property gets lost.

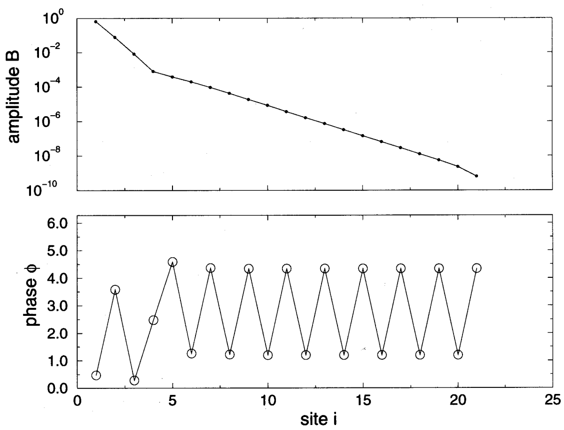

Fig. 2. The amplitude and the phase (relative to the phase of driving) for the lattice with parameters as in Fig. 1 and the coupling $c = 0.035$. One can see clearly the transition in the phase and the associated bend in the amplitude vs. space curve.

zero stable periodic solution (with frequency $\omega_0/2$) exists in the larger region $\omega_{0,1} < \omega_0 < \omega_{0,3}$. Here the asymptotic formulae for $\omega_{0,i}$ are

$$\omega_{0,1}^2 = 4 - 2\sqrt{h^2 - 16d^2},$$
$$\omega_{0,2}^2 = 4 + 2\sqrt{h^2 - 16d^2},$$
$$\omega_{0,3}^2 = \frac{h^2}{4d^2}$$

in the leading order of h and d. Thus, for certain parameter values we have a bistable situation in the region $\omega_{0,2} < \omega_0 < \omega_{0,3}$: both the zero fixed point and the periodic solution are stable.

3. Results

We solved the Eqs. (1) numerically with a Runge-Kutta method, using periodic boundary conditions $q_L = q_0$ (L is the chain length). Since we want to consider a symmetric situation we use always an odd number of oscillators. To produce a localized solution, we take the parameters in the region of bistability described above, and initialize the oscillator at zero with a finite amplitude while the rest of the chain is set to have zero velocity and displacement. Skipping the first 1000 periods of the external force as transients, we calculate the spatial oscillation profile in the chain for different coupling constants.

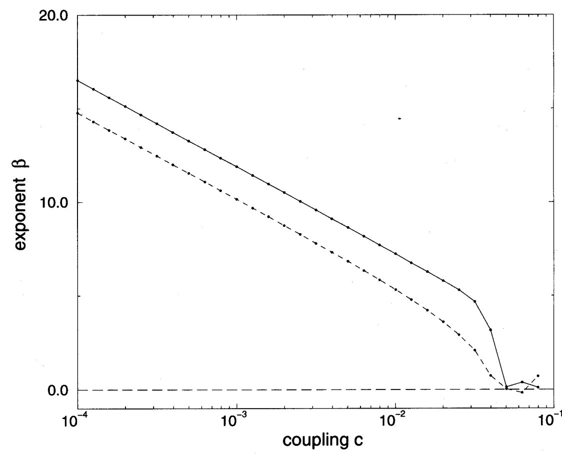

Fig. 3. Dependence of the decay exponent on the coupling for the lattice with parameters as in Fig. 1. The full line corresponds to the exponent for strong (small values of $|i - i_c|$), the dashed line for weak decay (large values of $|i - i_c|$).

The results are displayed in Figs. 1 and 3. One can see that the localization is exponential $B_i \sim \exp(-\beta|i - i_c|)$ where $B_i = \langle q_i^2 \rangle - \langle q_i \rangle^2$ is the variance of the oscillations at site i. Remarkably, the exponent β is not a constant but has different values for small and large distances from the central point i_c. This effect is clearer shown in Figure 2. Here we display together with the amplitude also the phase of oscillations. One can see the crossover between two states:

for small $i - i_c$ the phase takes the values ϕ_0, $\phi_0 + \pi$ and the exponent β is large, while for larger $i - i_c$ the phases ϕ_1 and $\phi_1 + \pi$ are observed and the exponent β is small. To explain this, we write the approximate equations for the lattice using the asymptotic method (method of averaging) [7]. For the complex amplitude, defined as

$$q_i = A_i e^{i\frac{\omega_0}{2}t} + \text{c.c.}$$

we then obtain

$$\dot{A}_i = -dA_i - \frac{i}{\omega_0}(\frac{\omega_0^2}{4} - 1)A_i - \frac{ih}{2\omega_0}A_i^* + \frac{i3\gamma}{\omega_0}|A_i|^2 A_i - \frac{ic}{\omega_0}(A_{i+1} + A_{i-1} - 2A_i).$$

Consider first the case of a single parametrically excited oscillator. Then, representing $A = ae^{i\phi}$ one gets the following relation for the stable solution (it is assumed that $\gamma > 0$):

$$\sin 2\phi_s = -\frac{2\omega_0 d}{h}, \quad \cos 2\phi_s = (1 - \frac{4d^2\omega_0^2}{h^2})^{1/2} \quad (2)$$

Consider next the oscillations in a linear lattice ($\gamma = 0$), assuming the phase shift to be π between neighboring sites and the exponential decay of the amplitude $a_{i+1}/a_i = \exp(-\beta)$. We get then for the phase

$$\sin 2\phi_1 = -\frac{2\omega_0 d}{h}, \quad \cos 2\phi_1 = \frac{2}{h}(1 - \frac{\omega_0^2}{4} + c(e^{\beta} + e^{-\beta} + 2)). \quad (3)$$

One can see from (2) and (3) that for $\omega_0 > 2$ and small enough c the relation $\phi_s = \pi - \phi_1$ holds. This is exactly the case shown in Figure 2. We can identify the phase of the central oscillator ϕ_0 with ϕ_s (doing this we neglect the influence of the neighbors), and the phase far away from the center ϕ_1 with the linear phase ϕ_1. Because $\phi_s \neq \phi_1$, the crossover from one regime to another one is observed.

The approach above allows one also to estimate the critical value of coupling at which the localized state disappears. Assuming $\beta = 0$ in (3), we obtain

$$c_{cr} = \frac{\omega_0^2 - 4}{16} - \frac{1}{8}(h^2 - 4\omega_0^2 d^2)^{1/2} .$$

For the parameters used this gives $c_{cr} = 0.042$ to be compared with our numerical estimate $c_{cr} = 0.04$.

4. Discussion

We have described a way to excite a stable localized periodic excitation (breather) in a nonlinear lattice with dissipation. The parametric forcing allows one to compensate dissipative losses, while not destroying the localized nature of the solution. Clearly, in contrast to the purely conservative case, we do not obtain a family of solutions, but an isolated solution completely determined by the parameters of the lattice and of the driving. The interesting feature of the parametrically excited breather is the existence of two localization exponents, with a crossover between them. We have explained this phenomenon as caused by different phases of the lattice oscillations in the nonlinear and linear regimes. A possibility to observe more complex (may be even chaotic) breathers remains a subject for further investigations.

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